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AXISYMMETRIC FLOW OF NAVIER-STOKES FLUID IN THE WHOLE SPACE WITH NON-ZERO ANGULAR VELOCITY COMPONENT

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We study axisymmetric solutions to the Navier-Stokes equations in the whole three-dimensional space. We find conditions on the radial and angular components of the velocity field which are sufficient for proving the regularity of weak solutions.

Keywords: axisymmetric flow, Navier-Stokes equations, regularity of systems of PDE's

MSC 2000: 35Q35, 35J35

1. Introduction

Let the incompressible fluid fill up the whole three-dimensional space. Then its flow is described by the Navier-Stokes system

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div}\mathbf{u} = 0 \end{aligned}\right\} \text{ in } (0,T)\times\mathbb{R}^3 \tag{1.1}$$

with the initial condition

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}^0(\mathbf{x}). \tag{1.2}$$

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The symbols $\mathbf{u}$ and p denote the unknown velocity and pressure, $\mathbf{f}$ is an external body force and $\nu > 0$ is the viscosity coefficient.

While the existence of a weak solution to the system (1.1)–(1.2) for the right-hand side $\mathbf{f} \in L^2(0,T;W^{-1,2}(\mathbb{R}^3))$ and $\mathbf{u}^0 \in L^2(\mathbb{R}^3)$ has been known for a long time (see J. Leray [7]) its uniqueness and regularity are fundamental open questions.

On the other hand, positive results are known provided additional conditions are imposed on the regularity of solutions. Thus, for example the uniqueness has been proved if the weak solution belongs also to the class $L^{r,s}(Q_T)$ with $\frac{2}{r} + \frac{3}{s} \leqslant 1$, $r \in [2,+\infty]$, $s \in [3,+\infty]$ (see G. Prodi [11], J. Serrin [12], H. Sohr, W. von Wahl [13], H. Kozono, H. Sohr [4], H. Kozono [3]). Moreover, if the weak solution is in $L^{r,s}(Q_T)$ with $\frac{2}{r} + \frac{3}{s} \leqslant 1$, $r \in [2,+\infty]$, $s \in (3,+\infty]$ and the input data are "smooth enough" then it is already a strong solution (see G. P. Galdi [1], Y. Giga [2]). The question whether the weak solution which is in $L^{\infty,3}(Q_T)$ is a strong solution is still open.

Let us further note that under a strong solution we understand a weak solution which has the maximal possible regularity, i.e. for our data the function $\mathbf{u}$ and the corresponding pressure field p are such that $\mathbf{u} \in L^2(0,T;W^{3,2}(\mathbb{R}^3)) \cap L^\infty(0,T;W^{2,2}(\mathbb{R}^3))$, $\frac{\partial \mathbf{u}}{\partial t}, \nabla p \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$. It is an easy matter to show (see e.g. [1]) that such a solution is unique in the class of all weak solutions satisfying the energy inequality.

The situation is much simpler in the case of planar (i.e. two-dimensional) flows where the existence of strong solutions and their uniqueness is known. (See J. Leray [7].) A natural question arises whether the same can also be proved for axisymmetric flows. However, the affirmative answer has up to now been given only in the case that the data are axisymmetric with zero angular components (see O. A. Ladyzhenskaya [5], M. R. Uchovskii, B. I. Yudovich [14] and S. Leonardi, J. Málek, J. Nečas, M. Pokorný [6]).

The question whether the components of velocity are coupled in such a way that some information about a higher regularity of one of them already implies the higher regularity of all of them was dealt with in the papers of J. Neustupa, P. Penel [9] and J. Neustupa, A. Novotný, P. Penel [8]. In [8], the authors proved that if $\mathbf{u}$ is a so called suitable weak solution in the sense of Caffarelli, Kohn, Nirenberg with one velocity component belonging to $L^{r,s}(D)$ with $\frac{2}{r} + \frac{3}{s} \leqslant \frac{1}{2}$, $r \in [4,+\infty]$, $s \in (6,+\infty]$ then the solution is, for the right-hand side sufficiently smooth, necessarily a strong one. Although the results were shown on bounded subdomains of the time-space cylinder and for suitable weak solutions, the method can also be easily applied to the Cauchy problem in the case of a weak solution satisfying the energy inequality.

This paper deals with a similar problem as the above mentioned papers [8] and [9], however, we study an axisymmetric flow (see Definition 1 below.)

R e m a r k 1. We will often use the so called cylindrical coordinates; the relations between the cartesian and the cylindrical coordinates of a vector field read as follows:

$$
\begin{aligned}
w_\varrho &= w_1 \cos\theta + w_2 \sin\theta,\\
w_\theta &= -w_1 \sin\theta + w_2 \cos\theta,\\
w_z &= w_3.
\end{aligned}
$$

Definition 1. A flow is called axisymmetric if the pressure p and the cylindrical velocity components u_ϱ, u_θ, u_z are independent of the angular variable θ.

The main theorems proved in this paper are

Theorem 1. *Let* $(\mathbf{u}, p)$ *be a weak solution to problem* (1.1)–(1.2) *satisfying the energy inequality with* $\mathbf{f} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$ *and* $\mathbf{u}^0 \in W^{2,2}(\mathbb{R}^3)$. *Let* $\mathbf{u}^0$ *and* $\mathbf{f}$ *be axisymmetric. Suppose further that the radial component* u_ϱ *of* $\mathbf{u}$ *belongs*[1] *to* $L^{r,s}(Q_T)$ *for some* $r \in [2,+\infty]$, $s \in (3,+\infty]$, $\frac{2}{r} + \frac{3}{s} \leqslant 1$. *Then* $(\mathbf{u}, p)$ *is an axisymmetric strong solution to problem* (1.1)–(1.2) *which is unique in the class of all weak solutions satisfying the energy inequality.*

Theorem 2. *Let* $(\mathbf{u}, p)$ *be a weak solution to problem* (1.1)–(1.2) *satisfying the energy inequality with* $\mathbf{f} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$ *and* $\mathbf{u}^0 \in W^{2,2}(\mathbb{R}^3)$. *Let* $\mathbf{u}^0$ *and* $\mathbf{f}$ *be axisymmetric.*

(i) *Suppose further that the angular component* u_θ *of* $\mathbf{u}$ *belongs to* $L^{r,s}(Q_T)$ *for some* $r \in [\frac{20}{7},+\infty]$, $s \in [6,+\infty]$, $\frac{2}{r} + \frac{3}{s} \leqslant \frac{7}{10}$. *Then* $(\mathbf{u}, p)$ *is an axisymmetric strong solution to problem* (1.1)–(1.2).

(ii) *Let* $\varrho u_\theta^0 \in L^\infty(\mathbb{R}^3)$ *and* $f_\theta \varrho \in L^{1,\infty}(Q_T)$. *Suppose further that the angular component* u_θ *of* $\mathbf{u}$ *belongs to* $L^{r,s}(Q_T)$ *for some* $r \in (10,+\infty]$, $s \in (\frac{24}{5},6)$, $\frac{2}{r} + \frac{3}{s} \leqslant 1 - \frac{9}{5s}$. *Then* $(\mathbf{u}, p)$ *is an axisymmetric strong solution to problem* (1.1)–(1.2).

In both cases the solution is unique in the class of all weak solutions satisfying the energy inequality.

Note that our conditions on the radial and the angular velocity component, respectively, are less restrictive than the conditions required for one velocity component in the case of the whole Navier-Stokes system (cf. [8]), but (for the angular component) more restrictive than the Prodi-Serrin condition which, on the contrary, has to be fulfilled by all velocity components. The condition on the radial component is then exactly the same as the Prodi-Serrin condition.

[1] See also Remark 2.

2. Auxiliary results

We use the standard notation for the Lebesgue spaces $L^p(\mathbb{R}^3)$ equipped with the standard norm $\|\cdot\|_{p,\mathbb{R}^3}$, the Sobolev spaces $W^{k,p}(\mathbb{R}^3)$ equipped with the standard norm $\|\cdot\|_{k,p,\mathbb{R}^3}$, $W^{-1,2}(\mathbb{R}^3)$ for the dual space to $W^{1,2}(\mathbb{R}^3)$. By $L^{r,s}(Q_T)$, $Q_T = (0,T)\times\mathbb{R}^3$ we denote the anisotropic Lebesgue space $L^r(0,T;L^s(\mathbb{R}^3))$. If no confusion can arise then we skip writing $\mathbb{R}^3$ and Q_T, respectively.

Vector-valued functions are printed boldfaced. Nonetheless, we do not distinguish between $L^q(\mathbb{R}^3)^3$ and $L^q(\mathbb{R}^3)$.

In order to keep a simple notation, all generic constants will be denoted by C; thus C can have different values from term to term, even in the same formula.

We will need the following result about the existence of local-in-time regular solutions.

Lemma 1. *Let* $\mathbf{f} \in L^2_{\mathrm{loc}}(0,T;W^{1,2})$, $\mathbf{u}^0 \in W^{2,2}$. *Then there exists* $t_0 > 0$ *and* $(\mathbf{u},p)$, *a weak solution to system* (1.1)–(1.2), *which is a strong solution on the time interval* $(0,t_0)$. *Moreover, if* $\mathbf{f}$ *and* $\mathbf{u}^0$ *are axisymmetric then also the strong solution is axisymmetric.*

Proof. The lemma is classical and it is based on the Banach fixed point theorem and regularity properties of the non-stationary Stokes system. Moreover, if the data to the Stokes system are axisymmetric then the solution is also axisymmetric and thus the same holds for the fixed point. □

Now let $\mathbf{f}$ and $\mathbf{u}^0$ be as in Lemma 1. We define

$$t^* = \sup\big\{t > 0;\ \ \text{there exists an axisymmetric strong solution to (1.1)–(1.2) on } (0,t)\big\}.$$

It follows from Lemma 1 that $t^* > 0$. Now, let $(\mathbf{u},p)$ be a weak solution to the Navier-Stokes system as in Theorem 1 and Theorem 2. Due to the uniqueness property, it coincides with the strong solution from Lemma 1 on any compact subinterval of $[0,t^*)$. There are two possibilities. Either $t^* = T$ (T may also be equal to ∞) and we have the global-in-time regular solution, or $t^* < T$. In the latter case, after redefining $\mathbf{u}$ on a set of zero measure, we necessarily have

$$\|\mathbf{u}(t)\|_{1,2} \to \infty \text{ for } t \to t^*.$$

(Note that $\mathbf{u} \in C([0,t];W^{2,2})$ for $t < t^*$.)

However, we will exclude this possibility by showing that

$$\|\mathbf{u}\|_{L^\infty(0,t;W^{1,2})} \leqslant C,$$

where C remains bounded for $t \to t^*$. To this aim we will essentially use both the information about the better regularity of one velocity component and the fact that the solution is axisymmetric.

Now, let $0 < \bar{t} < t^*$. Then on $(0,\bar{t})$, $(\mathbf{u},p)$ is in fact a strong solution to the Navier-Stokes system. It is convenient to write the Navier-Stokes system in the cylindrical coordinates for our purpose.

Thus, u_ϱ, u_θ, u_z and p satisfy a.e. in $(0,\bar{t}) \times \mathbb{R}^3$ the system

$$
\begin{aligned}
&\frac{\partial u_\varrho}{\partial t} + u_\varrho \frac{\partial u_\varrho}{\partial \varrho} + u_z \frac{\partial u_\varrho}{\partial z} - \frac{1}{\varrho} u_\theta^2 + \frac{\partial p}{\partial \varrho} - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} (\varrho \frac{\partial u_\varrho}{\partial \varrho}) + \frac{\partial^2 u_\varrho}{\partial z^2} - \frac{u_\varrho}{r^2} \Big] = f_\varrho, \\
&\frac{\partial u_\theta}{\partial t} + u_\varrho \frac{\partial u_\theta}{\partial \varrho} + u_z \frac{\partial u_\theta}{\partial z} + \frac{1}{\varrho} u_\theta u_\varrho - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} (\varrho \frac{\partial u_\theta}{\partial \varrho}) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{\varrho^2} \Big] = f_\theta, \\
&\frac{\partial u_z}{\partial t} + u_\varrho \frac{\partial u_z}{\partial \varrho} + u_z \frac{\partial u_z}{\partial z} + \frac{\partial p}{\partial z} - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} (\varrho \frac{\partial u_z}{\partial \varrho}) + \frac{\partial^2 u_z}{\partial z^2} \Big] = f_z, \\
&\frac{\partial u_\varrho}{\partial \varrho} + \frac{u_\varrho}{\varrho} + \frac{\partial u_z}{\partial z} = 0 .
\end{aligned}
\tag{2.1}
$$

Moreover, if we put $\omega_\varrho = -\frac{\partial u_\theta}{\partial z}$, $\omega_\theta = \frac{\partial u_\varrho}{\partial z} - \frac{\partial u_z}{\partial \varrho}$, $\omega_z = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho u_\theta)$ (ω_ϱ, ω_θ and ω_z are the cylindrical components of $\operatorname{curl}\mathbf{u}$), then we also have a.e. in $(0,\bar{t}) \times \mathbb{R}^3$

$$
\begin{aligned}
&\frac{\partial \omega_\varrho}{\partial t} + u_\varrho \frac{\partial \omega_\varrho}{\partial \varrho} + u_z \frac{\partial \omega_\varrho}{\partial z} - \frac{\partial u_\varrho}{\partial \varrho} \omega_\varrho - \frac{\partial u_\varrho}{\partial z} \omega_z \\
&\qquad\qquad - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} \Big(\varrho \frac{\partial \omega_\varrho}{\partial \varrho} \Big) + \frac{\partial^2 \omega_\varrho}{\partial z^2} - \frac{\omega_\varrho}{\varrho^2} \Big] = g_\varrho \\
&\frac{\partial \omega_\theta}{\partial t} + u_\varrho \frac{\partial \omega_\theta}{\partial \varrho} + u_z \frac{\partial \omega_\theta}{\partial z} - \frac{u_\varrho}{\varrho} \omega_\theta + \frac{2}{\varrho} u_\theta \omega_\varrho \\
&\qquad\qquad - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} \Big(\varrho \frac{\partial \omega_\theta}{\partial \varrho} \Big) + \frac{\partial^2 \omega_\theta}{\partial z^2} - \frac{\omega_\theta}{\varrho^2} \Big] = g_\theta \\
&\frac{\partial \omega_z}{\partial t} + u_\varrho \frac{\partial \omega_z}{\partial \varrho} + u_z \frac{\partial \omega_z}{\partial z} - \frac{\partial u_z}{\partial z} \omega_z - \frac{\partial u_z}{\partial \varrho} \omega_\varrho \\
&\qquad\qquad - \nu \Big[\frac{1}{\varrho} \frac{\partial}{\partial \varrho} \Big(\varrho \frac{\partial \omega_z}{\partial \varrho} \Big) + \frac{\partial^2 \omega_z}{\partial z^2} \Big] = g_z,
\end{aligned}
\tag{2.2}
$$

where g_ϱ, g_θ and g_z denote the components of $\operatorname{curl}\mathbf{f}$ in the cylindrical coordinates.

We close this section by showing several properties of axisymmetric functions which will be used later. In what follows, Dg denotes the cartesian components of the gradient of g while ∇g denotes $(\frac{\partial g}{\partial \varrho}, \frac{\partial g}{\partial z})$. Moreover, by $\mathbf{w}$ we denote $\operatorname{curl}\mathbf{v}$ while ω_ϱ, ω_θ and ω_z will be the cylindrical components of $\mathbf{w}$.

Lemma 2. *Let* $\mathbf{v}$ *be a sufficiently smooth vector field. Then there exists* $C(p) > 0$, *independent of* $\mathbf{v}$, *such that for* $1 < p < \infty$

$$\|D\mathbf{v}\|_p \leqslant C(p)(\|\mathbf{w}\|_p + \|\operatorname{div}\mathbf{v}\|_p).$$

Proof. This is a classical result based on the Fourier transform and the Marcinkiewicz multiplier theorem. $\square$

Lemma 3. *Let* $\mathbf{v}$ *be a sufficiently smooth divergence-free axisymmetric vector field. Then there exist constants* $C_1(p) > 0$ *and* $C_2 > 0$, *independent of* $\mathbf{v}$, *such that for* $1 < p < \infty$ *we have*

$$\|\nabla v_\varrho\|_p + \Big\|\frac{v_\varrho}{\varrho}\Big\|_p \leqslant C_1(p)\|\omega_\theta\|_p,$$
$$\Big\|\frac{\partial}{\partial\varrho}\Big(\frac{v_\varrho}{\varrho}\Big)\Big\|_p \leqslant C_2\|D^2\mathbf{v}\|_p.$$

Proof. Since both $\operatorname{div}\mathbf{v}$ and $(\operatorname{curl}\mathbf{v})_\theta$ are independent of v_θ (because $\mathbf{v}$ is axisymmetric), we can assume without loss of generality that $v_\theta = 0$. The result now follows from the direct calculation of each term. $\square$

Lemma 4. *Let* $\mathbf{v}$ *be a sufficiently smooth axisymmetric vector field. Then there exists* $C > 0$, *independent of* $\mathbf{v}$, *such that for* $1 \leqslant p \leqslant \infty$

$$\|\nabla v_\theta\|_p + \Big\|\frac{v_\theta}{\varrho}\Big\|_p \leqslant C\|D\mathbf{v}\|_p,$$
$$\Big\|\frac{\partial}{\partial\varrho}\Big(\frac{v_\theta}{\varrho}\Big)\Big\|_p \leqslant C\|D^2\mathbf{v}\|_p.$$

Proof. These estimates can be obtained by direct calculation of all terms on the left-hand sides. $\square$

Lemma 5. *Let* $\mathbf{v}$ *be a sufficiently smooth divergence-free axisymmetric vector field. Then there exist* $C_1(p)$, C_2, *independent of* $\mathbf{v}$, *such that for* $1 < p < \infty$

$$C_1(p)\|D^2\mathbf{v}\|_p \leqslant \Big\|\frac{\omega_\varrho}{\varrho}\Big\|_p + \Big\|\frac{\omega_\theta}{\varrho}\Big\|_p + \|\nabla\omega_\varrho\|_p + \|\nabla\omega_\theta\|_p + \|\nabla\omega_z\|_p \leqslant C_2\|D^2\mathbf{v}\|_p.$$

Proof. We can proceed as above and use Lemma 2. $\square$

Lemma 6. *Let* $\mathbf{v}$ *be a sufficiently smooth axisymmetric vector field. Then there exists* $C > 0$, *independent of* $\mathbf{v}$, *such that for* $1 \leqslant p \leqslant \infty$ *we have*

$$\Big\| \frac{\partial}{\partial \varrho}\Big(\frac{\omega_\theta}{\varrho}\Big)\Big\|_p \leqslant C\|D^2\mathbf{w}\|_p.$$

P r o o f. It can be done by analogy with Lemma 4. □

Lemma 7. *Let* $\mathbf{v}$ *be a sufficiently smooth axisymmetric vector function. Then for every* $\varepsilon \in (0,1]$ *there exists* $C(\varepsilon) > 0$, *independent of* $\mathbf{v}$, *such that*

$$\Big\|\frac{\omega_\theta}{\varrho^{2-\varepsilon}}\Big\|_2 + \Big\|\frac{1}{\varrho^{1-\varepsilon}}\frac{\partial \omega_\theta}{\partial \varrho}\Big\|_2 \leqslant C(\varepsilon)\|D\mathbf{w}\|_{1,2}.$$

Moreover,

$$\lim_{\varrho_0\to 0^+}\int_{-\infty}^{\infty}\Big(\frac{\partial\omega_\theta}{\partial\varrho}\frac{\omega_\theta}{\varrho_0^{1-\varepsilon}}\Big)(\varrho_0, z)\,\mathrm{d}z = 0.$$

P r o o f. It can be done in the same way as the proof of an analogous result in [6] where v_θ is assumed to be zero. □

Lemma 8. *Let* $\mathbf{v}$ *be a sufficiently smooth divergence-free axisymmetric vector field. Let* $V_R = \{(x_1,x_2,x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 < R^2\}$. *Then for any* $p \in (1,\infty)$ *there exists* $C(p)$, *independent of* $\mathbf{v}$, *such that*

$$\Big\|\frac{v_\varrho}{\varrho}\Big\|_{p,V_1} \leqslant C(p)\big(\|\omega_\theta\|_{p,V_2} + \|v_\varrho\|_{p,V_2} + \|v_z\|_{p,V_2}\big).$$

P r o o f. Using an appropriate cut-off function, we can show the inequality by combining Lemmas 2 and 4. □

3. Proof of Theorem 1

In what follows, $L^{p,q}$ will denote $L^p(0,\overline{t};L^q)$ with $\overline{t} < t^*$. Our aim is to get an estimate of $\operatorname{curl}\mathbf{u}$ in $L^{\infty,2}$ independent of $\overline{t}$ for $\overline{t} \to t^*$ which, due to Lemma 2, implies the desired estimate of $\mathbf{u}$ in $L^\infty(0,\overline{t};W^{1,2})$. We proceed in three steps.

S t e p 1: Take $q > 2$, multiply $(2.1)_2$ by $|u_\theta|^{q-2}u_\theta$ and integrate over $\mathbb{R}^3$. In what follows $\int\ldots$ will denote $\int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi}\ldots\varrho\,\mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}\varrho$. Then, using the divergence-free condition $(2.1)_4$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\theta\|_q^q + \frac{4(q-1)}{q}\nu\int\Bigl(\Bigl|\frac{\partial}{\partial\varrho}(|u_\theta|^{\frac{q}{2}})\Bigr|^2+\Bigl|\frac{\partial}{\partial z}(|u_\theta|^{\frac{q}{2}})\Bigr|^2\Bigr)+\nu q\int\frac{|u_\theta|^q}{\varrho^2} = -q\int\frac{u_\varrho}{\varrho}|u_\theta|^q + q\int|u_\theta|^{q-2}u_\theta f_\theta. \tag{3.1}$$

Note that in order to treat the convective term we have integrated by parts and the boundary integrals have vanished—at $\varrho = 0$ due to the boundedness of $\frac{u_\theta}{\varrho}$ while near $\varrho=\infty$ due to the standard density argument.

We can estimate the first term on the right-hand side of (3.1):

$$\int\frac{|u_\varrho|}{\varrho}|u_\theta|^q \leqslant \frac{\nu}{2}\int\frac{|u_\theta|^q}{\varrho^2}+\frac{\nu}{q}\int\Bigl(\Bigl|\frac{\partial}{\partial\varrho}(|u_\theta|^{\frac{q}{2}})\Bigr|^2+\Bigl|\frac{\partial}{\partial z}(|u_\theta|^{\frac{q}{2}})\Bigr|^2\Bigr) + C(\nu)q^{\frac{3}{s-3}}\|u_\varrho\|_s^{\frac{2s}{s-3}}\|u_\theta\|_q^q.$$

R e m a r k 2. Note that in order to estimate the first term on the right-hand side of (3.1) it is enough to assume that only the negative part of u_ϱ belongs to $L^{r,s}$; the positive part has good sign and can be put to the left-hand side of (3.1).

The other term can be estimated in a standard way. So, using the Gronwall inequality, we get

$$\|u_\theta\|^q_{L^{\infty,q}} + C\int_0^t\int\Bigl(|\nabla(|u_\theta|^{\frac{q}{2}})|^2+\frac{|u_\theta|^q}{\varrho^2}\Bigr)\leqslant C(q,\|u_\varrho\|_{L^{\frac{2s}{s-3},s}},\|u_\theta^0\|_q,\|f_\theta\|_{L^{2,6}}).$$

Unfortunately, the constant on the right-hand side depends exponentially on q^a for some $a>1$ and thus we cannot pass with q to infinity.

S t e p 2 : We multiply equation $(2.2)_2$ by $\frac{\omega_\theta}{\varrho^{2-\varepsilon}}$ for some $\varepsilon>0$ and integrate over $\mathbb{R}^3$. In fact we would like to take $\varepsilon=0$, but it is impossible because we could not control the convergence of several integrals. The situation is exactly the same as in [6].

Now, with $\varepsilon>0$, we can apply the Green identity and the "boundary" terms vanish (the most delicate term due to Lemma 7) and we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigl\|\frac{\omega_\theta}{\varrho^{1-\frac{\varepsilon}{2}}}\Bigr\|_2^2+\nu\int\Bigl(\Bigl|\nabla\Bigl(\frac{\omega_\theta}{\varrho^{1-\frac{\varepsilon}{2}}}\Bigr)\Bigr|^2+\Bigl(\varepsilon-\frac{\varepsilon^2}{4}\Bigr)\Bigl|\frac{\omega_\theta}{\varrho^{2-\frac{\varepsilon}{2}}}\Bigr|^2\Bigr) = \int g_\theta\frac{\omega_\theta}{\varrho^{2-\varepsilon}}+\frac{\varepsilon}{2}\int\frac{u_\varrho}{\varrho}\frac{\omega_\theta^2}{\varrho^{2-\varepsilon}}+2\int\frac{\omega_\varrho}{\varrho}u_\theta\frac{\omega_\theta}{\varrho^{2-\varepsilon}}.$$

We estimate the first and the second term exactly as in [6].

Recalling that $\omega_\varrho = -\frac{\partial u_\theta}{\partial z}$, after having integrated by parts and having used the Hölder inequality we estimate the last term by

$$\frac{\nu}{4}\int\Big|\frac{\partial}{\partial z}\frac{\omega_\theta}{\varrho^{1-\frac{\varepsilon}{2}}}\Big|^2 + C(\nu)\int\Big|\frac{u_\theta}{\varrho^{1-\frac{\varepsilon}{4}}}\Big|^4.$$

Thus, applying the Gronwall inequality ($0<\tau<\overline{t}$) and passing with ε to 0 we get

$$\Big\|\frac{\omega_\theta}{\varrho}\Big\|_2^2(\tau)+\nu\int_0^\tau\int\Big|\nabla\Big(\frac{\omega_\theta}{\varrho}\Big)\Big|^2 \leqslant C(\|\mathbf{f}\|_{L^2(0,\tau;W^{1,2})},\|\mathbf{u}^0\|_{2,2})+C\int_0^\tau\int\Big|\frac{u_\theta}{\varrho}\Big|^4. \tag{3.2}$$

In order to estimate the last term on the right-hand side we use the equation for u_θ; namely, we multiply it by $\frac{u_\theta^3}{\varrho^2}$ and we get

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{u_\theta^4}{\varrho^2}+\nu\int\Big(\Big|\frac{\partial u_\theta}{\partial\varrho}\Big|^2+3\Big|\frac{\partial u_\theta}{\partial z}\Big|^2\Big)\frac{u_\theta^2}{\varrho^2}+2\nu\int\frac{u_\theta^2}{\varrho}\Big|\frac{\partial}{\partial\varrho}\Big(\frac{u_\theta}{\varrho^{\frac12}}\Big)\Big|^2+\frac{\nu}{2}\int\frac{u_\theta^4}{\varrho^4}\\ =\int f_\theta\frac{u_\theta^3}{\varrho^2}-\frac{3}{2}\int\frac{u_\varrho}{\varrho}\frac{u_\theta^4}{\varrho^2}.\end{aligned}$$

(Note that all terms are finite.) The first term on the right-hand side can be easily estimated. Further, we apply the Gronwall inequality, multiply the resulting inequality by a sufficiently large constant and sum it up with estimate (3.2). So we get

$$\begin{aligned}\Big\|\frac{\omega_\theta}{\varrho}\Big\|_2^2(\tau)+\Big\|\frac{u_\theta}{\varrho^{\frac12}}\Big\|_4^4(\tau)+C(\nu)\int_0^\tau\Big(\Big\|\frac{u_\theta}{\varrho}\Big\|_4^4+\Big\|\nabla\Big(\frac{\omega_\theta}{\varrho}\Big)\Big\|_2^2\Big)\\ \leqslant C(\mathbf{u}^0,\mathbf{f})+C\int_0^\tau\int\frac{|u_\varrho|}{\varrho}\frac{u_\theta^4}{\varrho^2}.\end{aligned} \tag{3.3}$$

We divide the last integral into two parts; the integral over $V_1=\{(x_1,x_2,x_3)\in\mathbb{R}^3; x_1^2+x_2^2<1\}$ and over $\mathbb{R}^3\setminus V_1$. In the latter one we use that $\varrho\geqslant 1$, while to V_1 we apply Lemma 8 and use the boundedness of u_θ in $L^{\infty,36}$. The Gronwall inequality then yields the desired estimate

$$\Big\|\frac{\omega_\theta}{\varrho}\Big\|_{L^{\infty,2}}+\Big\|\frac{u_\theta}{\varrho^{\frac12}}\Big\|_{L^{\infty,4}}+\int_0^t\int\Big(\Big|\nabla\Big(\frac{u_\theta}{\varrho^{\frac12}}\Big)^2\Big|^2+\Big|\nabla\Big(\frac{\omega_\theta}{\varrho}\Big)\Big|^2\Big)\leqslant C(\mathbf{u}^0,\mathbf{f}).$$

S t e p 3: First we derive an estimate of ω_θ in $L^{\infty,2}$. We multiply $(2.2)_2$ by ω_θ and integrate over $\mathbb{R}^3$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta\|_2^2+\nu\int\Big(|\nabla\omega_\theta|^2+\Big|\frac{\omega_\theta}{\varrho}\Big|^2\Big)=\int\frac{u_\varrho}{\varrho}\omega_\theta^2+2\int\frac{u_\theta}{\varrho}\omega_\varrho\omega_\theta+\int g_\theta\omega_\theta.$$

Due to the boundedness of $\frac{\omega_\theta}{\varrho}$ in $L^{\infty,2}$, we can easily estimate each term on the right-hand side obtaining

$$\|\omega_\theta\|^2_{L^{\infty,2}} + \Big\|\frac{\omega_\theta}{\varrho}\Big\|^2_{L^{2,2}} + \|\nabla\omega_\theta\|^2_{L^{2,2}} \leqslant C(\mathbf{u}^0, \mathbf{f}).$$

Before starting to estimate the other vorticity components, let us recall that due to the divergence-free condition, we have for $1 < p < \infty$

$$\|\nabla u_z\|_p \leqslant C\|\omega_\theta\|_p.$$

Thus, multiplying equation $(2.2)_1$ by ω_ϱ, equation $(2.2)_3$ by ω_z and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega_\varrho\|_2^2 + \|\omega_z\|_2^2) + \nu\int\Big(|\nabla\omega_\varrho|^2 + |\nabla\omega_z|^2 + \Big|\frac{\omega_\varrho}{\varrho}\Big|^2\Big) \\ = \int g_\varrho\omega_\varrho + \int g_z\omega_z + \int\Big[\frac{\partial u_\varrho}{\partial\varrho}\omega_\varrho^2 + \Big(\frac{\partial u_\varrho}{\partial z} + \frac{\partial u_z}{\partial\varrho}\Big)\omega_\varrho\omega_z + \frac{\partial u_z}{\partial z}\omega_z^2\Big].\end{aligned}$$

Using the estimate of ω_θ, we can again easily control all terms on the right-hand side and we end up with

$$\|\omega_\varrho\|^2_{L^{\infty,2}} + \|\omega_z\|^2_{L^{\infty,2}} + \|\nabla\omega_\varrho\|^2_{L^{2,2}} + \|\nabla\omega_z\|^2_{L^{2,2}} + \Big\|\frac{\omega_\varrho}{\varrho}\Big\|_{L^{2,2}} \leqslant C(\mathbf{u}^0, \mathbf{f}).$$

The theorem is proved.

4. Proof of Theorem 2

We have to proceed slightly differently because, unlike the previous case, we cannot get the same estimate of u_θ.

Step 1: We can derive inequality (3.3) in the same way as in the previous section.

Step 2: We multiply equation $(2.2)_2$ by $|\omega_\theta|^{\frac{2}{5}}\omega_\theta$ and integrate over $\mathbb{R}^3$. We have

$$\begin{aligned}\frac{5}{12}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta\|^{\frac{12}{5}}_{\frac{12}{5}} + \frac{35}{36}\nu\int\big|\nabla(|\omega_\theta|^{\frac{6}{5}})|^2 + \nu\int\frac{|\omega_\theta|^{\frac{12}{5}}}{\varrho^2} \\ = \int\frac{u_\varrho}{\varrho}|\omega_\theta|^{\frac{12}{5}} + 2\int\frac{u_\theta}{\varrho}|\omega_\theta|^{\frac{2}{5}}\omega_\varrho\omega_\theta + \int g_\theta|\omega_\theta|^{\frac{2}{5}}\omega_\theta.\end{aligned} \tag{4.1}$$

The first term on the right-hand side can be easily estimated. The second term can be treated as follows:

$$\begin{aligned}\Big|\int\frac{u_\theta}{\varrho}|\omega_\theta|^{\frac{2}{5}}\omega_\theta\omega_\varrho\Big| &\leqslant C\int\frac{u_\theta^2}{\varrho}|\omega_\theta|^{\frac{1}{5}}\Big|\frac{\partial|\omega_\theta|^{\frac{6}{5}}}{\partial z}\Big| \\ &\leqslant \frac{\nu}{4}\|\nabla(|\omega_\theta|^{\frac{6}{5}})\|_2^2 + \Big\|\frac{u_\theta}{\varrho^{\frac{1}{2}}}\Big\|^4_{12} + C(\nu)\Big\|\frac{u_\theta}{\varrho^{\frac{1}{2}}}\Big\|^4_4\|\omega_\theta\|_2^{\frac{4}{7}}.\end{aligned}$$

Since

$$\Big\|\frac{u_\theta}{\varrho^{\frac12}}\Big\|_{12}^4 \leqslant C\int\Big[\nabla\Big(\Big|\frac{u_\theta}{\varrho^{\frac12}}\Big|^2\Big)\Big]^2$$

due to the Sobolev imbedding theorem, we can also estimate the second term. Thus we are left with the last, the most delicate term:

$$I_1 \equiv \int\frac{|u_\varrho|}{\varrho}|\omega_\theta|^{\frac{12}{5}} \leqslant \Big\|\frac{\omega_\theta}{\varrho}\Big\|_2\|u_\varrho\|_\infty\|\omega_\theta\|_{\frac{14}{5}}^{\frac75}.$$

In order to estimate the two norms on the right-hand side, we use the following three interpolation inequalities; the last two can be found in Nirenberg [10]:

$$\|\omega_\theta\|_{\frac{14}{5}}^{\frac75} \leqslant \|\omega_\theta\|_2^{\frac{11}{13}}\|\omega_\theta\|_{\frac{36}{5}}^{\frac{36}{65}}, \tag{4.2}$$

$$\|u_\varrho\|_\infty \leqslant C\|u_\varrho\|_{12}^{\frac{7}{10}}\|\nabla u_\varrho\|_{\frac{36}{5}}^{\frac{3}{10}} \leqslant C\|\omega_\theta\|_{\frac{12}{5}}^{\frac{7}{10}}\|\omega_\theta\|_{\frac{36}{5}}^{\frac{3}{10}}, \tag{4.3}$$

$$\|u_\varrho\|_\infty \leqslant C\|u_\varrho\|_6^{\frac{7}{13}}\|\nabla u_\varrho\|_{\frac{36}{5}}^{\frac{6}{13}} \leqslant C\|\omega_\theta\|_2^{\frac{7}{13}}\|\omega_\theta\|_{\frac{36}{5}}^{\frac{6}{13}}. \tag{4.4}$$

We raise inequality (4.3) to the power $\frac{12}{35}$, inequality (4.4) to the power $\frac{23}{35}$, we also apply the Young inequality and we get

$$I_1 \leqslant \delta\|\omega_\theta\|_{\frac{36}{5}}^{\frac{12}{5}} + C(\delta)\Big(\Big\|\frac{\omega_\theta}{\varrho}\Big\|_2^2 + \|\omega_\theta\|_{\frac{12}{5}}^{\frac{12}{5}}\Big)\|\omega_\theta\|_2^2,$$

δ being a sufficiently small positive number. Thus summing all these estimates up with (3.3) and employing the Gronwall inequality we get

$$\begin{aligned}\int\Big(\frac{\omega_\theta^2}{\varrho^2}+\frac{u_\theta^4}{\varrho^2}+|\omega_\theta|^{\frac{12}{5}}\Big)(\tau)+\nu\int_0^\tau\int\Big(|\nabla u_\theta|^2\frac{u_\theta^2}{\varrho^2}+\frac{u_\theta^2}{\varrho}\Big|\frac{\partial}{\partial\varrho}\Big(\frac{u_\theta}{\varrho^{\frac12}}\Big)\Big|^2+\frac{u_\theta^4}{\varrho^4}\Big)\\ +\nu\int_0^\tau\int\Big(|\nabla(|\omega_\theta|^{\frac65})|^2+\frac{|\omega_\theta|^{\frac{12}{5}}}{\varrho^2}+\Big|\nabla\Big(\frac{\omega_\theta}{\varrho}\Big)\Big|^2\Big)\leqslant C(\mathbf{u}^0,\mathbf{f})+C\int_0^\tau\int\frac{|u_\varrho|}{\varrho}\frac{u_\theta^4}{\varrho^2}.\end{aligned}$$

S t e p 3 : Let $s \geqslant 6$. Then

$$I_2 = \int\frac{|u_\varrho|}{\varrho}\frac{u_\theta^4}{\varrho^2} \leqslant \Big\|\frac{u_\varrho}{\varrho}\Big\|_q\Big\|\frac{u_\theta}{\varrho}\Big\|_4^\alpha\Big\|\frac{u_\theta}{\varrho^{\frac12}}\Big\|_4^\beta\|u_\theta\|_s^\gamma,$$

where $\alpha+\beta+\gamma=4$, $\alpha+\frac{\beta}{2}=2$, $\frac1q+\frac{\alpha}{4}+\frac{\beta}{4}+\frac{\gamma}{s}=1$. Now, interpolating the L^q norm between $L^{\frac{12}{5}}$ and $L^{\frac{36}{5}}$, we get

$$I_2 \leqslant \delta_1\|\omega_\theta\|_{\frac{36}{5}}^{\frac{12}{5}} + \delta_2\Big\|\frac{u_\theta}{\varrho}\Big\|_4^4 + C(\delta_1,\delta_2)\|u_\theta\|_s^{\frac{8\gamma q}{3q+12-2q\alpha}}\Big(\Big\|\frac{u_\varrho}{\varrho}\Big\|_{\frac{12}{5}}^{\frac{12}{5}} + \Big\|\frac{u_\theta}{\varrho^{\frac12}}\Big\|_4^{\frac{12\beta}{7-3\alpha}}\Big).$$

Thus, we require $\frac{12\beta}{7-3\alpha} = 4$, i.e. $\alpha + \beta = \frac{7}{3}$.

Now $\gamma = \frac{5}{3}$, $\alpha = \frac{5}{3}$, $\beta = \frac{2}{3}$, $q = \frac{12s}{5(s-4)}$. So $\frac{12}{5} \leqslant \frac{12s}{5(s-4)} \leqslant \frac{36}{5} \iff s \geqslant 6$, $\frac{8\gamma q}{3q+12-2q\alpha} = \frac{20s}{7s-30}$ and under the assumptions that $u_\theta \in L^{r,s}$, $\frac{2}{r} + \frac{3}{s} = \frac{7}{10}$ and $s \geqslant 6$ we get the desired estimate

$$\left\| \frac{\omega_\theta}{\varrho} \right\|_{L^{\infty,2}} + \left\| \frac{u_\theta}{\varrho^{\frac{1}{2}}} \right\|_{L^{\infty,4}} + \|\omega_\theta\|_{L^{\infty,\frac{12}{5}}} \leqslant C(\mathbf{u}^0, \mathbf{f}). \tag{4.5}$$

S t e p 4 : If $s < 6$ then we have to modify our method. First, let us make the following observation. If we formally multiply the equation for u_θ by $|u_\theta|^{q-2}u_\theta\varrho^q$, integrate over $\mathbb{R}^3$ and formally use the integration by parts, we finally get

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\theta\varrho\|_q^q + \nu(q-1)\int |\nabla(u_\theta\varrho)|^2|u_\theta\varrho|^{q-2} \leqslant \int |f_\theta\varrho||u_\theta\varrho|^{q-1}.$$

Thus, using the Hölder and Gronwall inequalities and passing with q to infinity we obtain

$$\|u_\theta\varrho\|_{L^{\infty,\infty}} \leqslant C(\mathbf{f}, \mathbf{u}^0). \tag{4.6}$$

Although this result holds generally in every axisymmetric case, its application in previous situations does not lead to better results; this only illustrates the fact that the only possible singularities must be concentrated on the z-axis.

As mentioned above, the proof of (4.6) was only formal because it was not clear whether during the integration by parts some boundary terms vanish when approaching infinity. However, the proof can be done rigorously by means of the standard cut-off technique. We leave this straightforward but slightly technical calculations to the kind reader.

Let us now estimate I_2 for $s < 6$. Having in mind the boundedness of $u_\theta\varrho$ in $L^{\infty,\infty}$, we get

$$I_2 \leqslant \left\| \frac{u_\varrho}{\varrho} \right\|_{\frac{36}{5}} \left\| \frac{u_\theta}{\varrho} \right\|_4^\alpha \left\| \frac{u_\theta}{\varrho^{\frac{1}{2}}} \right\|_4^\beta \|u_\theta\|_s^\gamma \|u_\theta\varrho\|_\infty^\delta$$

with $\alpha + \beta + \gamma + \delta = 4$, $\alpha + \frac{\beta}{2} - \delta = 2$, $\frac{\alpha}{4} + \frac{\beta}{4} + \frac{\gamma}{s} = \frac{31}{36}$.

Further we proceed as above obtaining

$$I_2 \leqslant \delta_1\|\omega_\theta\|_{\frac{36}{5}}^{\frac{12}{5}} + \delta_2\left\| \frac{u_\theta}{\varrho} \right\|_4^4 + C_1(\delta_1,\delta_2)\|u_\theta\|_s^{\frac{12\gamma}{7-3\alpha}} \left\| \frac{u_\theta}{\varrho^{\frac{1}{2}}} \right\|_4^{\frac{12\beta}{7-3\alpha}}.$$

Thus, we require again $\alpha + \beta = \frac{7}{3}$ and we obtain $\alpha = 5(1 - \frac{s}{9})$, $\beta = \frac{5s}{9} - \frac{8}{3}$ $\gamma = \frac{5s}{18}$, $\delta = \frac{5}{3}(1 - \frac{s}{6})$. Since α, β must be non-negative, we get $\frac{24}{5} < s \leqslant 6$. Then $\frac{12\gamma}{7-3\alpha} = \frac{10s}{5s-24}$ and under the assumptions that $s \in (\frac{24}{5}, 6)$, $u_\theta \in L^{r,s}$, $\frac{2}{r} + \frac{3}{s} = 1 - \frac{9}{5s}$ we get estimate (4.5). Let us note that the case $s = \frac{24}{5}$ must be excluded.

S t e p 5 : The rest of the proof can be done exactly as in the proof of Theorem 1.

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