ON MULTIFUNCTIONS WITH CLOSED GRAPHS

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Abstract. The set of points of upper semicontinuity of multi-valued mappings with a closed graph is studied. A topology on the space of multi-valued mappings with a closed graph is introduced.

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1. Introduction

In what follows let X, Y be Hausdorff topological spaces. A (multi-valued) mapping $F\colon Y \to X$ is said to be upper semicontinuous (USC) at y_0, if for every open set O which includes $F(y_0)$ there exists a neighbourhood V of y_0 such that $O \supset F(V)(=\bigcup\{F(y)\colon y \in V\}$.

The outer part of F at y_0 is the mapping $\widetilde{F}\colon Y \to X$ given by $\widetilde{F}(y) = F(y) \setminus F(y_0)$ (see [4]). By the (outer) active boundary of F at y_0 ($\operatorname{Frac} F(y_0)$) we understand

$$\operatorname{Frac} F(y_0) = \bigcap\{\operatorname{Cl} \widetilde{F}(V)\colon V \in \mathcal{B}(y_0)\},$$

where $\mathcal{B}(y_0)$ stands for a neighbourhood base at y_0 (see [3]).

A mapping $F\colon Y \to X$ is said to be c-upper semicontinuous at y_0 [1] if for every open set V containing $F(y_0)$ and having a compact complement, there is an open neighbourhood U of y_0 such that $F(U) \subset V$. We say that $F\colon Y \to X$ is c-upper semicontinuous if it is c-upper semicontinuous at every $y \in Y$.

As usual, we say that $F\colon Y \to X$ has a closed graph if $G(F) = \{(y,x)\colon x \in F(y);\ y \in Y\}$ is a closed subset of $Y \times X$ with the product topology. Of course if $F\colon Y \to X$ has a closed graph then F is c-upper semicontinuous and also $\operatorname{Frac} F(y) \subset$

$F(y)$ for every $y \in Y$. We will see later that in general the opposite is not true. Let us mention here also papers [7], [8], [9], [11], which deal with the above properties of multifunctions.

Let $(X, \mathcal{U})$ be a uniform space. We say that a multi-valued mapping F is totally bounded at y_0, if to every $U \in \mathcal{U}$ there corresponds a neighbourhood V of y_0 and a finite set C such that $F(V \setminus \{y_0\}) \subset U[C]$, where $U[C] = \bigcup\{U[c]\colon\ c \in C\}$.

Combining Theorems 8.1 and 10.1 from [4] we obtain the following result:

Theorem A. *Let Y be first countable at y_0, let $(X, \mathcal{U})$ be a complete uniform space and $F\colon\ Y \to X$ a multi-valued mapping. Suppose that $F(y_0)$ is closed. Then F is USC at y_0 if and only if* (i) $\operatorname{Frac} F(y_0) \subset F(y_0)$ *and* (ii) *the outer part of F at y_0 is totally bounded at y_0.*

1. It is easy to see that one implication of the above theorem holds under weaker conditions.

Lemma 1.1. *Let $(X, \mathcal{U})$ be a complete uniform space, Y a topological space and $F\colon\ Y \to X$ a multi-valued mapping. If* $\operatorname{Frac} F(y_0) \subset F(y_0)$ *and the outer part of F at y_0 is totally bounded at y_0 then F is USC at y_0.*

P r o o f. Suppose F is not USC at y_0. There is an open set $O \subset X$ such that for every neighbourhood V of y_0 we have $F(V) \setminus O \neq \emptyset$. Let $\mathcal{B}(y_0)$ denote the family of all open neighbourhoods of y_0. For every $V \in \mathcal{B}(y_0)$ choose $x_V \in F(V) \setminus O$. Now the set $\bigcap\limits_{V \in \mathcal{B}(y_0)} \overline{\{x_G\colon\ G \subset V\}}$ is nonempty, since $(X, \mathcal{U})$ is complete and for every $U \in \mathcal{U}$ there is $V \in \mathcal{B}(y_0)$ such that $\overline{\{x_G\colon\ G \subset V\}} \subset U[C]$, where C is a finite set in X.

$\square$

The completeness of X in the previous lemma is essential.

Proposition 1.2. *Suppose $(X, \mathcal{U})$ is not complete. Then there are a topological space Y, a point $y_0 \in Y$ and a mapping $F\colon\ Y \to X$ such that*

(i) $\operatorname{Frac} F(y_0) \subset F(y_0)$,
(ii) *the outer part of F at y_0 is totally bounded at y_0,*
(iii) *F is not USC at y_0.*

P r o o f. There is a Cauchy net $\{x_\alpha\colon\ \alpha \in A\}$ with no cluster point in X. Let X^* be a completion of X. There is a point $x \in X^*$ such that $\{x_\alpha\colon\ \alpha \in A\}$ converges to x. Put $Y = \{x_\alpha\colon\ \alpha \in A\} \cup \{x\}$ and for each $\alpha \in A$ put $V_\alpha = \{x_\beta\colon\ \beta \geqslant \alpha\} \cup \{x\}$. The system $\tau = \{K \subset Y\colon\ x \notin K\} \cup \{V_\alpha\colon\ \alpha \in A\}$ is a base of a topology $\mathcal{G}$. Define a mapping $F\colon\ (Y, \mathcal{G}) \to X$ as follows: $F(x_\alpha) = x_\alpha$ for each $\alpha \in A$ and $F(x) = x_0$, where x_0 is a point from X. It is easy to see that F is totally bounded at x. (Let

U be a symmetric element from $\mathcal{U}$. There is $\alpha \in A$ such that $(x_\alpha, x_\beta) \in U$ for each $\beta \geqslant \alpha$. Thus V_α is a neighbourhood of x for which $F(V_\alpha \setminus \{x\}) \subset U[x_\alpha])$.

Since $\{x_\alpha\colon \alpha \in A\}$ has no cluster point in X, $\operatorname{Frac} F(x) = \emptyset$. But F is not upper semicontinuous. (There are an open set V in X and $\alpha \in A$ such that $x_0 \in V$ and $x_\beta \notin V$ for each $\beta \geqslant \alpha$. Thus the set $\{z \in Y\colon F(z) \subset V\}$ is not open in Y). □

Proposition 1.3. *Let X, Y be topological spaces. Then $F\colon Y \to X$ has a closed graph if and only if for every $y \in Y$ the following conditions are satisfied:*

(i) $\operatorname{Frac} F(y) \subset F(y)$,

(ii) *$F(y)$ is a closed set in X.*

Proof. Of course the "only if" implication is trivial. To prove the other one, we will suppose that (i) and (ii) are satisfied for all $y \in Y$ and F does not have a closed graph. Thus there is $(y_0, x) \in \overline{G(F)} \setminus G(F)$. We claim that $x \in \operatorname{Frac} F(y_0)$. Let $V \in \mathcal{B}(y_0)$ and $G \in \mathcal{B}(x)$ with $G \subset X \setminus F(y_0)$, where $\mathcal{B}(y_0)$, and $\mathcal{B}(x)$ are the families of all neighbourhoods of y_0, x, respectively. Since $(y_0, x) \in \overline{G(F)}$ we have $(V \times G) \cap G(F) \neq \emptyset$. Choose $(v, g) \in V \times G$ and $(v, g) \in G(F)$; $g \in G$ implies that $g \notin F(y_0)$, i.e. $g \in \widetilde{F}(v)$. Thus $x \in \operatorname{Frac} F(y_0)$ and, by the assumption, $x \in F(y_0)$, a contradiction. □

Proposition 1.4. *Let X a locally compact space and Y be a topological space. Let $F\colon Y \to X$ be c-upper semicontinuous with closed values. Then F has a closed graph.*

Proposition 1.5. *Let X, Y be first countable topological spaces. Let $F\colon Y \to X$ be c-upper semicontinuous with closed values. Then F has a closed graph.*

Proof. Suppose the implication is not true. Thus there is $(y, x) \in \overline{G(F)} \setminus G(F)$; i.e. $x \notin F(y)$. Since X, Y are first countable, there are sequences $\{y_n\}_n$ in Y and $\{x_n\}_n$ in X such that $\{(y_n, x_n)\}_n$ converges to (y, x) and $(y_n, x_n) \in G(F)$ for every $n \in \mathbb{N}$. The openess of $X \setminus F(y)$ and the convergence of $\{x_n\}_n$ to x imply that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin F(y)$ for every $n \geqslant n_0$. Now $K = \{x_n\colon n \geqslant n_0\} \cup \{x\}$ is a compact set in X with $F(y) \cap K = \emptyset$. The c-upper semicontinuity of F at y implies that there is an open neighbourhood V of y with $F(V) \cap K = \emptyset$, a contradiction. □

The following example shows that the first countability of X in the above proposition is essential.

Example 1.6. *For every $n \in \mathbb{N}$ let $\{y_k^n\}_k$ be a sequence of different points from $(1/(n+1), 1/n)$ convergent to $1/(n+1)$. Consider $Y = \{0\} \cup \{y_k^n\colon k, n \in \mathbb{N}\}$ and let τ be the inherited euclidean topology on Y. Let further $\mathcal{G}$ be the following topology* *on Y: all points y_k^n are isolated. For every $n \in \mathbb{N}$ put $E_n = \{y_k^n\colon k \in \mathbb{N}\}$. The basic neighbourhoods of* 0 *in $\mathcal{G}$ are sets obtained from Y by removing finitely many E_n's and a finite number of points in all the remaining E_n's. It is easy to verify that the function $f\colon (Y,\tau) \to (Y,\mathcal{G})$ defined by $f(0) = 1$ and $f(y) = y$ for all $y \neq 0$ is c-upper semicontinuous, since compact sets in $(Y,\mathcal{G})$ are finite, but the graph of f is not closed since $(0,0) \in \overline{G(f} \setminus G(f)$.*

2. Let $(X,\mathcal{U})$ be a uniform space. The smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for $\mathcal{U}$, is called the weight of the uniformity $\mathcal{U}$ and is denoted by $w(\mathcal{U})$. ($|\mathcal{B}|$ denotes the cardinality of $\mathcal{B}$).

A system is a synonym for an indexed family. If m is a cardinal number, then an m-system is a system whose index set has cardinality m.

A subset G of a space Y is said to be a $G(m)$-subset of Y, if it is the intersection of an open m-system in Y (see [6]), where an open family of a space Y is a family consisting of open subsets of Y.

Now we state the main theorem:

Theorem 2.1. *Let $(X,\mathcal{U})$ a complete uniform space and Y be a topological space. Let $F\colon Y \to X$ be a multi-valued mapping with totally bounded values and such that $\operatorname{Frac} F(y) \subset F(y)$ for every $y \in Y$. Then the set of points of upper semicontinuity of F is a (possibly empty) $G(m)$-subset of Y, where $m = w(\mathcal{U})$.*

P r o o f. Put $\Omega = \{y \in Y\colon F$ is totally bounded at $y\}$. By Lemma 1.1, F is USC at each point from Ω. Now let $y \in Y$ be such that F is USC at y. Let $U \in \mathcal{U}$. There is an element U_1 from $\mathcal{U}$ such that $U_1 \odot U_1 \subset U$. The upper semicontinuity of F at y implies that there is a neighbourhood V of y such that $F(V) \subset U_1[F(y)]$. The total boundedness of $F(y)$ implies that there is a finite subset K of X such that $F(y) \subset U_1[K]$, thus $F(V) \subset U[K]$, i.e. $y \in \Omega$.

Now let $\mathcal{B}$ be a base for $\mathcal{U}$ such that $|\mathcal{B}| = w(\mathcal{U})$. For each $B \in \mathcal{B}$ put $G_B = \{y \in Y\colon$ there are $V \in \mathcal{B}(y)$ and a finite set $K \subset X$ such that $F(V) \subset B[K]\}$, where $\mathcal{B}(y)$ denotes a family of all neighbourhoods of y. It is easy to verify that G_B is open for each $B \in \mathcal{B}$. Thus $\Omega = \bigcap\{G_B\colon B \in \mathcal{B}\}$, i.e. the set of points of upper semicontinuity of F is a $G(m)$-subset of Y, where $m = w(\mathcal{U})$. $\square$

Corollary 2.2. *Let (X,ϱ) be a complete metric space and Y a topological space. Let $F\colon Y \to X$ be a multi-valued mapping with totally bounded values and such that $\operatorname{Frac} F(y) \subset F(y)$ for every $y \in Y$. Then the set of points of upper semicontinuity of F is a (possibly empty) G_δ-subset of Y.*

Recall that a (completely regular) space is metric topologically complete if its topology admits a complete metric.

Theorem 2.3. *Let (X,ϱ) be a metric space. X is metric topologically complete if and only if for each topological space Y and for each compact-valued mapping $F\colon Y\to X$ with a closed graph the set of points of upper semicontinuity of F is a G_δ-subset of Y.*

P r o o f. $\Rightarrow$ Suppose that a metric space (X,ϱ) is metric topologically complete and $F\colon Y\to X$ is a compact-valued mapping with a closed graph. There is a complete metric d in X topologically equivalent to ϱ. It is easy to see that the multi-valued mapping $F\colon Y\to(X,d)$ satisfies the assumptions of Corollary 2.2. Thus the set of points of the upper semicontinuity of $F\colon Y\to(X,\varrho)$ is a G_δ-subset of Y.

$\Leftarrow$ Let (X^*,ϱ^*) be a completion of (X,ϱ). Put $Y=(X^*,\varrho^*)$. Choose $y_0\in X$ and define the multi-valued mapping $F\colon Y\to X$ as follows: $F(y)=\{y,y_0\}$ if $y\in X$ and $F(y)=\{y_0\}$ otherwise.

Then the set of points of upper semicontinuity of F is the set X. (Let $y\notin X$. There are open sets U,V in Y such that $y\in U$, $y_0\in V$ and $U\cap V=\emptyset$. For each $z\in U\cap X$, we have $F(z)\cap(X\setminus V)\neq\emptyset$, i.e. F is not USC at $y\notin X$). It is easy to verify that F is a compact-valued mapping with a closed graph. By assumption X is a G_δ-subset of $Y=(X^*,\varrho^*)$, i.e. X is metric topologically complete. □

3. In this part we assume that the range space of multi-valued mappings possesses a complete m-system of open coverings.

A centered family is a family of sets having the finite intersection property.

Definition 3.1 (See [6]). A system $\{\mathcal{B}_i\colon i\in I\}$ of open coverings of a space X is said to be complete if the following condition is satisfied: If $\mathcal{U}$ is an open centered family in X such that $\mathcal{U}\cap\mathcal{B}_i\neq\emptyset$ for each $i\in I$ then $\bigcap\limits_{U\in\mathcal{U}}\overline{U}\neq\emptyset$.

It follows from Theorem 2.8 in [6] that if X is a Čech-complete space then X possesses a complete countable system of open coverings of X.

Proposition 3.2 (See [6]). *Let $\{\mathcal{B}_i\colon i\in I\}$ be a complete system of open coverings of a regular space X. Suppose that $\mathcal{M}$ is a centered family of subsets of X such that for each $i\in I$ there exists an $M\in\mathcal{M}$ and a finite subfamily $\mathcal{U}_i$ of $\mathcal{B}_i$ which covers M. Then $\bigcap\limits_{M\in\mathcal{M}}\overline{M}\neq\emptyset$.*

Theorem 3.3. *Let X, Y be topological spaces and let X be a regular space with a complete m-system of open coverings. Let $F\colon Y\to X$ be a compact-valued mapping with a closed graph. The set of points of upper semicontinuity of F is a (possibly empty) $G(m)$-subset of Y.*

P r o o f. First denote by $\Omega(F)$ the set of points of Y at which F is upper semicontinuous. Let $\{\mathcal{B}_i\colon i\in I\}$ be a complete m-system of open coverings of X.

For each $i \in I$ put $G_i = \{y \in Y\colon$ there are $V \in \mathcal{B}(y)$ and a finite subfamily $\mathcal{U}_i$ of $\mathcal{B}_i$ which covers $F(V)\}$.

Then of course $G(i)$ is open for every $i \in I$. Now we prove that $\Omega(F) = \bigcap_{i\in I} G(i)$. Let $y \in \Omega(F)$. Let $i \in I$. The compactness of $F(y)$ implies that there is a finite subfamily $\mathcal{U}_i$ of $\mathcal{B}_i$ such that $F(y) \subset \bigcup_{U\in\mathcal{U}_i} U$. The upper semicontinuity of F at y implies that there is $V \in \mathcal{B}(y)$ such that $F(V) \subset \bigcup_{U\in\mathcal{U}_i} U$; i.e. $y \in G_i$. Thus we have proved that $\Omega(F) \subset \bigcap_{i\in I} G(i)$.

Now let $y \in \bigcap_{i\in I} G(i)$ and suppose that F is not upper semicontinuous at y. There is an open set H in Y such that $F(y) \subset H$ and $F(V) \cap (X \setminus H) \neq \emptyset$ for every $V \in \mathcal{B}(y)$.

For every $V \in \mathcal{B}(y)$ choose $y_V \in V$ and $x_V \in F(y_V) \setminus H$. For every $V \in \mathcal{B}(y)$ put $M(V) = \overline{\{x_U\colon U \subset V\}}$. By Proposition 3.2 we have $\bigcap_{V\in\mathcal{B}(y)} M(V) \neq \emptyset$; choose a point x from this intersection. Then of course $x \notin H$ and it is easy to verify that $(y,x) \in \overline{G(F)}$; i.e. $(y,x) \in G(F)$, a contradiction.

$\square$

Corollary 3.4 (See [10]). *Let X, Y be topological spaces and let X be Čech-complete. Let $F\colon Y \to X$ be a compact-valued mapping with a closed graph. Then the set of points of upper semicontinuity of F is a G_δ subset of Y.*

Corollary 3.5. *Let X, Y be topological spaces and let X be locally compact. Let $F\colon Y \to X$ be a compact-valued mapping with a closed graph. Then the set of points of upper semicontinuity of F is open.*

Notice that even if Y is compact and X locally compact hemicompact, the set of points of upper semicontinuity of a compact-valued mapping $F\colon Y \to X$ with a closed graph can be empty. (See Example 2 in [1]).

4. Denote by $M(Y,X)$ the space of all closed-valued mappings F from Y to X such that $F(y) \neq \emptyset$ for every $y \in Y$. If we equip the space of all nonempty closed subsets $CL(X)$ of X with a uniformity, we can consider the uniform topology on compacta on $M(Y,X)$. Of course we have many possibilities to do this; if X is a metric space, we can take the Hausdorff metric on $CL(X)$, or a uniformity of the Wijsman topology, if X is a locally compact space, then also the Fell topology is uniformizable [2]. In this paper we will consider a uniformity of the Fell topology.

For a subset A of X put

$$A^- = \{B \in CL(X)\colon B \cap A \neq \emptyset\} \text{ and } A^+ = \{B \in CL(X)\colon B \subset A\}.$$

Definition 4.1 (See [2]). Let X be a Hausdorff topological space. The Fell topology τ_F on $CL(X)$ has as a subbase all sets of the form V^-, where V is a nonempty open subset of X, plus all sets of the form W^+, where W is a nonempty open subset of X with a compact complement.

It is known [2] that if X is a locally compact space, then the Fell topology on $CL(X)$ is uniformizable and the following family of sets forms a base for a uniformity $\mathcal{F}$ compatible with the Fell topology on $CL(X)$. Let $\mathcal{U}$ be a compatible uniformity on X and let $K(X)$ stand for the family of all nonempty compact subsets of X. For $K \in K(X)$ and $U \in \mathcal{U}$, write

$$[K,U] = \{(A_1,A_2) \in CL(X) \times CL(X)\colon\ A_1 \cap K \subset U[A_2] \text{ and } A_2 \cap K \subset U[A_1]\}.$$

Then the family $\mathcal{F} = \{[K,U]\colon\ K \in K(X), U \in \mathcal{U}\}$ forms a base for a uniformity compatible with the Fell topology on $CL(X)$.

Now the basic open sets for the uniform topology on compacta τ_K on $M(Y,X)$ are the sets

$$\langle F, A, [K,U]\rangle = \{G \in M(Y,X)\colon\ (G(y),F(y)) \in [K,U] \text{ for all } y \in A\},$$

where $F \in M(Y,X)$, $A \in K(Y)$, $K \in K(X)$, $U \in \mathcal{U}$.

Theorem 4.2. *Let X, Y be locally compact Hausdorff topological spaces. Then the space $U(Y,X)$ of multi-valued mappings with closed graphs is a closed subset of $(M(Y,X),\tau_K)$.*

Proof. We show that $M(Y,X) \setminus U(Y,X)$ is open in τ_K. Since X is a locally compact Hausdorff topological space, by Proposition 1.4 the notion of a multi-valued mapping with a closed graph is equivalent to the notion of a c-upper semicontinuous multifunction.

Let $F \in M(Y,X) \setminus U(Y,X)$. There is a point $p \in Y$ such that F is not c-upper semicontinuous at p. Let A be a compact set in Y with $p \in \operatorname{Int} A$. There is $K \in K(X)$ such that $F(p) \in (K^c)^+$ and $F(V) \cap K \neq \emptyset$ for every $V \in \mathcal{B}(p)$, where $\mathcal{B}(p)$ stands for the family of all neighbourhoods of p.

For every $V \in \mathcal{B}(p)$ such that $V \subset A$ choose $y_V \in V$ and $x_V \in F(y_V) \cap K$. The compactness of K implies that there is a cluster point $x \in K$ of $\{x_V\colon\ V \in \mathcal{B}(p), V \subset A\}$. Let $H \in \mathcal{U}$ be such that $H[F(p)] \cap H[K] = \emptyset$ and $\overline{H[K]}$ is compact. Let $H_1 \in \mathcal{U}$ be such that $H_1 \circ H_1 \subset H$. We claim that the set $L = \{G \in M(Y,X)\colon\ (G(y),F(y)) \in [\overline{H_1[K]}, H_1]$ for every $y \in A\} \cap \{G \in M(Y,X)\colon\ (G(y),F(y)) \in [K,H_1]$ for every $y \in A\}$ is a subset of $M(Y,X) \setminus U(Y,X)$.

Let $G \in L$. Then $G(p) \cap \overline{H_1[K]} = \emptyset$ and for each $V \in \mathcal{B}(p)$ with $V \subset A$ there is $z_v \in G(y_V) \cap H_1[K]$; i.e. G is not c-upper semicontinuous at p. □

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