# THE FORCING DIMENSION OF A GRAPH

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*Abstract.* For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex $v$ in a connected graph $G$, the (metric) representation of $v$ with respect to $W$ is the $k$-vector $r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$, where $d(x,y)$ represents the distance between the vertices $x$ and $y$. The set $W$ is a resolving set for $G$ if distinct vertices of $G$ have distinct representations. A resolving set of minimum cardinality is a basis for $G$ and the number of vertices in a basis is its (metric) dimension $\dim(G)$. For a basis $W$ of $G$, a subset $S$ of $W$ is called a forcing subset of $W$ if $W$ is the unique basis containing $S$. The forcing number $f_G(W, \dim)$ of $W$ in $G$ is the minimum cardinality of a forcing subset for $W$, while the forcing dimension $f(G, \dim)$ of $G$ is the smallest forcing number among all bases of $G$. The forcing dimensions of some well-known graphs are determined. It is shown that for all integers $a, b$ with $0 \leqslant a \leqslant b$ and $b \geqslant 1$, there exists a nontrivial connected graph $G$ with $f(G) = a$ and $\dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$.

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## 1. Introduction

The *distance* $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex $v$ of $G$, we refer to the $k$-vector

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$$

as the (*metric*) *representation of $v$ with respect to $W$*. The set $W$ is called a *resolving set* for $G$ if distinct vertices have distinct representations. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for $G$.

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The (*metric*) *dimension* $\dim(G)$ is the number of vertices in a basis for $G$. For example, the graph $G$ of Figure 1 has the basis $W = \{u, z\}$ and so $\dim(G) = 2$. The representations for the vertices of $G$ with respect to $W$ are $r(u|W) = (0, 1)$, $r(v|W) = (2, 1)$, $r(x|W) = (1, 2)$, $r(y|W) = (1, 1)$, $r(z|W) = (1, 0)$.

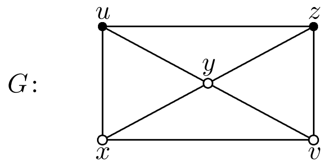


Figure 1. A graph $G$ with $\dim(G) = 2$

The example just presented also illustrates an important point. When determining whether a given set $W$ of vertices of a graph $G$ is a resolving set for $G$, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of $G$ whose distance from $w$ is 0. The following lemma will be used on several occasions. The proof of this lemma is routine and is therefore omitted.

**Lemma 1.1.** *Let $G$ be a nontrivial connected graph. For $u, v \in V(G)$, if $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $u$ and $v$ belong to every resolving set of $G$.*

The inspiration for these concepts stems from chemistry. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1, 2]. The dimension of directed graphs has been studied in [5, 6].

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [14] and later in [15], Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph $G$ as its location number. Independently, Harary and Melter [11] investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted.

For a basis $W$ of $G$, a subset $S$ of $W$ with the property that $W$ is the unique basis containing $S$ is called a *forcing subset* of $W$. The *forcing number* $f_G(W, \dim)$ of $W$ in $G$ is the minimum cardinality of a forcing subset for $W$, while the *forcing dimension* $f(G, \dim)$ of $G$ is the smallest forcing number among all bases of $G$. Since the parameter dimension is understood in this context, we write $f_G(W)$ for $f_G(W, \dim)$

and $f(G)$ for $f(G, \dim)$. Hence if $G$ is a graph with $f(G) = a$ and $\dim(G) = b$, then $0 \leqslant a \leqslant b$ and there exists a basis $W$ of cardinality $b$ containing a forcing subset of cardinality $a$. Forcing concepts have been studied for a various of subjects in graph theory, including such diverse parameters as the chromatic number [9], the graph reconstruction number [12], and geodetic concepts in graphs [3, 7, 8]. Also, many invariants arising from the study of forcing in graph theory offer abundant new subjects for new and applicable research. A survey of graphical forcing parameters is discussed in [10].

To illustrate these concepts, we consider the graph $G$ of Figure 2. The graph $G$ has dimension 2 and so $f(G) \leqslant 2$. Let $W = \{x, z\}$ and $W' = \{v, z\}$. Since $r(s|W) = (2,1)$, $r(t|W) = (1,2)$, $r(u|W) = (1,3)$, $r(v|W) = (2,2)$, and $r(y|W) = (1,1)$, it follows that $W$ is a basis of $G$. Also, since $r(s|W') = (1,1)$, $r(t|W') = (1,2)$, $r(u|W') = (1,3)$, $r(x|W') = (2,2)$, and $r(y|W') = (3,1)$, the set $W'$ is a basis of $G$. Hence $1 \leqslant f(G) \leqslant 2$ by Lemma 1.2. Next we show that $f_G(W) = 1$ and $f_G(W') = 2$. Let $S_1 = \{x, s\}$, $S_2 = \{x, t\}$, $S_3 = \{x, u\}$, $S_4 = \{x, v\}$, and $S_5 = \{x, y\}$. Observe that $r(u|S_1) = r(y|S_1) = (1,2)$, $r(s|S_2) = r(v|S_1) = (2,1)$, $r(t|S_3) = r(y|S_3) = (1,2)$, $r(t|S_4) = r(u|S_4) = (1,1)$, and $r(u|S_5) = r(t|S_5) = (1,2)$. Hence $W$ is the unique basis containing $x$ and so $f_G(W) = 1$. Certainly, $W'$ is not the unique basis containing $z$ since $z \in W$. Moreover, $W'' = \{v, s\}$ is a basis in $G$ containing $v$ and so $W'$ is not the unique basis containing $v$. Hence $W'$ is not the unique basis containing any of its proper subset and so $f_G(W') = 2$. Now the forcing dimension $f(G)$ of $G$ is the smallest forcing number among all bases of $G$ an so $f(G) = 1$.

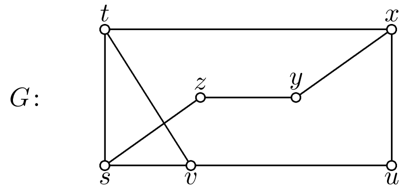

Figure 2. A graph $G$ with $\dim(G) = 2$ and $f(G) = 1$

It is immediate that $f(G) = 0$ if and only if $G$ has a unique basis. If $G$ has no unique basis but contains a vertex belonging to only one basis, then $f(G) = 1$. Moreover, if for every basis $W$ of $G$ and every proper subset $S$ of $W$, the set $W$ is not the unique basis containing $S$, then $f(G) = \dim(G)$. We summarize these observations below.

**Lemma 1.2.** *For a graph $G$, the forcing dimension $f(G) = 0$ if and only if $G$ has a unique basis, $f(G) = 1$ if and only if $G$ has at least two distinct bases but some* *vertex of $G$ belongs to exactly one basis, and $f(G) = \dim(G)$ if and only if no basis of $G$ is the unique basis containing any of its proper subsets.*

## 2. Forcing dimensions of certain graphs

The following three theorems (see [2], [11], [14], [15]) give the dimensions of some well-known classes of graphs. In this section, we determine the forcing dimensions of these graphs.

**Theorem A.** *Let $G$ be a connected graph of order $n \geqslant 2$.*

(a) *Then $\dim(G) = 1$ if and only if $G = P_n$.*
(b) *Then $\dim(G) = n - 1$ if and only if $G = K_n$.*
(c) *For $n \geqslant 3$, $\dim(C_n) = 2$.*
(d) *For $n \geqslant 4$, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geqslant 1)$, $G = K_r + \overline{K_s}$ $(r \geqslant 1,\ s \geqslant 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$.*

A vertex of degree at least 3 in a tree $T$ is called a *major vertex*. An end-vertex $u$ of $T$ is said to be a *terminal vertex of a major vertex* $v$ of $T$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $T$. The *terminal degree* $\operatorname{ter}(v)$ of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $T$ is an *exterior major vertex* of $T$ if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of $T$ and let $\operatorname{ex}(T)$ denote the number of exterior major vertices of $T$.

**Theorem B.** *If $T$ is a tree that is not a path, then $\dim(G) = \sigma(T) - \operatorname{ex}(T)$.*

**Theorem C.** *Let $T$ be a tree of order $n \geqslant 3$ that is not a path having $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of $v_i$, and let $P_{ij}$ be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$. Suppose that $W$ is a set of vertices of $T$. Then $W$ is a basis of $T$ if and only if $W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p)$ with exactly one exception for each $i$ with $1 \leqslant i \leqslant p$ and $k_i \geqslant 2$, and $W$ contains no other vertices of $T$.*

**Proposition 2.1.** *Let $G$ be a nontrivial connected graph. If $G$ is a complete graph, cycle, or tree, then $f(G) = \dim(G)$.*

P r o o f. First assume that $G$ is the complete graph $K_n$ of order $n \geqslant 2$. Since every set $W$ of $n - 1$ vertices in $K_n$ is a basis of $K_n$, it follows that $W$ is not the unique basis containing any of its proper subset. By Lemma 1.2, $f(K_n) = \dim(K_n)$. Next assume that $G$ is a cycle $C_n$ of order $n \geqslant 4$. If $n$ is odd, then every pair of vertices forms a basis of $C_n$. If $n$ is even, then every pair $u, v$ of vertices with $d(u, v) \neq n/2$ forms a basis of $C_n$. So in either cases, there is no basis of $C_n$ that is the unique basis containing any of its proper subset. Again, it then follows from Lemma 1.2 that $f(C_n) = \dim(C_n)$.

Now let $T$ be a tree. First assume that $T$ is the path $P_n$ of order $n \geqslant 2$. Since each end-vertex of $P_n$ forms a basis for $P_n$, it follows that $f(P_n) \geqslant 1 = \dim(P_n)$ by Lemma 1.2. Hence $f(P_n) = \dim(P_n) = 1$. Next assume that $T$ is a tree of order $n \geqslant 4$ that is not a path and $T$ has $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of $v_i$, and let $P_{ij}$ be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$. Let $W$ be a basis of $G$. It then follows from Theorem C that $W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p)$ with exactly one exception for each $i$ with $1 \leqslant i \leqslant p$ and $k_i \geqslant 2$, and $W$ contains no other vertices of $G$. Let $S$ be a proper subset of $W$ and let $x \in W - S$. Then there exist $i, j$ with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$ such that $x$ is a vertex from the path $P_{ij} - v_i$, say $x$ is a vertex from $P_{11} - v_1$. Since $x \in W$, it follows that $\operatorname{ter}(v_1) = k_1 \geqslant 2$. Assume, without loss of generality, that for each $j$ with $1 \leqslant j \leqslant k_1 - 1$, there is a vertex $x_j$ from $P_{1j} - v_1$ that belongs to $W$ and there is no vertex of $P_{1,k_1} - v_1$ that belongs to $W$. So $x_1 = x$. Let $x_{k_1}$ be a vertex of the path $P_{1,k_1} - v_1$. Then $W' = (W - \{x_1\}) \cup \{x_{k_1}\}$ is a basis of $T$ by Theorem C. Since $W'$ contains $S$ and $W' \neq W$, it follows that $W$ is not the unique basis containing $S$. Therefore, $f(T) = \dim(T)$ by Lemma 1.2. $\square$

**Proposition 2.2.** *Let $G$ be a connected graph of order $n \geqslant 2$ with $\dim(G) = n-2$. If $G = K_{r,s}$ $(r, s \geqslant 1)$ or $G = K_r + \overline{K_s}$ $(r \geqslant 1, s \geqslant 2)$, then $f(G) = \dim(G)$. If $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$, then $f(G) = \dim(G) - 1$.*

P r o o f. By Theorem A, if $\dim(G) = n-2$, then $G = K_{r,s}$ $(r, s \geqslant 1)$, $G = K_r + \overline{K_s}$ $(r \geqslant 1, s \geqslant 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$. First let $G = K_{r,s}$ whose the partite sets are $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$. Then by Lemma 1.1 every basis $W$ of $G$ has the form $W = W_1 \cup W_2$, where $W_i \subseteq V_i$ $(1 = 1, 2)$ with $|W_1| = r-1$ and $|W_2| = s-1$. Assume, without loss of generality, that $W = V(G) - \{u_r, v_s\}$. Let $S$ be a proper subset of $W$. Then $S = S_1 \cup S_2$, where $S_i \subseteq W_i$ $(1 = 1, 2)$ and $|S_1| \leqslant r-2$ or $|S_2| \leqslant s-2$, say $|S_1| \leqslant r-2$. Thus there is $u_i \in W$, where $1 \leqslant i \leqslant r-1$, such that $u_i \notin S_1$. Then $W' = (W - \{u_i\}) \cup \{u_r\}$ is a basis of $G$ containing $S$. Since $W' \neq W$, it follows that $W$ is not the unique basis containing $S$. Therefore, $f(G) = \dim G$. If $G = K_r + \overline{K_s}$, let $V_1 = V(K_r) = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = V(\overline{K_s}) = \{v_1, v_2, \ldots, v_s\}$. Since every basis $W$ of $G$ has the form $W = W_1 \cup W_2$, where $W_i \subseteq V_i$ $(1 = 1, 2)$ with $|W_1| = r-1$ and $|W_2| = s-1$, a similar argument shows that $f(G) = \dim G$.

Now let $G = K_r + (K_1 \cup K_s)$. Assume that $V_1 = V(K_r) = \{u_1, u_2, \ldots, u_r\}$, $V_2 = V(K_s) = \{v_1, v_2, \ldots, v_s\}$, and $V(K_1) = \{x\}$. Then by Lemma 1.1 it can be verified that every basis of $G$ has the form $W = W_1 \cup W_2 \cup \{x\}$, where $W_i \subseteq V_i$ $(i = 1, 2)$ and $|W_1| = r - 1$ and $|W_2| = s - 1$. Since the vertex $x$ belongs to every basis, $f(G) \leqslant |W| - 1 = \dim(G) - 1$. On the other hand, let $W$ be a basis of $G$, say $W = V(G) - \{u_r, v_s\}$, and let $S$ be a subset of $W$ with $|S| \leqslant |W| - 2$. Then there is a vertex $y \in W - S$ such that $y \neq x$. We may assume that $y \in V_1$. Then $W' = (W - \{y\}) \cup \{u_r\}$ is a basis of $G$ containing $S$. So $W$ is not the unique basis containing $S$. Thus $f(G) \geqslant |W| - 1 = \dim(G) - 1$. Therefore, $f(G) = \dim(G) - 1$.

□

## 3. Graphs with prescribed dimensions and forcing dimensions

We have already noted that if $G$ is a graph with $f(G) = a$ and $\dim(G) = b$, then $0 \leqslant a \leqslant b$ and $b \geqslant 1$. We now determine which pairs $a, b$ of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 1$ are realizable as the forcing dimension and dimension of some nontrivial connected graph. In order to do this, we state the following result obtained in [1].

**Theorem D.** *For $k \geqslant 2$, there exists a connected graph of dimension $k$ with a unique basis.*

**Theorem 3.1.** *For all integers $a$, $b$ with $0 \leqslant a \leqslant b$ and $b \geqslant 1$, there exists a nontrivial connected graph $G$ with $f(G) = a$ and $\dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$.*

Proof. By Theorem A, the path $P_n$ of order $n \geqslant 2$ is the only nontrivial connected graph of order $n$ with dimension 1. However, $f(P_n) = 1$ for all $n \geqslant 2$ by Proposition 2.1. Hence there is no nontrivial connected graph $G$ with $f(G) = 0$ and $\dim(G) = 1$.

We now verify the converse. Let $a = 0$ and $b \geqslant 2$. By Theorem D there is a connected graph $G$ of dimension $b$ with a unique basis. Thus $f(G) = 0$ by Lemma 1.2 and $\dim(G) = b$. Hence the result is true for $a = 0$ and $b \geqslant 2$. So we may assume that $a > 0$. First assume that $b = a$. When $b = a = 1$, each path $P_n$ $(n \geqslant 2)$ has the desired property. When $b = a = 2$, the star $K_{1,3}$ has the desired property. When $b = a \geqslant 3$, then the complete graph $K_{a+1}$ has the desired property. So we now assume that $a < b$. We consider two cases.

Case 1. $b = a + 1$. Let $G$ be the graph obtained from the 4-cycle $u_1$, $u_2$, $u_3$, $u_4$, $u_1$ by adding a new edge $u_2u_4$ and then joining $b$ new vertices $v_1, v_2, \ldots, v_b$ to $u_2$ and $u_3$. The graph $G$ is shown in Figure 3. First note every basis of $G$ contains at least

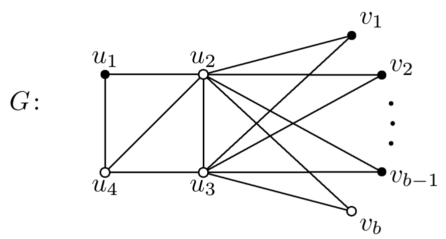


Figure 3. A graph $G$ with $\dim(G) = b$ and $f(G) = b - 1$

$b-1$ vertices from $\{v_1, v_2, \ldots, v_b\}$ by Lemma 1.1. However, it can be verified that if $W$ is a basis of $G$, then $W$ contains exactly $b-1$ vertices from $\{v_1, v_2, \ldots, v_b\}$ and the vertex $u_1$. Hence $\dim(G) = b$. Next we show that $f(G) = b-1$. Let $W$ be a basis of $G$, say $W = \{u_1, v_1, v_2, \ldots, v_{b-1}\}$. Since $u_1$ belongs to every basis of $G$, it follows that $W$ is the unique basis containing the subset $\{v_1, v_2, \ldots, v_{b-1}\}$, which implies that $f_G(W) \leqslant b-1$. On the other hand, if $S$ is a subset of $W$ with $|S| \leqslant b-2$, then, without loss of generality, we assume that $v_{b-1} \notin S$. Then $W' = (W - \{v_{b-1}\}) - \{v_b\}$ is a basis of $G$ containing $S$. Thus $W$ is not the unique basis containing $S$ and so $f_G(W) \geqslant b-1$. Hence $f_G(W) = b-1$ for every basis $W$ of $G$ and so $f(G) = b-1 = a$.

Case 2. $b \geqslant a+2$. Let $r = b-a$. Then $2 \leqslant r \leqslant b-1$. First we construct a graph $H$ of order $r + 2^r$ with $V(H) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_{2^r-1}\}$ and the ordered set $V = \{v_{r-1}, v_{r-2}, \ldots, v_0\}$ are disjoint. The induced subgraph $\langle U \rangle$ of $H$ is complete, while $V$ is independent. It remains to define the adjacencies between $V$ and $U$. Let each integer $j$ $(0 \leqslant j \leqslant 2^r - 1)$ be expressed in its base 2 (binary) representation. Thus, each such $j$ can be expressed as a sequence of $r$ coordinates, that is, an $r$-vector, where the rightmost coordinate represents the value (either 0 or 1) in the $2^0$ position, the coordinate to its immediate left is the value in the $2^1$ position, etc. For integers $i$ and $j$, with $0 \leqslant i \leqslant r-1$ and $0 \leqslant j \leqslant 2^r - 1$, we join $v_i$ and $u_j$ if and only if the value in the $2^i$ position in the binary representation of $j$ is 1. The structure of $H$ is based on one given in the proof of Theorem D (see [1]), where it was shown that $H$ has dimension $r$ and $V$ is the unique basis of $H$. Now the graph $G$ is obtained from $H$ by adding the $a$ new vertices $x_1, x_2, \ldots, x_a$ such that each $x_i$ $(1 \leqslant i \leqslant a)$ has the same neighborhood as $u_0$ in $V$ and the induced subgraph $\langle U \cup \{x_1, x_2, \ldots, x_a\} \rangle$ is complete.

We first show that $\dim G = b$. Let $T = \{u_0, x_1, x_2, \ldots, x_a\}$. Note that if $t_1, t_2 \in T$ and $v \in V(G)$, then $d(t_1, v) = d(t_2, v)$. Hence every resolving set of $G$ must contain at least $a$ vertices from $T$ by Lemma 1.1. Let $W = V \cup \{x_1, x_2, \ldots, x_a\}$. We show that $W$ is a resolving set of $G$. It suffices to show that the metric representations of vertices in $U$ are distinct. Observe that the first $r$ coordinates of the metric representation for each $u_j$ $(0 \leqslant j \leqslant 2^r - 1)$ can be expressed as $r(u_j|V)$. Since $V$

is the basis of $H$, the metric representations $r(u_j|V)$ $(0 \leqslant j \leqslant 2^r - 1)$ of $u_j$ with respect to $V$ are distinct. In fact, $r(u_j|V) = (2 - a_{r-1}, 2 - a_{r-2}, \ldots, 2 - a_0)$, where $a_m$ $(0 \leqslant m \leqslant r - 1)$ is the value in the $2^m$ position of the binary representation of $j$. Since the binary representations $a_{r-1}a_{r-2}\ldots a_1a_0$ are distinct for the vertices of $U$, their metric representations $(2 - a_{r-1}, 2 - a_{r-2}, \ldots, 2 - a_0)$ (with respect to $V$) are distinct. This implies that the metric representations $r(u_j|W)$ are distinct as well. Hence $W$ is a resolving set of $G$ and so $\dim G \leqslant |W| = (b - a) + a = b$. Next we show that $\dim G \geqslant b$. Assume, to the contrary, that $\dim(G) \leqslant b - 1$. Let $S$ be a basis of $G$ with $|S| = \dim(G)$. Let $S = S' \cup X$, where $X \subseteq T$ and $S' \subseteq V(G) - T$. Then $|X| \geqslant a$ by Lemma 1.1. Let $S^* = S' \cup \{u_0\}$. Hence $|S^*| = |S| - |X| + 1 \leqslant (b - 1) - a + 1 = b - a$. Since $V$ is the unique basis of $H$ and $u_0 \notin V$, it follows that $S^*$ is not a basis of $H$. Thus there exist $z, z' \in V(H) - \{u_0\}$ such that $r(z|S^*) = r(z'|S^*)$ and so $d(z, u_0) = d(z', u_0)$. Thus $d(z, x_i) = d(z', x_i)$ for all $i$. This implies that $r(z|S) = r(z'|S)$ and so $S$ is not a basis, which is a contradiction. Therefore, $\dim(G) \leqslant b$ and so $\dim(G) = b$.

In order to determine $f(G)$, we first show that $V$ belongs to every basis of $G$. Assume, to the contrary, there exists a basis $W$ of $G$ such that $V \nsubseteq W$. If $T \subseteq W$, then $W' = (W - T) \cup \{u_0\} \neq V$ and so $W'$ is not a basis of $H$. Thus there exist $z, z' \in V(H) - \{u_0\}$ such that $r(z|W') = r(z'|W')$. This implies that $r(z|W) = r(z'|W)$ and so $W$ is not a basis, a contradiction. Hence $W$ contains exactly $a$ vertices from $T$. Assume, without loss of generality, that $W = S \cup X$, where $X = T - \{u_0\}$ and $S \subseteq V(H) - T$. A similar argument to the one employed in the proof of Theorem D [1] shows that there exist two vertices $z$ and $z'$ in $U = V(H) - V$ such that $r(z|S) = r(z'|S)$. Since the distance between every two vertices in $U \cup T$ is 1, it follows that $r(z|W) = r(z'|W)$. This contradicts the fact that $W$ is a basis. Therefore, $V$ belongs to every basis $W$ of $G$.

We are now prepared to show that $f(G) = a$. Let $W$ be a basis of $G$. Since $V$ must belong to $W$, it follows that $W$ is the unique basis containing $W - V$. Thus $f_G(W) \leqslant |W - V| = b - (b - a) = a$. This is true for every basis $W$ of $G$ and so $f(G) \leqslant a$. On the other hand, let $W$ be a basis and $S$ be a subset of $W$ with $|S| \leqslant a - 1$. Without loss of generality, assume that $W = V \cup X$ with $X = \{x_1, x_2, \ldots, x_a\}$. Since $|S| \leqslant a - 1$, there exists $x \in W \cap X$ such that $x \notin S$. Then $W' = (W - \{x\}) \cup \{u_0\}$ is a basis of $G$ that contains $S$. Hence $W$ is not the unique basis containing $S$ and so $f_G(W) \geqslant |S| + 1 = a$. Again, this is true for every basis $W$ in $G$ and so $f(G) \leqslant a$. Therefore, $f(G) = a$ and $\dim(G) = b$, as desired. $\square$

## 4. Open problem

While the forcing dimension $f(G)$ of a graph $G$ is the minimum forcing number among all bases of $G$, we define the *upper forcing dimension* $f^+(G)$ as the maximum forcing number among all bases of $G$. Hence

$$0 \leqslant f(G) \leqslant f^+(G) \leqslant \dim(G).$$

If a graph $G$ has a unique basis, then $f(G) = f^+(G) = 0$. Also, there are numerous examples of graphs $G$, such as complete graphs and trees, with $f(G) = f^+(G) = \dim(G)$. On the other hand, as we have seen, the graph $G$ of Figure 1 contains two bases with distinct forcing numbers and so $f(G) = 1$ and $f^+(G) = 2$. Hence $f(G) < f^+(G)$. We close with the following open problem.

P r o b l e m 4.1. For which pairs $a$, $b$ of integers with $0 \leqslant a \leqslant b$, does there exist a nontrivial connected graph $G$ with $f(G) = a$ and $f^+(G) = b$?

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