PURE SUBGROUPS

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Abstract. Let λ be an infinite cardinal. Set $\lambda_0 = \lambda$, define $\lambda_{i+1} = 2^{\lambda_i}$ for every $i = 0, 1, \ldots$, take μ as the first cardinal with $\lambda_i < \mu$, $i = 0, 1, \ldots$ and put $\kappa = (\mu^{\aleph_0})^+$. If F is a torsion-free group of cardinality at least κ and K is its subgroup such that F/K is torsion and $|F/K| \leqslant \lambda$, then K contains a non-zero subgroup pure in F. This generalizes the result from a previous paper dealing with F/K p-primary.

Keywords: torsion-free abelian groups, pure subgroup, P-pure subgroup

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By the word "module" we mean a unital left R-module over an associative ring R with an identity element. Dualizing the notion of the injective envelope of a module [4] H. Bass [1] investigated the projective cover of a module and characterized the class of rings R over which every module has a projective cover. By a projective cover of a module M we mean an epimorphism $\varphi\colon F \to M$ with F projective and such that the kernel K of φ is superfluous in F in the sense that $K + L = F$ implies $L = F$ whenever L is a submodule of F. Recently, the general theory of covers has been studied intensively. If $\mathcal{G}$ is an abstract class of modules (i.e. $\mathcal{G}$ is closed under isomorphic copies) then a homomorphism $\varphi\colon G \to M$ with $G \in \mathcal{G}$ is called a *$\mathcal{G}$-precover* of the module M if for each homomorphism $f\colon F \to M$ with $F \in \mathcal{G}$ there is $g\colon F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover of M is said to be a *$\mathcal{G}$-cover* if every endomorphism f of G with $\varphi f = \varphi$ is an automorphism of G. It is well-known (see e.g. [8]) that an epimorphism $\varphi\colon F \to M$, F projective, is a projective cover of the module M if and only if it is a $\mathcal{P}$-cover of M, where $\mathcal{P}$ denotes the class of all

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projective modules. Denoting by $\mathcal{F}$ the class of all flat modules, Enochs' conjecture [5] whether every module over any associative ring has an $\mathcal{F}$-cover remains still open.

In the general theory of precovers several types of purities have been used. In some cases (see e.g. [8], [7]) the existence of pure submodules in the kernels of some homomorphisms plays an important role. Using the general theory of covers, in [3] the main result of this note appears as a corollary. However, the direct proof presented here is of some interest because the existence of non-zero pure submodules of "large" flat modules contained in submodules with "small" factors is sufficient for the existence of flat covers (see [3]). The purity here is meant in the sense of P. M. Cohn.

All groups are additively written abelian groups. If P is any set of primes then the P-purity of a subgroup S of a group F means its p-purity for each prime $p \in P$. Other notation and terminology are essentially the same as in [6].

We start with three preliminary statements, the first extending the result of [2].

Lemma 1. *Let λ be an infinite cardinal. If F is a torsion-free group of cardinality greater than 2^λ and K is its subgroup such that the factor-group F/K is p-primary for some prime p and $|F/K| \leqslant \lambda$, then K contains a subgroup S pure in F such that $|F/S| \leqslant 2^\lambda$.*

P r o o f. By [2; Theorem 1] K contains a non-zero subgroup pure in F. Since pure subgroups are closed under unions of ascending chains, there is a maximal subgroup S of K pure in F. If $|F/S| > 2^\lambda$ then $F/S/K/S \cong F/K$ is p-primary and so K/S contains a non-zero subgroup T/S pure in F/S. Thus $T \subseteq K$ is pure in F, and so the proper inclusion $S \subset T$ contradicts the maximality of S. □

Lemma 2. *If K is a subgroup of a torsion-free group F, then for any prime p the subgroup $F(p)$ consisting of all elements $x \in F$ such that $p^k x \in K$ for some non-negative integer k is p-pure in F.*

P r o o f. Obvious. □

Lemma 3. *Let λ be an infinite cardinal and let $P \subseteq \Pi$ be any subset of the set Π of all primes. Further, let F be a torsion-free group of cardinality greater than 2^λ and K its subgroup such that K is P-pure in F and $|F/K| \leqslant \lambda$. Then for each prime $p \in \Pi \setminus P$ there is a subgroup S of K such that S is $\overline{P}$-pure in F, $\overline{P} = P \cup \{p\}$, and $|F/S| \leqslant 2^\lambda$.*

P r o o f. If $(F/K)_p = 0$, then it clearly suffices to take $S = K$. In the opposite case we set $F(p) = \{x \in F \mid p^k x \in K \text{ for a non-negative integer } k\}$ and we obviously have $2^\lambda < |K| \leqslant |F(p)| \leqslant |F|$, $F(p)/K$ is p-primary and $|F(p)/K| \leqslant \lambda$. By Lemma 1,

K contains a subgroup S pure in $F(p)$ such that $|F(p)/S| \leqslant 2^{\lambda}$. The transitivity of p-purity and Lemma 2 now yield that S is p-pure in F. Morever, S is pure in $F(p)$, hence in K and consequently it is P-pure in F by the hypothesis and the transitivity of P-purity. Finally, F/S is an extension of $F(p)/S$ by $F/F(p)$, where $|F(p)/S| \leqslant 2^{\lambda}$, $|F/F(p)| \leqslant |F/K| \leqslant \lambda < 2^{\lambda}$, which yields $|F/S| \leqslant 2^{\lambda}$. □

Let λ be an infinite cardinal. Set $\lambda_0 = \lambda$, define $\lambda_{i+1} = 2^{\lambda_i}$ for every $i = 0, 1, \ldots$, take μ as the first cardinal with $\lambda_i < \mu$, $i = 0, 1, \ldots$, and put $\kappa = (\mu^{\aleph_0})^+$. Now we are ready to prove our main result.

Theorem. *Let λ be an infinite cardinal. If F is a torsion-free group of cardinality at least κ and K is its subgroup such that the factor-group F/K is a torsion group of cardinality at most λ, then K contains a non-zero subgroup pure in F.*

P r o o f. Let $\Pi = \{p_1, p_2, \ldots\}$ be any list of elements of the set Π of all primes and for every $i = 1, 2, \ldots$ let $P_i = \{p_1, p_2, \ldots, p_i\}$. By Lemma 3 there is a subgroup S_1 of K P_1-pure in F such that $|F/S_1| \leqslant 2^{\lambda_0} = \lambda_1$. Continuing by induction let us suppose that the subgroups $S_1, S_2, \ldots, S_k$ of K have been already constructed in such a way that

$$\begin{aligned}
&S_{i+1} \leqslant S_i,\ i = 1, \ldots, k-1,\\
&S_i \text{ is } P_i\text{-pure in } F,\ i = 1, 2, \ldots k,\\
&|F/S_i| \leqslant \lambda_i,\ i = 1, 2, \ldots, k.
\end{aligned}$$

An application of Lemma 3 yields the existence of $S_{k+1} \subseteq S_k$ such that S_{k+1} is P_{k+1}-pure in F and $|F/S_{k+1}| \leqslant 2^{\lambda_k} = \lambda_{k+1}$. Setting $S = \bigcap\limits_{i=1}^{\infty} S_i$ and assuming that the equation $p_j^l x = s$, $s \in S$, $p_j \in \Pi$ is solvable in F we see that $s \in S_k$ for all $k \geqslant j$. However, S_k is p_j-pure in F, which means that $x \in \bigcap\limits_{k=j}^{\infty} S_k = \bigcap\limits_{i=1}^{\infty} S_i = S$, showing the purity of S in F. It remains now to show that S in non-zero. However, there is a natural embedding $\varphi\colon F/S \to \prod\limits_{i=1}^{\infty} F/S_i$ given by the formula $\varphi(x+S) = (x + S_1, x + S_2, \ldots)$. Now the inequalities $|F/S_i| \leqslant \lambda_i < \mu$ yield $|F/S| \leqslant \mu^{\aleph_0} < \kappa$, hence $|S| = |F| \geqslant \kappa$ and the proof is complete. □

Corollary 1. *Under the same hypotheses as in Theorem, K contains a subgroup S pure in F such that $|F/S| \leqslant \mu^{\aleph_0}$.*

P r o o f. It runs along the same lines as that of Lemma 1. □

Corollary 2. *If F is a torsion-free group of cardinality at least κ and K is a subgroup of F such that $|F/K| \leqslant \lambda$ then K contains a non-zero subgroup S pure in F.*

P r o o f. Let L/K be the torsion part of F/K. Since $|L| \geqslant |K| = |F| \geqslant \kappa$ and $|L/K| \leqslant |F/K| \leqslant \lambda$, K contains a non-zero subgroup S pure in L by Theorem. Hence S is pure in F, L being so by its choice. □

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