ESSENTIAL NORMS OF THE NEUMANN OPERATOR OF THE ARITHMETICAL MEAN

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Abstract. Let $K \subset \mathbb{R}^m$ $(m \geqslant 2)$ be a compact set; assume that each ball centered on the boundary B of K meets K in a set of positive Lebesgue measure. Let $\mathcal{C}_0^{(1)}$ be the class of all continuously differentiable real-valued functions with compact support in $\mathbb{R}^m$ and denote by σ_m the area of the unit sphere in $\mathbb{R}^m$. With each $\varphi \in \mathcal{C}_0^{(1)}$ we associate the function

$$W_K\varphi(z) = \frac{1}{\sigma_m} \int_{\mathbb{R}^m \setminus K} \operatorname{grad} \varphi(x) \cdot \frac{z - x}{|z - x|^m} \, \mathrm{d}x$$

of the variable $z \in K$ (which is continuous in K and harmonic in $K \setminus B$). $W_K\varphi$ depends only on the restriction $\varphi|_B$ of φ to the boundary B of K. This gives rise to a linear operator W_K acting from the space $\mathcal{C}^{(1)}(B) = \{\varphi|_B; \varphi \in \mathcal{C}_0^{(1)}\}$ to the space $\mathcal{C}(B)$ of all continuous functions on B. The operator $\mathcal{T}_K$ sending each $f \in \mathcal{C}^{(1)}(B)$ to $\mathcal{T}_K f = 2W_K f - f \in \mathcal{C}(B)$ is called the Neumann operator of the arithmetical mean; it plays a significant role in connection with boundary value problems for harmonic functions. If p is a norm on $\mathcal{C}(B) \supset \mathcal{C}^{(1)}(B)$ inducing the topology of uniform convergence and $\mathcal{G}$ is the space of all compact linear operators acting on $\mathcal{C}(B)$, then the associated p-essential norm of $\mathcal{T}_K$ is given by

$$\omega_p \mathcal{T}_K = \inf_{Q \in \mathcal{G}} \sup\{p[(\mathcal{T}_K - Q)f];\ f \in \mathcal{C}^{(1)}(B),\ p(f) \leqslant 1\}.$$

In the present paper estimates (from above and from below) of $\omega_p \mathcal{T}_K$ are obtained resulting in precise evaluation of $\omega_p \mathcal{T}_K$ in geometric terms connected only with K.

Keywords: double layer potential, Neumann's operator of the arithmetical mean, essential norm

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In what follows $\mathbb{R}^m$ will be the Euclidean space of dimension $m \geqslant 2$. The Euclidean norm of a vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ will be denoted by $|x|$. If $M \subset \mathbb{R}^m$, then the symbols $\overline{M}$, M° and ∂M will denote the closure, the interior and the boundary of M, respectively. $B_r(z) \colon = \{x \in \mathbb{R}^m; |x - z| < r\}$ is the open ball of radius $r > 0$ centered at $z \in \mathbb{R}^m$. The symbol λ_k will denote the outer k-dimensional Hausdorff measure with the usual normalization (so that λ_m coincides with the outer Lebesgue measure in $\mathbb{R}^m$). We put

$$\sigma_m \colon = \lambda_{m-1}(\partial B_1(0)) = \frac{2\pi^{m/2}}{\Gamma(m/2)},$$

where Γ is the Euler gamma function. For fixed $z \in \mathbb{R}^m$ the symbol h_z will denote the fundamental harmonic function with a pole at z, whose values at any $x \in \mathbb{R}^m \setminus \{z\}$ are given by

$$h_z(x) \colon = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x - z|} & \text{if } m = 2, \\ \dfrac{1}{(m-2)\sigma_m} |x - z|^{2-m} & \text{if } m > 2; \end{cases}$$

we put $h_z(z) = +\infty$. Let $\mathcal{C}_0^{(1)}$ be the space of all continuously differentiable compactly supported real-valued functions on $\mathbb{R}^m$. We fix a compact set $K \subset \mathbb{R}^m$ and put $G = \mathbb{R}^m \setminus K$, $B = \partial K$. With any $\varphi \in \mathcal{C}_0^{(1)}$ we associate the function $W_K\varphi \equiv W\varphi$ on K defined by

$$W\varphi(z) = \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h_z(x) \, \mathrm{d}\lambda_m(x), \ z \in K.$$

It is not difficult to verify that $W\varphi$ is continuous in K and harmonic in K°; besides, $W\varphi$ depends only on the restriction $\varphi|_B$ of $\varphi \in \mathcal{C}_0^{(1)}$ to B (cf. §2 in [9]). Denote by

$$\mathcal{C}^{(1)}(B) \colon = \{\varphi|_B; \ \varphi \in \mathcal{C}_0^{(1)}\}$$

the vectorspace (over the reals) of all restrictions to B of functions in $\mathcal{C}_0^{(1)}$ and let $\mathcal{C}(K)$ be the vectorspace of all finite continuous real-valued functions in K; then W gives rise to a linear operator acting from $\mathcal{C}^{(1)}(B)$ to $\mathcal{C}(K)$. In connection with boundary value problems it is natural to inquire about conditions on K guaranteeing the continuity of the operator W with respect to the topologies of uniform convergence in $\mathcal{C}^{(1)}(B)$ and in $\mathcal{C}(K)$ (compare [3], [15], [8], [9]). For simplicity, we will always assume that K is massive in the sense that

(1) $$\lambda_m(B_r(z) \cap K) > 0 \quad \text{for each } z \in K, \ r > 0,$$

which does not cause any lack of generality (cf. the observation on p. 27 in [9]). Geometric conditions, which enable us to extend W to a bounded linear operator from $\mathcal{C}(B) \supset \mathcal{C}^{(1)}(B)$ to $\mathcal{C}(K)$ (equipped with the sup-norm), can be conveniently described in terms of the so-called essential boundary $\partial_e K \equiv B_e$ defined by

$$B_e \colon = \Big\{x \in \mathbb{R}^m;\ \limsup_{r\searrow 0} \lambda_m(B_r(x)\cap K)r^{-m} > 0, \limsup_{r\searrow 0} \lambda_m(B_r(x)\cap G)r^{-m} > 0\Big\}$$

(cf. [4]). For any $z \in \mathbb{R}^m$ and $\theta \in \partial B_1(0)$ consider the half-line

$$H_z(\theta) \colon = \{z + t\theta;\ t > 0\}$$

and denote by $n(z,\theta)$ $(0 \leqslant n(z,\theta) \leqslant +\infty)$ the total number of points in

$$H_z(\theta) \cap B_e.$$

It appears that, for fixed $z \in \mathbb{R}^m$, the function

$$\theta \mapsto n(z,\theta)$$

is λ_{m-1}-measurable on $\partial B_1(0)$ so that we may introduce the integral

$$v(z) \colon = \frac{1}{\sigma_m} \int\limits_{\partial B_1(0)} n(z,\theta)\, \mathrm{d}\lambda_{m-1}(\theta)$$

(compare §2 in [9], Lemma 3 in [11] and [4]). With this notation

$$\sup_{z\in B} v(z) < +\infty \tag{2}$$

is a necessary and sufficient condition guaranteeing that for any uniformly convergent (on B) sequence $\varphi_n \in \mathcal{C}^{(1)}(B)$, the correspondig sequence $W\varphi_n \in \mathcal{C}(K)$ is uniformly convergent on K (which is equivalent to continuous extendability of W, defined so far only on $\mathcal{C}^{(1)}(B)$, to a bounded linear operator acting from $\mathcal{C}(B) \supset \mathcal{C}^{(1)}(B)$ to $\mathcal{C}(K)$, where $\mathcal{C}(B)$ and $\mathcal{C}(K)$ are equipped with the usual maximum norm). In what follows we always assume (2), which implies that

$$\sup_{z\in\mathbb{R}^m} v(z) < +\infty$$

(cf. Theorem 2.16 in [9]) and guarantees the existence of a well-defined density

$$d_K(z) \colon = \lim_{r\searrow 0} \frac{\lambda_m(B_r(z)\cap K)}{\lambda_m(B_r(z))}$$

for any $z \in \mathbb{R}^m$ (cf. Lemma 2.1 in [9]). For any $f \in \mathcal{C}(B)$ the corresponding $Wf \in \mathcal{C}(K)$ is harmonic in K° and admits an integral representation reminding one of the classical double layer potential with momentum density f. For this purpose let us recall that a unit vector $n \in \partial B_1(0)$ is termed the exterior normal of K at $y \in \mathbb{R}^m$ in the sense of Federer provided

$$\begin{aligned} &\lim_{r \searrow 0} r^{-m} \lambda_m(\{x \in B_r(y) \cap K;\ (x-y)\cdot n > 0\}) = 0, \\ &\lim_{r \searrow 0} r^{-m} \lambda_m(\{x \in B_r(y) \cap G;\ (x-y)\cdot n < 0\}) = 0. \end{aligned} \tag{3}$$

For any fixed $y \in \mathbb{R}^m$ there exists at most one vector $n \in \partial B_1(0)$ with the property (3) and it will be denoted by $n^K(y) \equiv n$ provided it is available; if there is no such $n \in \partial B_1(0)$ with (3), then we put $n^K(y) = 0$ $(\in \mathbb{R}^m)$. The vector-valued function $y \mapsto n^K(y)$ is Borel measurable and

$$\widehat{B} \equiv \widehat{\partial K} \colon = \{y \in \mathbb{R}^m;\ |n^K(y)| > 0\}$$

is a Borel set which is termed the reduced boundary of K (cf. [6]). Clearly,

$$\widehat{B} \subset \{y \in \mathbb{R}^m;\ d_K(y) = \tfrac{1}{2}\} \subset B_e$$

and under our assumption (2) we have

$$\lambda_{m-1}(B_e) < +\infty$$

and

$$\lambda_{m-1}(B_e \setminus \widehat{B}) = 0$$

(cf. Section 4.5 in [5], 5.6 in [17] and 2.12 in [9]). If $f \in \mathcal{C}(B)$, then Wf can be represented by

$$Wf(z) = \begin{cases} d_G(z)f(z) + \displaystyle\int_{\widehat{B}} f(y)n^K(y)\cdot \operatorname{grad} h_z(y)\,\mathrm{d}\lambda_{m-1}(y) & \text{for } z \in B \\ \displaystyle\int_{\widehat{B}} f(y)n^K(y)\cdot \operatorname{grad} h_z(y)\,\mathrm{d}\lambda_{m-1}(y) & \text{for } z \in K^\circ \end{cases}$$

where, of course, $d_G(z) = 1 - d_K(z)$ is the density of $G = \mathbb{R}^m \setminus K$ at z (cf. [9], Proposition 2.8 and Lemmas 2.9, 2.15).

For $\alpha \in \mathbb{R}$ we denote by W^α the operator on $\mathcal{C}(B)$ sending $f \in \mathcal{C}(B)$ to $W^\alpha f \in \mathcal{C}(B)$ attaining the value $W^\alpha f(y)\colon = Wf(y) - \alpha f(y)$ at any $y \in B$. Given a boundary condition $g \in \mathcal{C}(B)$ then an attempt to solve the corresponding Dirichlet problem for

K° (at least in the case $B \subset \overline{K^\circ}$) in the form of a Wf with an unknown $f \in \mathcal{C}(B)$ leads to the equation

$$(4) \qquad \big(\alpha I + W^\alpha\big) f = g,$$

where I denotes the identity operator on $\mathcal{C}(B)$.

The space $\mathcal{C}'(B)$ dual to $\mathcal{C}(B)$ can be identified with the space of all finite signed Borel measures with support contained in B. For any $\nu \in \mathcal{C}'(B)$ the potential

$$(5) \qquad \mathcal{U}\nu(y) = \int_B h_y(x)\,\mathrm{d}\nu(x), \;\; y \in G$$

represents a harmonic function in G whose weak normal derivative can be properly interpreted (cf. §1 in [9], [15]). Given a $\mu \in \mathcal{C}'(B)$ then an attempt to solve the corresponding Neumann problem for G (with the Neumann boundary condition given by μ) in the form of a potential (5) with an unknown $\nu \in \mathcal{C}'(B)$ leads to the equation

$$(6) \qquad (\alpha I + W^\alpha)'\nu = \mu$$

which is dual to (4).

Let us agree to denote by $\mathcal{G}$ the space of all compact linear operators acting on $\mathcal{C}(B)$. If p is a norm on $\mathcal{C}(B)$ and T is a bounded linear operator acting on $\mathcal{C}(B)$ then its norm $p(T)$ is defined in the usual way and the p-essential norm $\omega_p T$ is given by

$$\omega_p T\colon = \inf\{p(T-Q);\; Q \in \mathcal{G}\}.$$

In connection with the applicability of the Fredholm-Radon theory to the pair of dual equations (4), (6) it is important to have estimates of the essential spectral radius of the operator W^α. According to the theorem of Gohberg and Markus (cf. [7]), this radius coincides with

$$\inf_p \omega_p W^\alpha,$$

where p ranges over all equivalent norms on $\mathcal{C}(B)$ inducing the topology of uniform convergence in $\mathcal{C}(B)$. Let us recall that simple examples are known showing that for the usual maximum norm p_1, where $p_1(f) = \sup\{|f(y)|; y \in B\}$, $f \in \mathcal{C}(B)$, it may occur that

$$\omega_{p_1} W^\alpha > |\alpha| \quad \text{for all } \alpha \neq 0,$$

while

$$\omega_p W^{\frac12} < \tfrac12$$

for a suitable norm p on $\mathcal{C}(B)$ topologically equivalent to p_1 (cf. [13], [1]; note that $2W^{\frac12}$ is the so-called Neumann operator of the arithmetical mean as mentioned on

p. 72 in [9]). Accordingly, it is useful to investigate estimates of $\omega_p W^\alpha$ for general norms p topologically equivalent to p_1, which is the subject of the present paper. Given such a norm p on $\mathcal{C}(B)$ inducing the topology of uniform convergence in $\mathcal{C}(B)$ we put

$$\overline{p}(y) = \sup\{g(y);\ g \in \mathcal{C}(B),\ p(g) \leqslant 1\} \tag{7}$$

for $y \in B$. The function

$$\overline{p}\colon\ y \mapsto \overline{p}(y)$$

defined by (7) is lower-semicontinuous on B.

Given a bounded non-negative lower-semicontinuous function ψ on B we put for $z \in \mathbb{R}^m$, $r > 0$ and $\theta \in \partial B_1(0)$

$$n_r^\psi(z,\theta) = \sum_\xi \psi(\xi), \quad \xi \in H_z(\theta) \cap B_e \cap B_r(z), \tag{8}$$

the sum on the right-hand side of (8) counting, with the weight $\psi(\xi)$, all points ξ in $B_e \cap \{z + \varrho\theta; 0 < \varrho < r\}$ $(0 \leqslant n_r^\psi(z,\theta) \leqslant +\infty)$. We shall see that, for fixed $z \in \mathbb{R}^m$ and $r > 0$, the function $\theta \mapsto n_r^\psi(z,\theta)$ is λ_{m-1}-measurable on $\partial B_1(0)$, which justifies the definition

$$v_r^\psi(z) = \frac{1}{\sigma_m} \int\limits_{\partial B_1(0)} n_r^\psi(z,\theta)\,\mathrm{d}\lambda_{m-1}(\theta), \quad z \in \mathbb{R}^m,\ 0 < r \leqslant \infty. \tag{9}$$

(Observe that this quantity reduces to $v(z)$ in the case $r = \infty$ and $\psi \equiv 1$.) We are going to establish upper and lower estimates of $\omega_p W^\alpha$ with help of the functions

$$y \mapsto v_r^{\overline{p}}(y), \quad y \in B.$$

In particular, for suitable weighted norms p on $\mathcal{C}(B)$ these estimates permit to prove the equality

$$\omega_p W^\alpha = |\tfrac{1}{2} - \alpha| + \inf_{r>0} \sup_{y \in B} \frac{v_r^{\overline{p}}(y)}{\overline{p}(y)},$$

extending Theorem 4.1 in [9].

1. Lemma. *Let p be a norm on $\mathcal{C}(B)$ inducing the topology of uniform convergence and define the function $\overline{p}\colon B \to \mathbb{R}$ by (7). Then $\overline{p}$ is lower-semicontinuous on B and there are constants $0 < k_p \leqslant K_p < \infty$ such that*

$$k_p \leqslant \overline{p} \leqslant K_p \tag{10}$$

on B.

P r o o f. The definition (7) shows that $\overline{p}$ is a (pointwise) supremum of a class of continuous functions on B; hence $\overline{p}$ is lower-semicontinuous in B. Since the identity operator acting from $\mathcal{C}(B)$ normed by p to $\mathcal{C}(B)$ normed by the maximum norm p_1 is bounded, there is a $K_p \in (0,\infty)$ such that $\overline{p} \leqslant K_p$ on B. Since also the identity operator acting inversely from $(\mathcal{C}(B), p_1)$ into $(\mathcal{C}(B), p)$ is bounded, there is a $c \in (0,+\infty)$ such that the implication

$$(g \in \mathcal{C}(B),\ |g| \leqslant 1) \Longrightarrow p\Big(\frac{g}{c}\Big) \leqslant 1$$

is valid. This together with the definition of $\overline{p}$ shows that

$$\overline{p}(y) \geqslant \frac{1}{c}$$

for any $y \in B$, so that (10) holds with $k_p = \frac{1}{c}$. $\square$

2. R e m a r k. As a consequence of our assumption (1) we have

$$\lambda_{m-1}(B_r(y) \cap \widehat{B}) > 0, \quad \forall y \in B,\ \forall r > 0.$$

This follows from the relative isoperimetric inequality concerning sets of locally finite perimeter (cf. Section 4.5 in [5] and p. 50 in [9]).

3. Lemma. *If ψ is a non-negative λ_{m-1}-measurable function defined λ_{m-1}-a.e. on $\widehat{B}$ we denote by*

$$\widehat{\psi}(y) \colon = \lambda_{m-1}\text{-}\operatorname*{ess\,lim\,inf}_{x\to y, x\in\widehat{B}} \psi(x)$$

the λ_{m-1}-essential lower limit of ψ at $y \in B$ which is defined as the least upper bound of all $\gamma \in \mathbb{R}$ for which there is an $r > 0$ such that

$$\lambda_{m-1}(\{x \in B_r(y) \cap \widehat{B};\ \psi(x) < \gamma\}) = 0. \tag{11}$$

Then the function $\widehat{\psi}\colon\ y \mapsto \widehat{\psi}(y)$ is lower-semicontinuous on B and

$$\lambda_{m-1}(\{y \in \widehat{B};\ \psi(y) < \widehat{\psi}(y)\}) = 0.$$

P r o o f. For the sake of completeness we include the following argument occurring in [12] in connection with Lemma 8. Consider an arbitrary $y \in B$ and $c < \widehat{\psi}(y)$. Then there are $\gamma \in (c, \widehat{\psi}(y)]$ and $r > 0$ such that (11) holds. If $z \in B \cap B_{r/2}(y)$ then $B_{r/2}(z) \subset B_r(y)$ and, consequently,

$$\lambda_{m-1}(\{x \in B_{r/2}(z) \cap \widehat{B};\ \gamma(x) < \gamma\}) = 0,$$

which shows that $\widehat{\psi}(z) \geqslant \gamma > c$. We have thus shown that, given $c < \widehat{\psi}(y)$, the inequality $c < \widehat{\psi}(z)$ holds for all $z \in B$ sufficiently close to y and the lower-semicontinuity of $\widehat{\psi}$ at y is established. Admitting

$$\lambda_{m-1}(\{y \in \widehat{B};\ \psi(y) < \widehat{\psi}(y)\}) > 0$$

we get, by Lusin's theorem, that there is a compact set $C \subset \{y \in \widehat{B}; \psi(y) < \widehat{\psi}(y)\}$ with $\lambda_{m-1}(C) > 0$ such that the restriction $\psi|_C$ is continuous. There is a $z \in C$ such that

$$\lambda_{m-1}(B_\varrho(z) \cap C) > 0, \quad \forall \varrho > 0. \tag{12}$$

Since $\psi(z) < \widehat{\psi}(z)$, there are $\gamma \in (\psi(z), \widehat{\psi}(z)]$ and $r > 0$ such that

$$\lambda_{m-1}(\{y \in B_r(z) \cap \widehat{B};\ \psi(y) < \gamma\}) = 0. \tag{13}$$

Continuity of $\psi|_C$ guarantees the validity of the implication

$$y \in B_\varrho(z) \cap C \Longrightarrow \psi(y) < \gamma$$

for sufficiently small $\varrho \in (0, r)$ which, in view of the inclusion $B_\varrho(z) \cap C \subset B_r(z) \cap \widehat{B}$, together with (12) contradicts (13). This completes the proof. $\square$

4. Lemma. *If $\psi \geqslant 0$ is a lower-semicontinuous function on B, then $\widehat{\psi}$ (defined as in Lemma 3) satisfies $\widehat{\psi} \geqslant \psi$ on B; moreover, $\widehat{\psi}$ is the greatest lower-semicontinuous majorant of ψ on B coinciding with ψ almost everywhere (λ_{m-1}) on $\widehat{B}$.*

P r o o f. Let $\widetilde{\psi}$ be a lower-semicontinuous majorant of ψ coinciding with ψ almost everywhere (λ_{m-1}) on $\widehat{B}$. We are going to verify that $\widehat{\psi} \geqslant \widetilde{\psi}$ on B. Admit that there is a $y \in B$ with $\widehat{\psi}(y) < \widetilde{\psi}(y)$ and fix a $c \in \mathbb{R}$ such that

$$\widehat{\psi}(y) < c < \widetilde{\psi}(y). \tag{14}$$

Since $\widetilde{\psi}$ is lower-semicontinuous, we have

$$z \in B_r(y) \cap B \Longrightarrow \widetilde{\psi}(z) > c$$

for sufficiently small $r > 0$, whence

$$\lambda_{m-1}(\{z \in B_r(y) \cap \widehat{B};\ \psi(z) \leqslant c\}) = 0,$$

because $\psi = \widetilde{\psi}$ almost everywhere (λ_{m-1}) on $\widehat{B}$. We conclude that $\widehat{\psi}(y) \geqslant c$, which contradicts (14). Letting $\widetilde{\psi} = \psi$ we get from Lemma 3 that $\widehat{\psi} = \psi$ almost everywhere (λ_{m-1}) on $\widehat{B}$ and the proof is complete. $\square$

5. Lemma. *Let $\mathcal{C}_+(B)$ denote the class of all non-negative functions in $\mathcal{C}(B)$ and let $\mathcal{C}_+^{\uparrow}(B)$ denote the class of all non-negative lower-semicontinuous functions on B. Let $f \in \mathcal{C}_+(B)$, $\psi \in \mathcal{C}_+^{\uparrow}(B)$ and put $\varphi = f + \psi$. Then $\widehat{\varphi} = f + \widehat{\psi}$. In particular, $\widehat{f} = f$ for each $f \in \mathcal{C}_+(B)$.*

P r o o f. Observe that $f + \widehat{\psi}$ is a lower-semicontinuous majorant of φ on B such that $f + \widehat{\psi} = \varphi$ holds λ_{m-1}-a.e. in $\widehat{B}$. By Lemma 4 we get $\widehat{\varphi} \geqslant f + \widehat{\psi}$. We see that $\widehat{\varphi} - f \in \mathcal{C}_+^{\uparrow}(B)$ is a majorant of ψ on B coinciding with ψ almost everywhere (λ_{m-1}) on $\widehat{B}$. Using Lemma 4 again we arrive at the inequality $\widehat{\varphi} - f \leqslant \widehat{\psi}$, so that $\widehat{\varphi} = f + \widehat{\psi}$. Taking $\psi \equiv 0$ we get $\widehat{f} = f$, $\forall f \in \mathcal{C}_+(B)$. □

6. Lemma. *Let p be a norm on $\mathcal{C}(B)$ inducing the topology of uniform convergence in $\mathcal{C}(B)$ such that the implication*

$$|f| \leqslant |g| \quad \Longrightarrow \quad p(f) \leqslant p(g) \tag{15}$$

holds for any $f, g \in \mathcal{C}(B)$. Then we have

$$p(h) = \sup\{p(f);\ f \in \mathcal{C}(B),\ |f| \leqslant h\} \tag{16}$$

whenever $h \in \mathcal{C}_+(B)$, and (16) can be used to define $p(h)$ for any $h \in \mathcal{C}_+^{\uparrow}(B)$. Having extended p from $\mathcal{C}_+(B)$ to $\mathcal{C}_+^{\uparrow}(B)$ in this way we get for any $\alpha \in [0, +\infty)$ and $\psi_j \in \mathcal{C}_+^{\uparrow}(B)$ $(j = 0, 1, 2)$

$$p(\alpha\psi_0) = \alpha p(\psi_0), \tag{17}$$

$$p(\psi_1 + \psi_2) \leqslant p(\psi_1) + p(\psi_2). \tag{18}$$

P r o o f. The implication (15) $\Rightarrow$ (16) is evident and if (15) is used to define $p(h)$ for any $h \in \mathcal{C}_+^{\uparrow}(B)$ then (17) obviously holds for $\alpha \in [0, +\infty)$ and $\psi_0 \in \mathcal{C}_+^{\uparrow}(B)$. It is easy to verify (18) assuming first that $\psi_1, \psi_2 \in \mathcal{C}_+^{\uparrow}(B)$ satisfy

$$\psi_1 + \psi_2 > 0 \qquad \text{on } B. \tag{19}$$

We then have

$$p(\psi_1 + \psi_2) = \sup\{p(f);\ f \in \mathcal{C}(B), |f(y)| < \psi_1(y) + \psi_2(y),\ \forall y \in B\}.$$

Choose non-decreasing sequences $\{g_j^n\}_{n=1}^{\infty}$ in $\mathcal{C}_+(B)$ such that $g_j^n \nearrow \psi_j$ as $n \to \infty$ $(j = 1, 2)$. Fix $f \in \mathcal{C}(B)$ such that $|f| < \psi_1 + \psi_2$. If the compact sets

$K_n = \{x \in B; |f(x)| \geqslant g_1^n(x) + g_2^n(x)\}$ are nonempty then there is an $x \in \bigcap K_n$ and therefore $\psi_1(x) + \psi_2(x) \leqslant |f(x)|$, which is a contradiction. So, we have

$$|f| < g_1^n + g_2^n$$

for all sufficiently large $n \in N$. Defining for such n

$$f_j = f \frac{g_j^n}{g_1^n + g_2^n} \quad (j = 1, 2)$$

we get

$$|f_j| \leqslant |f| \frac{g_j^n}{g_1^n + g_2^n} < g_j^n \quad (j = 1, 2), \quad f_1 + f_2 = f,$$

whence

$$p(f) \leqslant p(f_1) + p(f_2) \leqslant p(\psi_1) + p(\psi_2).$$

Since $f \in \mathcal{C}(B)$ with $|f| < \psi_1 + \psi_2$ has been chosen arbitrarily, we get (18). It remains to observe that the additional assumption (19) can be omitted. Denote by $1_B \in \mathcal{C}(B)$ the constant function attaining the value 1 at any point in B. For any $\psi \in \mathcal{C}_+^{\uparrow}(B)$ and $\varepsilon > 0$ we then have

$$p(\psi) \leqslant p(\psi + \varepsilon 1_B) \leqslant p(\psi) + \varepsilon p(1_B),$$

so that

$$p(\psi + \varepsilon 1_B) \to p(\psi) \quad \text{as } \varepsilon \downarrow 0.$$

Consequently, for any $\psi_j \in \mathcal{C}_+^{\uparrow}(B)$ $(j = 1, 2)$ we get

$$p(\psi_1 + \psi_2) \leqslant p(\psi_1) + p(\psi_2 + \varepsilon 1_B) \to p(\psi_1) + p(\psi_2) \quad \text{as } \varepsilon \downarrow 0$$

and (18) follows. $\square$

7. Lemma. *Let $\psi \geqslant 0$ be a bounded lower-semicontinuous function on B and define for fixed $z \in \mathbb{R}^m$ and $r \in (0, \infty]$ the function $n_r^{\psi}(z, \theta)$ of the variable $\theta \in \partial B_1(0)$ by (8). This function is λ_{m-1}-integrable in $\partial B_1(0)$ and*

$$\int_{\partial B_1(0)} n_r^{\psi}(z, \theta) \, \mathrm{d}\lambda_{m-1}(\theta) = \int_{B \cap B_r(z)} \psi(x) |n^K(x) \cdot \operatorname{grad} h_z(x)| \, \mathrm{d}\lambda_{m-1}(x).$$

The function $v_r^{\psi}\colon z \mapsto v_r^{\psi}(z)$ defined by (9) is bounded and lower-semicontinuous on $\mathbb{R}^m$.

P r o o f. This is a consequence of Lemma 3 in [12]. $\square$

8. Lemma. *If*

$$(x,y) \mapsto g_y(x)$$

is a continuous (real-valued) function on $B \times B$ *then, for each* $f \in \mathcal{C}(B)$,

$$W(fg_y)(y)\colon = f(y)g_y(y)d_G(y) + \int_B f(x)g_y(x)n^K(x) \cdot \operatorname{grad} h_y(x)\, \mathrm{d}\lambda_{m-1}(x)$$

represents a continuous function of the variable $y \in B$.

P r o o f. As mentioned above, our assumption (2) guarantees that the operator W sending each $f \in \mathcal{C}(B)$ to

$$Wf\colon\ y \mapsto f(y)d_G(y) + \int_B f(x)n^K(x) \cdot \operatorname{grad} h_y(x)\, \mathrm{d}\lambda_{m-1}(x), \quad y \in B \tag{20}$$

is continuous on $\mathcal{C}(B)$ with respect to the topology of the uniform convergence (cf. Proposition 2.8 and Lemmas 2.9, 2.15 in [9]). Let now $\{y_n\}_{n=1}^{\infty}$ be an arbitrary convergent sequence of points in B, $\lim\limits_{n\to\infty} y_n = y_0$. Then, for each $f \in \mathcal{C}(B)$, the sequence of functions $\{fg_{y_n}\}_{n=1}^{\infty}$ converges uniformly on B to $fg_{y_0} \in \mathcal{C}(B)$ and $\{W(fg_{y_n})\}_{n=1}^{\infty}$ converges uniformly on B to $W(fg_{y_0})$ as $n \to \infty$, whence

$$\lim_{n\to\infty} W(fg_{y_n})(y_n) = W(fg_{y_0})(y_0)$$

and the continuity of $y \mapsto W(fg_y)(y)$ is established. $\square$

9. Lemma. *Let* $\psi \geqslant 0$ *be a bounded lower-semicontinuous function on* B *and let*

$$(x,y) \mapsto g_y(x)$$

be a continuous function on $B \times B$ *such that* $0 \leqslant g_y(x) \leqslant 1$. *Then*

$$F_g^{\psi}(y)\colon = \psi(y)g_y(y)\Big|d_G(y) - \frac{1}{2}\Big| + \int_B \psi(x)g_y(x)|n^K(x) \cdot \operatorname{grad} h_y(x)|\, \mathrm{d}\lambda_{m-1}(x)$$

is a lower-semicontinuous function of the variable y *on* B.

P r o o f. It follows from Lemma 8 that

$$\begin{aligned} H_g^f(y)\colon = \big(W - \tfrac{1}{2}I\big)(fg_y)(y) = f(y)g_y(y)[d_G(y) - \tfrac{1}{2}] \\ + \int_B f(x)g_y(x)n^K(x) \cdot \operatorname{grad} h_y(x)\, \mathrm{d}\lambda_{m-1}(x) \end{aligned}$$

is a continuous function of the variable y on B for each $f \in \mathcal{C}(B)$. It is therefore sufficient to verify that F_g^{ψ} is the (pointwise) supremum of the class

$$\mathcal{F} \colon = \{H_g^f;\ f \in \mathcal{C}(B),\ |f| \leqslant \psi\} \subset \mathcal{C}(B).$$

Clearly, any function in $\mathcal{F}$ is majorized by F_g^{ψ}. Fix now an arbitrary $\xi \in B$ and $\varepsilon > 0$. Since

$$\sup\left\{\int_B f(x) g_\xi(x) n^K(x) \cdot \operatorname{grad} h_\xi(x)\,\mathrm{d}\lambda_{m-1}(x);\ f \in \mathcal{C}(B),\ |f| \leqslant \psi,\ \operatorname{spt} f \subset B \setminus \{\xi\}\right\}$$
$$= \int_B \psi(x) g_\xi(x) |n^K(x) \cdot \operatorname{grad} h_\xi(x)|\,\mathrm{d}\lambda_{m-1}(x)$$

there is an $f_0 \in \mathcal{C}(B)$ such that $|f_0| \leqslant \psi$, $f_0 = 0$ on $B_\varrho(\xi) \cap B$ for sufficiently small $\varrho > 0$ and

$$\begin{aligned}\int_B f_0(x) g_\xi(x) n^K(x) \cdot \operatorname{grad} h_\xi(x)\,\mathrm{d}\lambda_{m-1}(x) \\ > \int_B \psi(x) g_\xi(x) |n^K(x) \cdot \operatorname{grad} h_\xi(x)|\,\mathrm{d}\lambda_{m-1}(x) - \varepsilon.\end{aligned} \tag{21}$$

Since

$$\int_B \psi(x) g_\xi(x) |n^K(x) \cdot \operatorname{grad} h_\xi(x)|\,\mathrm{d}\lambda_{m-1}(x) \leqslant v_\infty^{\psi}(\xi) < \infty,$$

we can assume that $\varrho > 0$ has been chosen small enough to have

$$\int\limits_{B \cap B_\varrho(\xi)} \psi(x) g_\xi(x) |n^K(x) \cdot \operatorname{grad} h_\xi(x)|\,\mathrm{d}\lambda_{m-1}(x) < \varepsilon. \tag{22}$$

Consider first the case when

$$\psi(\xi) g_\xi(\xi) |d_G(\xi) - \tfrac{1}{2}| > 0.$$

Clearly, we can assume that $0 < \varepsilon < \psi(\xi)$. Choose $f_1 \in \mathcal{C}(B)$ with $\operatorname{spt} f_1 \subset B_\varrho(\xi) \cap B$ such that $|f_1| \leqslant \psi$ and

$$|f_1(\xi)| > \psi(\xi) - \varepsilon, \quad \operatorname{sign} f_1(\xi) = \operatorname{sign}[d_G(\xi) - \tfrac{1}{2}].$$

Letting $f = f_0 + f_1$ we have $|f| \leqslant \psi$,

$$
\begin{aligned}
H_g^f(\xi) &= f_1(\xi) g_\xi(\xi)[d_G(\xi) - \tfrac{1}{2}] + \int\limits_{B_\varrho(\xi)\cap B} f_1(x) g_\xi(x) n^K(x) \cdot \operatorname{grad} h_\xi(x) \,\mathrm{d}\lambda_{m-1}(x) \\
&\quad + \int\limits_{B\setminus B_\varrho(\xi)} f_0(x) g_\xi(x) n^K(x) \cdot \operatorname{grad} h_\xi(x) \,\mathrm{d}\lambda_{m-1}(x) \\
&\geqslant \psi(\xi) g_\xi(\xi) |d_G(\xi) - \tfrac{1}{2}| - \varepsilon \\
&\quad - \int\limits_{B\cap B_\varrho(\xi)} \psi g_\xi |n^K \cdot \operatorname{grad} h_\xi| \,\mathrm{d}\lambda_{m-1} + \int_B \psi g_\xi |n^K \cdot \operatorname{grad} h_\xi| \,\mathrm{d}\lambda_{m-1} - \varepsilon \\
&> \psi(\xi) g_\xi(\xi) |d_G(\xi) - \tfrac{1}{2}| + \int_B \psi g_\xi |n^K \cdot \operatorname{grad} h_\xi| \,\mathrm{d}\lambda_{m-1} - 3\varepsilon
\end{aligned}
$$

by (21), (22). The inequality

$$H_g^f(\xi) > F_g^\psi(\xi) - 3\varepsilon$$

with arbitrarily small $\varepsilon > 0$ shows that

$$F_g^\psi(\xi) = \sup\{h(\xi);\ h \in \mathcal{F}\}. \tag{23}$$

If

$$\psi(\xi) g_\xi(\xi) |d_G(\xi) - \tfrac{1}{2}| = 0,$$

then (21) yields

$$H_g^{f_0}(\xi) > F_g^\psi(\xi) - \varepsilon$$

and (23) holds again. Since $\xi \in B$ was arbitrary, the proof is complete. $\square$

10. Corollary. *Let $\psi \geqslant 0$ be a bounded lower-semicontinuous fuction on B, $r \in (0, \infty]$ and define*

$$V_r^\psi(y) \colon = \psi(y) |d_G(y) - \tfrac{1}{2}| + v_r^\psi(y), \quad y \in B.$$

Then

$$V_r^\psi \colon\ y \mapsto V_r^\psi(y)$$

is lower-semicontinuous on B.

Proof. Let $h^n \geqslant 0$ be a nondecreasing sequence of continuous functions on $[0, \infty)$ such that

$$\lim_{n\to\infty} h^n(t) = \begin{cases} 1 & \text{for } t \in [0, r), \\ 0 & \text{elsewhere on } [0, \infty) \end{cases}$$

and put

$$g_x^n(y) = h^n(|x-y|), \quad x, y \in B.$$

Then

$$F_{g^n}^{\psi}(y) \nearrow \psi(y)|d_G(y) - \tfrac{1}{2}| + \int\limits_{B\cap B_r(y)} \psi(x)|n^K(x) \cdot \operatorname{grad} h_y(x)| \,\mathrm{d}\lambda_{m-1}(x) = V_r^{\psi}(y)$$

as $n \to \infty$. Since the functions $F_{g^n}^{\psi}$ are all lower-semicontinuous on B, the same holds of V_r^{ψ}. □

11. Definition. Let p be a norm on $\mathcal{C}(B)$ with the property (15), inducing the topology of uniform convergence; extend p to $\mathcal{C}_+^{\uparrow}(B)$ by (16) and for any $h \in \mathcal{C}_+^{\uparrow}(B)$ put

$$\widehat{p}(h) \colon = p(\widehat{h}), \quad h \in \mathcal{C}_+^{\uparrow}(B),$$

where $\widehat{h}$ is defined by Lemma 3.

Combining this definition with Lemmas 5 and 6 we arrive at

12. Remark. If $\varphi = f + \psi$, where $f \in \mathcal{C}_+(B)$ and $\psi \in \mathcal{C}_+^{\uparrow}(B)$, then $\widehat{p}(\varphi) \leqslant p(f) + \widehat{p}(\psi)$. In particular, $\widehat{p}(f) = p(f)$ whenever $f \in \mathcal{C}_+(B)$.

13. Theorem. *Let p be a norm on $\mathcal{C}(B)$ with (15) inducing the topology of uniform convergence, define $\overline{p}\colon\, y \mapsto \overline{p}(y)$ by (7) and for $r \in (0,\infty)$ put*

$$v_r^{\overline{p}}\colon\, y \mapsto v_r^{\overline{p}}(y), \quad y \in B,$$
$$V_r^{\overline{p}}\colon\, y \mapsto \overline{p}(y)|\tfrac{1}{2} - d_G(y)| + v_r^{\overline{p}}(y), \quad y \in B.$$

Then for each $\alpha \in \mathbb{R}$

$$\omega_p(W^{\alpha}) \leqslant |\alpha - \tfrac{1}{2}| + \inf_{r>0} \widehat{p}(v_r^{\overline{p}}) = |\alpha - \tfrac{1}{2}| + \inf_{r>0} \widehat{p}(V_r^{\overline{p}}). \tag{24}$$

Proof. Fix $r > 0$ and construct a function g^r on $\mathbb{R}^m$ satisfying the Lipschitz condition

$$x^1, x^2 \in \mathbb{R}^m \Longrightarrow |g^r(x^1) - g^r(x^2)| \leqslant \tfrac{1}{r}|x^1 - x^2|$$

and such that

$$0 \leqslant g^r \leqslant 1,\; g^r(B_r(0)) = \{1\},\; g^r(\mathbb{R}^m \setminus B_{2r}(0)) = \{0\}.$$

Put

$$g_y(x) = g^r(x-y), \quad x, y \in \mathbb{R}^m$$

and define an operator V on $\mathcal{C}(B)$ sending each $f \in \mathcal{C}(B)$ to Vf given by

$$Vf(y) = \int_B f(x)[1 - g_y(x)]n^K(x) \cdot \operatorname{grad} h_y(x) \, \mathrm{d}\lambda_{m-1}(x), \quad y \in B.$$

Elementary reasoning (described in detail in the proof of Theorem 4.1 in [9], pp. 104–111) shows that V is a compact linear operator acting in $\mathcal{C}(B)$. We are going to estimate $p(W^\alpha - V)$. Let $f \in \mathcal{C}(B)$, $p(f) \leqslant 1$. Consequently, $|f| \leqslant \overline{p}$ on B. By Proposition 2.8 and Lemmas 2.9 and 2.15 in [9] we have

$$(W^\alpha - V)f(y) = f(y)\big[d_G(y) - \alpha\big] + \int_B f(x) g_y(x) n^K(x) \cdot \operatorname{grad} h_y(x) \, \mathrm{d}\lambda_{m-1}(x), \quad y \in B.$$

Hence

$$\begin{aligned}
|(W^\alpha - V)f(y)| &\leqslant |(\tfrac{1}{2} - \alpha)f(y)| + \overline{p}(y)|d_G(y) - \tfrac{1}{2}| \\
&\quad + \int_B \overline{p}(x) g^r(x-y)|n^K(x) \cdot \operatorname{grad} h_y(x)| \, \mathrm{d}\lambda_{m-1}(x) \\
&= |(\tfrac{1}{2} - \alpha)f(y)| + F_g^{\overline{p}}(y),
\end{aligned}$$

where $F_g^{\overline{p}}$ is the lower-semicontinuous function on B defined in Lemma 9. Since $p(f) \leqslant 1$ implies $p(|f|) \leqslant 1$, in view of Remark 12 we get

$$p[(W^\alpha - V)f] \leqslant |\tfrac{1}{2} - \alpha| p(|f|) + \widehat{p}(F_g^{\overline{p}}) \leqslant |\tfrac{1}{2} - \alpha| + \widehat{p}(F_g^{\overline{p}}).$$

Observe that $F_g^{\overline{p}} \leqslant V_{2r}^{\overline{p}}$, where $V_{2r}^{\overline{p}}$ is a lower-semicontinuous function on B coinciding with $v_{2r}^{\overline{p}}$ on $\widehat{B}$, so that $\widehat{p}(V_{2r}^{\overline{p}}) = \widehat{p}(v_{2r}^{\overline{p}})$. Since $r > 0$ was arbitrary, we arrive at

$$\begin{aligned}
p(W^\alpha - V) &\leqslant |\tfrac{1}{2} - \alpha| + \widehat{p}(V_{2r}^{\overline{p}}), \\
\omega_p(W^\alpha) &\leqslant |\tfrac{1}{2} - \alpha| + \inf_{r>0} \widehat{p}(V_{2r}^{\overline{p}}) = |\tfrac{1}{2} - \alpha| + \inf_{r>0} \widehat{p}(v_{2r}^{\overline{p}})
\end{aligned}$$

and (24) is established. □

14. Corollary. *Let $q > 0$ be a bounded lower-semicontinuous function on B such that*

$$q(y) \geqslant \lambda_{m-1}\text{-}\operatorname*{ess\,lim\,inf}_{x \in \widehat{B}, x \to y} q(x), \quad \forall y \in B. \tag{25}$$

For $f \in \mathcal{C}(B)$ define

$$p_q(f) \colon = \sup_{y \in B} \frac{|f(y)|}{q(y)}. \tag{26}$$

Then p_q is a norm on $\mathcal{C}(B)$ inducing the topology of uniform convergence and for each $\alpha \in \mathbb{R}$ we have

$$\omega_{p_q} W^\alpha \leqslant |\alpha - \tfrac{1}{2}| + \inf_{r>0} \sup_{y \in \widehat{B}} \frac{v_r^q(y)}{q(y)}.$$

Proof. Let $\overline{p}_q$ correspond to p_q in the sense of Lemma 1. It is easy to see from (26) that $\overline{p}_q = q$ on B. In view of Theorem 13 it suffices to verify

$$\widehat{p_q}(v_r^q) = \sup_{x \in \widehat{B}} \frac{v_r^q(x)}{q(x)} \tag{27}$$

for any $r > 0$. Recalling Definition 11 we get

$$\widehat{p_q}(v_r^q) = \sup\{p_q(f);\ f \in \mathcal{C}(B),\ |f| \leqslant \widehat{v}_r^q\} = \sup_{y \in B} \frac{\widehat{v}_r^q(y)}{q(y)} \geqslant \sup_{x \in \widehat{B}} \frac{v_r^q(x)}{q(x)}.$$

In order to obtain the desired inequality

$$\sup_{y \in B} \frac{\widehat{v}_r^q(y)}{q(y)} \leqslant \sup_{x \in \widehat{B}} \frac{v_r^q(x)}{q(x)}, \tag{28}$$

consider an arbitrary $y \in B$ with $\widehat{v}_r^q(y) > 0$ and choose $\varepsilon \in (0, \widehat{v}_r^q(y))$. There is a $\varrho > 0$ such that

$$v_r^q(x) \geqslant \widehat{v}_r^q(y) - \varepsilon \quad \text{for } \lambda_{m-1}\text{-a.e. } x \in B_\varrho(y) \cap \widehat{B}.$$

Our assumption (25) guarantees that

$$\lambda_{m-1}(\{x \in B_\varrho(y) \cap \widehat{B};\ q(y) + \varepsilon > q(x)\}) > 0$$

(for otherwise we would have $\lambda_{m-1}\text{-}\operatorname{ess} \liminf\limits_{x \in \widehat{B}, x \to y} q(x) \geqslant q(y) + \varepsilon > q(y)$). As $\lambda_{m-1}(B_\varrho(y) \cap \widehat{B}) > 0$ (cf. Remark 2), there are $x \in B_\varrho(y) \cap \widehat{B}$ for which we have, simultaneously,

$$v_r^q(x) \geqslant \widehat{v}_r^q(y) - \varepsilon,\ q(x) < q(y) + \varepsilon,$$

so that

$$\frac{\widehat{v}_r^q(y) - \varepsilon}{q(y) + \varepsilon} \leqslant \frac{v_r^q(x)}{q(x)} \leqslant \sup_{x \in \widehat{B}} \frac{v_r^q(x)}{q(x)}.$$

Making $\varepsilon \downarrow 0$ we get (28), which completes the proof. □

15. Remark. Since q is lower-semicontinuous, we have

$$\lambda_{m-1}\text{-}\operatorname{ess}\liminf_{x\in\widehat{B},x\to y} q(x) \geqslant \liminf_{x\in\widehat{B},x\to y} q(x) \geqslant q(y),$$

which combined with (25) yields

$$q(y) = \lambda_{m-1}\text{-}\operatorname{ess}\liminf_{x\in\widehat{B},x\to y} q(x) = \liminf_{x\in\widehat{B},x\to y} q(x), \quad y \in B.$$

16. Lemma. *Let p be a norm defining the topology of uniform convergence in $\mathcal{C}(B)$ and define $\overline{p}$ by (7). Suppose that $q \geqslant 0$ is a bounded lower-semicontinuous function on B such that for each $\mu \in \mathcal{C}'(B)$,*

$$\sup\left\{\int_B f\,\mathrm{d}\mu;\ f \in \mathcal{C}(B),\ p(f) \leqslant 1\right\} \geqslant \int_B q\,\mathrm{d}|\mu|, \tag{29}$$

where $|\mu|$ is the indefinite total variation of μ. Then

$$\omega_p W^\alpha \geqslant \inf_{r>0}\sup_{y\in B}\left[|\tfrac{1}{2}-\alpha|\widehat{q}(y) + v_r^{\widehat{q}}(y)\right] / \,\widehat{\overline{p}}(y) \qquad \text{for } \alpha \in \mathbb{R}. \tag{30}$$

If

$$\overline{p}(y) = \liminf_{x\in\widehat{B}\setminus\{y\},x\to y} \overline{p}(x) \qquad \text{for each } y \in \widehat{B}, \tag{31}$$

then

$$\omega_p W^\alpha \geqslant \inf_{r>0}\sup_{y\in\widehat{B}}\left[|\tfrac{1}{2}-\alpha|q(y) + v_r^q(y)\right] / \,\overline{p}(y). \tag{32}$$

Proof. Fix an $\varepsilon > 0$ and denote by $\langle f,\nu\rangle$ ($\equiv \int_B f\,\mathrm{d}\nu$) the pairing between $f \in \mathcal{C}(B)$ and $\nu \in \mathcal{C}'(B)$. As explained in [9], pp. 107–108, there are $\varphi_1,\dots,\varphi_n \in \mathcal{C}(B)$ and $\nu_1,\dots,\nu_n \in \mathcal{C}'(B)$ such that

$$D\colon = \left\{y \in B;\ \sum_{k=1}^n |\nu_k|(y) > 0\right\}$$

is finite and the finite-dimensional operator V sending $f \in \mathcal{C}(B)$ to

$$Vf\colon = \sum_{k=1}^n \langle f,\nu_k\rangle\varphi_k$$

satisfies

$$p(W^\alpha - V) \leqslant \omega_p W^\alpha + \varepsilon.$$

For any $y \in B$ denote by $\delta_y \in \mathcal{C}'(B)$ the Dirac measure concentrated at y and by $\lambda_y \in \mathcal{C}'(B)$ the representing measure of the functional

$$f \mapsto Wf(y) = \int_B f(x)\,\mathrm{d}\lambda_y(x).$$

According to (20)

$$\mathrm{d}\lambda_y(x) = d_G(y)\,\mathrm{d}\delta_y(x) + n^K(x)\cdot \operatorname{grad} h_y(x)\,\mathrm{d}\lambda_{m-1}(x). \tag{33}$$

Observing that

$$p(g) \geqslant \sup_{y\in B} |g(y)|/\overline{p}(y), \quad \forall g \in \mathcal{C}(B),$$

we get

$$\begin{aligned} p(W^\alpha - V) &= \sup_{p(f)\leqslant 1, f\in\mathcal{C}(B)} p((W^\alpha - V)f) \\ &\geqslant \sup_{p(f)\leqslant 1} \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \left| \int_B f\,\mathrm{d}\Big(\lambda_y - \alpha\delta_y - \sum_{k=1}^n \varphi_k(y)\nu_k\Big)\right|. \end{aligned} \tag{34}$$

Now we decompose each ν_k into a continuous part ν_k^1 (not charging singletons) and a finite combination of the Dirac measures; we thus have $\nu_k = \nu_k^1 + \nu_k^2$ and

$$\nu_k^1(M) = \nu_k(M\setminus D), \nu_k^2(M) = \nu_k(M\cap D)$$

for each Borel set M. By virtue of (34) we obtain

$$\begin{aligned} \omega_p(W^\alpha) + \varepsilon &\geqslant \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \sup_{p(f)\leqslant 1} \left| \int_B f\,\mathrm{d}\Big(\lambda_y - \alpha\delta_y - \sum_{k=1}^n \varphi_k(y)\nu_k\Big)\right| \\ &\geqslant \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \int_B q\,\mathrm{d}\Big|\lambda_y - \alpha\delta_y - \sum_{k=1}^n \varphi_k(y)\nu_k\Big| \\ &= \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \Big[\int_B q\,\mathrm{d}\Big|\lambda_y - \alpha\delta_y - \sum_{k=1}^n \varphi_k(y)\nu_k^1\Big| + \int_B q\,\mathrm{d}\Big|\sum_{k=1}^n \varphi_k(y)\nu_k^2\Big|\Big] \\ &\geqslant \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \int_B q\,\mathrm{d}\Big|\lambda_y - \alpha\delta_y - \sum_{k=1}^n \varphi_k(y)\nu_k^1\Big| \\ &\geqslant \sup_{y\in B\setminus D} \frac{1}{\overline{p}(y)} \Big[\int_{B\cap B_r(y)} q\,\mathrm{d}|\lambda_y - \alpha\delta_y| \\ &\qquad - \sum_{k=1}^n \max_{x\in B} |\varphi_k(x)| \sup_{z\in B} q(z) |\nu_k^1|(B\cap B_r(y))\Big] \end{aligned}$$

for any $r>0$. Since $|\nu_k^1|$ does not charge singletons, we have

$$\lim_{r\downarrow 0}|\nu_k^1|(B_r(y)\cap B)=0 \quad \text{uniformly with respect to } y\in B.$$

We can thus choose an $r_0>0$ small enough to ensure the validity of the implication

$$0<r<r_0\Longrightarrow\sum_{k=1}^{n}\max|\varphi_k|(B)\sup q(B)|\nu_k^1|(B_r(y)\cap B)<\varepsilon,\ \forall y\in B.$$

Hence we get

$$\omega_p(W^\alpha)+2\varepsilon\geqslant\sup_{y\in B\setminus D}\frac{1}{\overline{p}(y)}\int\limits_{B\cap B_r(y)}q\,\mathrm{d}\big|\lambda_y-\alpha\delta_y\big|\geqslant\sup_{y\in\widehat{B}\setminus D}\frac{1}{\overline{p}(y)}\big[q(y)|\tfrac12-\alpha|+v_r^q(y)\big]$$

for any $r\in(0,r_0)$ by Lemma 3 in [12]. Recall that

$$H\colon=\{x\in B;\ \widehat{q}(x)\neq q(x)\}\cup D$$

has vanishing λ_{m-1}-measure. By Remark 2 we get for each $x\in B$ a sequence $x_n\in\widehat{B}\setminus H$ such that

$$x_n\to x\ \text{ and }\ \overline{p}(x_n)\to\widehat{\overline{p}}(x)\ \text{ as }\ n\to\infty.$$

Noting that the functions $v_r^{\widehat{q}}=v_r^q$ (cf. Remark 4 in [12]) and $\widehat{q}$ are lower-semicontinuous, we obtain

$$\frac{1}{\widehat{\overline{p}}(x)}\big[\widehat{q}(x)|\tfrac12-\alpha|+v_r^{\widehat{q}}(x)\big]\leqslant\liminf_{n\to\infty}\frac{1}{\overline{p}(x_n)}\big[q(x_n)|\tfrac12-\alpha|+v_r^q(x_n)\big]\leqslant\omega_pW^\alpha+2\varepsilon.$$

We have thus shown

$$\omega_pW^\alpha+2\varepsilon\geqslant\sup_{x\in B}\frac{1}{\widehat{\overline{p}}(x)}\big[\widehat{q}(x)|\tfrac12-\alpha|+v_r^{\widehat{q}}(x)\big]$$

for any $r\in(0,r_0)$, which proves (30), because $\varepsilon>0$ was arbitrary. Assuming (31) and noting that D is finite we get for any $x\in\widehat{B}$ a sequence $x_n\in\widehat{B}\setminus D$ such that

$$x_n\to x \text{ and } \overline{p}(x_n)\to\overline{p}(x) \text{ as } n\to\infty.$$

Hence

$$\frac{1}{\overline{p}(x)}\big[q(x)|\tfrac12-\alpha|+v_r^q(x)\big]\leqslant\liminf_{n\to\infty}\frac{1}{\overline{p}(x_n)}\big[q(x_n)|\tfrac12-\alpha|+v_r^q(x_n)\big]\leqslant\omega_pW^\alpha+2\varepsilon,$$

so that

$$\sup_{x\in\widehat{B}}\frac{1}{\overline{p}(x)}\big[q(x)|\tfrac12-\alpha|+v_r^q(x)\big]\leqslant\omega_pW^\alpha+2\varepsilon$$

and (32) follows. $\square$

17. Lemma. *Let μ be a finite signed Borel measure with support in B. Let $q > 0$ be a bounded lower-semicontinuous function on B and define the norm p_q on $\mathcal{C}(B)$ by (26). Then*

$$\sup\left\{\int_B f\,\mathrm{d}\mu;\ f \in \mathcal{C}(B),\ p_q(f) \leqslant 1\right\} = \int_B q\,\mathrm{d}|\mu|.$$

P r o o f. If $f \in \mathcal{C}(B)$, then $p_q(f) \leqslant 1$ means that $|f| \leqslant q$ on B, so that

$$\int_B f\,\mathrm{d}\mu \leqslant \int_B q\,\mathrm{d}|\mu| \text{ and } \sup\left\{\int_B f\,\mathrm{d}\mu;\ f \in \mathcal{C}(B),\ p_q(f) \leqslant 1\right\} \leqslant \int_B q\,\mathrm{d}|\mu|.$$

In order to prove the converse inequality we fix an arbitrary $\varepsilon > 0$ and consider a nondecreasing sequence $f_n \in \mathcal{C}_+(B)$ such that $f_n \nearrow q$ as $n \to \infty$. Since

$$\lim_{n\to\infty} \int_B f_n\,\mathrm{d}|\mu| = \int_B q\,\mathrm{d}|\mu|$$

we can fix $n \in N$ large enough to have

$$\int_B f_n\,\mathrm{d}|\mu| > \int_B q\,\mathrm{d}|\mu| - \varepsilon. \tag{35}$$

Consider the Hahn decomposition (cf. [14])

$$B = B_+ \cup B_-$$

corresponding to the signed measure μ formed by disjoint Borel sets B_+, B_- such that

$$\mu(B_+ \cap M) = |\mu|(B_+ \cap M), \mu(B_- \cap M) = -|\mu|(B_- \cap M)$$

for each Borel set M. Choose compact sets $Q_+ \subset B_+$ and $Q_- \subset B_-$ such that

$$\int_S q\,\mathrm{d}|\mu| < \varepsilon, \tag{36}$$

where $S = (B_+ \setminus Q_+) \cup (B_- \setminus Q_-)$. Construct a $\varphi \in \mathcal{C}(B)$ satisfying the conditions

$$\varphi(Q_+) = \{1\},\ \varphi(Q_-) = \{-1\},\ |\varphi| \leqslant 1$$

and put $f = \varphi f_n$, so that

$$f \in \mathcal{C}(B),\ p_q(f) \leqslant 1.$$

We then have

$$\int_B f\,\mathrm{d}\mu = \int_{Q_+} f_n\,\mathrm{d}|\mu| + \int_{Q_-} f_n\,\mathrm{d}|\mu| + \int_S \varphi f_n\,\mathrm{d}\mu = \int_B f_n\,\mathrm{d}|\mu| - \int_S f_n\,\mathrm{d}|\mu| + \int_S \varphi f_n\,\mathrm{d}\mu.$$

Noting that

$$\left|\int_S f_n\,\mathrm{d}|\mu|\right| \leqslant \int_S q\,\mathrm{d}|\mu|$$

and

$$\left|\int_S \varphi f_n\,\mathrm{d}\mu\right| \leqslant \int_S q\,\mathrm{d}\mu$$

we conclude from (36), (35) that

$$\int_B f\,\mathrm{d}\mu > \int_B q\,\mathrm{d}|\mu| - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we arrive at

$$\sup\left\{\int_B f\,\mathrm{d}\mu;\ f \in \mathcal{C}(B),\ p_q(f) \leqslant 1\right\} \geqslant \int_B q\,\mathrm{d}|\mu|,$$

which completes the proof. □

18. Theorem. *Let $q > 0$ be a bounded lower-semicontinuous function on B satisfying (25) and define the norm p_q on $\mathcal{C}(B)$ by (26). Then p_q induces the topology of uniform convergence in $\mathcal{C}(B)$ and, for each $\alpha \in \mathbb{R}$,*

$$\omega_{p_q} W^\alpha = |\alpha - \tfrac{1}{2}| + \inf_{r>0} \sup_{y \in \widehat{B}} \frac{v_r^q(y)}{q(y)}.$$

P r o o f. This follows from Corollary 14 and Lemma 16 combined with (27) together with Lemma 17. □

19. R e m a r k. Theorem 18 shows that, for the norm p_q defined on $\mathcal{C}(B)$ by (26), the optimal choice of the parameter α in the equation (4) is $\alpha = \frac{1}{2}$ (compare also 4.2 in [9]), which leads to the Neumann operator $\mathcal{T} = 2W^{1/2}$. Simple examples of domains "built of bricks" in $\mathbb{R}^3$ demonstrate that $\omega_{p_1}\mathcal{T} > 1$ may occur for the maximum norm p_1 while, as shown in [1], [13], for such domains an elementary construction of another norm p topologically equivalent to p_1 such that $\omega_p\mathcal{T} < 1$ is always possible.

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