HOMOGENIZATION OF DIFFUSION EQUATION WITH SCALAR HYSTERESIS OPERATOR

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. The paper deals with a scalar diffusion equation $c\,u_t = (\mathscr{F}[u_x])_x + f$, where $\mathscr{F}$ is a Prandtl-Ishlinskii operator and c, f are given functions. In the diffusion or heat conduction equation the linear constitutive relation is replaced by a scalar Prandtl-Ishlinskii hysteresis spatially dependent operator. We prove existence, uniqueness and regularity of solution to the corresponding initial-boundary value problem. The problem is then homogenized by considering a sequence of equations of the above type with spatially periodic data c^ε and η^ε when the spatial period ε tends to zero. The homogenized characteristics c^* and η^* are identified and the convergence of the corresponding solutions to the solution of the homogenized equation is proved.

Keywords: hysteresis, Prandtl-Ishlinskii operator, material with periodic structure, nonlinear diffusion equation, homogenization

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Introduction

In [5] the homogenization problem for a scalar wave equation

$$\varrho\, u_{tt} = (\mathscr{F}[u_x])_x + f$$

was studied. In the equation $\varrho = \varrho(x)$, $f = f(x,t)$ are given functions and $\mathscr{F}$ is a Prandtl-Ishlinskii scalar hysteresis operator determined by a spatially dependent distribution function $\eta = \eta(x,r)$. The problem can be interpreted as the vibration

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of a plastic rod; the hysteresis operator $\mathscr{F}$ represents the constitutive relation of the plastic material.

In this paper we study the homogenization problem for a diffusion equation

$$c\,u_t = (\mathscr{F}[u_x])_x + f.$$

In the diffusion or heat conduction equation the linear constitutive relation is replaced by a nonlinear Prandtl-Ishlinskii scalar hysteresis operator $\mathscr{F}$.

Hysteresis. The Prandtl-Ishlinskii hysteresis operator is used for modelling constitutive relations. A systematic mathematical investigation of these operators started relatively recently (see for instance [8], [3], [4], [9], [11]), although the model itself had been introduced much earlier ([10], [7]). In [3], [4], [11] existence and uniqueness of a solution to the heat equation for a homogeneous medium is proved.

We consider an equation with space dependent constitutive relations, i.e. $c = c(x)$ and a distribution function $\eta = \eta(x, r)$ of the Ishlinskii operator $\mathscr{F}$. The equation is completed by initial and boundary conditions. We shall deal with existence, uniqueness and regularity of the solution.

Homogenization. Composite materials with periodic structure are efficiently modeled by a mathematical method called homogenization. If the spatial period is very small, rapidly varying coefficients should be replaced by constant ones corresponding to an idealized homogeneous material, which at the macroscopic level exhibits a similar behaviour. The approach proposed in [1] consists in considering a sequence of problems with spatially periodic constitutive laws with a diminishing period. We look for a limit homogeneous constitutive relation in the "homogenized" problem such that its solution is a limit of solutions to the original "periodic" problems. Basic information can be found e.g. in [2] and many others.

We shall deal with the homogenization problem. We consider a sequence of "periodic" problems with diminishing period ε, i.e. with $c^\varepsilon(x) = c(x/\varepsilon)$, $\eta^\varepsilon(x) = \eta(x/\varepsilon)$ while $\varepsilon \to 0$. The problems determine a sequence of solutions u^ε. We shall prove that the solutions converge to a function u^* which is a solution of the "homogenized" problem. In this case the homogenized problem is of the same type and we give explicit formulae for the homogenized coefficients c^*, $\eta^*(r)$ in the constitutive relations.

The paper is organized as follows. In Section 1 we deal with hysteresis operators, beside definitions we give a survey of results needed in the subsequent sections. We follow the notation of [5], where also the proofs can be found. Section 2 deals with an analysis of the solution to the space dependent problem and the last Section 3 deals with the sequence of problems and with homogenization.

Hysteresis operators are nonlinear causal functional operators; they have a memory character: the value at some time t depends on inputs up to the time t. We confine ourselves to scalar hysteresis operators of Prandtl-Ishlinskii type. They have some "good" properties: they are rate independent, causal, Lipschitz continuous, and locally monotone.

The operators are introduced using the following elementary stop and play operators. We state the results without proofs, for details see e.g. [5], [4], [11], [5], [6].

Stop and play operators.

Let $r > 0$ be a parameter, $u(t)$ a continuous input function on the time interval $I = [0, T]$ and $s_r^0 \in [-r, r]$ an initial state. We consider a variational inequality

$$\begin{aligned} & s(t) \in [-r, r] && t \in I, \\ & (\dot{s}(t) - \dot{u}(t))(\varphi - s(t)) \geqslant 0 && \text{for a.e. } t \in I \quad \forall\, \varphi \in [-r, r], \\ & s(0) = s_r^0 \end{aligned} \tag{1.1}$$

for the unknown $s(t)$. For an input $u \in W^{1,1}(I)$ the problem admits a unique solution $s \in W^{1,1}(I)$. The solution $s(t)$ defines two complementary operators: $\mathscr{S}_r$ is called the *stop* operator and $\mathscr{P}_r$ the *play* operator with the threshold r. They are defined by relations

$$\mathscr{S}_r[u, s_r^0] := s, \qquad \mathscr{P}_r[u, s_r^0] := u - \mathscr{S}_r[u, s_r^0]. \tag{1.2}$$

In each interval of monotonicity $[t_0, t_1]$ of the input function $u(t)$ the relations

$$\begin{aligned} & \mathscr{P}_r[u, s_r^0](t) = \max\{u(t) - r,\ \min\{u(t) + r,\ \mathscr{P}_r[u, s_r^0](t_0)\}\}, \\ & \mathscr{S}_r[u, s_r^0](t) = u(t) - \mathscr{P}_r[u, s_r^0](t) \qquad t \in (t_0, t_1) \end{aligned}$$

are often used for an alternative definition of the play and the stop operator for piecewise monotone inputs, see [4], [8], [11].

In the following we use the "unperturbed" or "virgin" initial state s_r^0 defined by

$$s_r^0 = \min\{r, \max\{-r, u(0)\}\} \tag{1.3}$$

and write only $\mathscr{S}_r[u]$, $\mathscr{P}_r[u]$. We put the input u into square brackets to indicate the functional dependence—the operators map a function to a function.

R e m a r k . The stop operator can be physically interpreted as a dependence of stress on strain of the rheological elastic spring element and the friction element combined in series. Both operators have also a geometric “piston-in-cylinder” interpretation. Let us consider a cylinder of length $2r$ with a piston moving inside. If the input $u(t)$ denotes the position of the piston moving in the cylinder, then the relative position of the piston with respect to the cylinder represents the stop operator and the position of the cylinder yields the play operator.

We survey properties of the stop and the play operator. We will use the notation $\|u\|_{[0,t]} = \max\{|u(\tau)|,\ \tau \in [0,t]\}$.

Proposition 1.1.

(a) *The operators $\mathscr{S}_r$ and $\mathscr{P}_r$ map the following spaces:*

$$\mathscr{S}_r, \mathscr{P}_r \colon\ W^{1,\infty}(I) \to W^{1,\infty}(I) \qquad \text{and} \qquad \mathscr{S}_r, \mathscr{P}_r \colon\ W^{1,1}(I) \to W^{1,1}(I).$$

(b) *The stop operator $\mathscr{S}_r$ yields hysteresis loops with two slopes $0, 1$ and with concave increasing branches and convex decreasing branches while the play operator $\mathscr{P}_r$ yields hysteresis loops also with two slopes $0, 1$ but with convex increasing branches and concave decreasing branches.*

(c) *The operators $\mathscr{S}_r$, $\mathscr{P}_r$ are Lipschitz continuous. Let $u_i \in W^{1,1}(I)$ be input functions. Denoting $s_i := \mathscr{S}_r[u_i]$, $p_i := \mathscr{P}_r[u_i]$, $i = 1, 2$ we have*

$$\begin{aligned} |p_1(t) - p_2(t)| &\leqslant \|u_1 - u_2\|_{[0,t]} \qquad t \in I, \\ |s_1(t) - s_2(t)| &\leqslant 2\,\|u_1 - u_2\|_{[0,t]} \qquad t \in I. \end{aligned}$$

(d) *The operators $\mathscr{S}_r, \mathscr{P}_r$ are locally monotone in the following sense:*

$$\frac{\mathrm{d}\mathscr{S}_r[u]}{\mathrm{d}t} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \geqslant 0, \qquad \frac{\mathrm{d}\mathscr{P}_r[u]}{\mathrm{d}t} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \geqslant 0, \qquad \frac{\mathrm{d}\mathscr{S}_r[u]}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathscr{P}_r[u]}{\mathrm{d}t} = 0.$$

(e) *The operators are monotone on subintervals of monotone inputs:*
let $u_i \in W^{1,1}(I)$, $i = 1, 2$, be input functions. Denoting $s_i := \mathscr{S}_r[u_i]$, $p_i := \mathscr{P}_r[u_i]$, $(i = 1, 2)$ we have

$$(\dot p_1(t) - \dot p_2(t))\,(s_1(t) - s_2(t)) \geqslant 0 \qquad \text{for a.e. } t \in I.$$

(f) *Adding a multiple of the identity operator $\mathscr{I}$ the stop and the play operators are mutually inverse: for any $a, b, r > 0$ there exist $c, d, s > 0$ such that the operator $c\mathscr{I} + d\mathscr{P}_s$ is inverse to the operator $a\mathscr{I} + b\mathscr{S}_r$, where $c = 1/(a+b)$, $d = 1/a - 1(a+b)$, $s = (a+b)r$.*

Prandtl-Ishlinskii operators.

The stop and the play operators yield hysteresis loops with two slopes only. By a linear combination of the operators with different r, i.e. $\sum_i \alpha_i \mathscr{S}_{r_i}[u]$ we obtain hysteresis loops with more slopes; using an integral of the operators even smooth branches of loops are attained. To include both the discrete and continuous combinations we replace the sum or integral of the operators by the Stieltjes integral with respect to a distribution function η. To obtain invertible operators we add a multiple of the identity operator $\mathscr{I}$.

Since the operators will be mutually inverse we denote the input of the stop type operator $\mathscr{F}$ by e and its output by σ. Conversely, we denote the input of the inverse play type operator $\mathscr{G}$ by σ and the output by e. Thus $\sigma = \mathscr{F}[e]$ will be equivalent to $e = \mathscr{G}[\sigma]$.

Definition 1.2. We say that a function $\eta\colon [0,\infty] \to (0,\infty)$ is an *admissible distribution function of stop type* if it is bounded, nonincreasing and right-continuous. It determines the *Prandtl-Ishlinskii operator of stop type* by the integral

$$\mathscr{F}[e] := \eta(\infty)\, e - \int_0^\infty \mathscr{S}_r[e]\, \mathrm{d}\eta(r).$$

Similarly, a function $\zeta\colon [0,\infty] \to (0,\infty)$ is an *admissible distribution function of play type* if it is bounded, nondecreasing and right-continuous. It determines the *Prandtl-Ishlinskii operator of play type* by the integral

$$\mathscr{G}[\sigma] := \zeta(0)\sigma + \int_0^\infty \mathscr{P}_r[\sigma]\, \mathrm{d}\zeta(r).$$

Further we say that a pair of admissible functions (η,ζ) of stop and play type forms a *pair of adjoint Prandtl-Ishlinskii distribution functions* if the functions $\varphi,\psi\colon [0,\infty) \to [0,\infty)$ defined by

$$\varphi(r) = \int_0^r \eta(h)\mathrm{d}h, \qquad \psi(s) = \int_0^s \zeta(h)\mathrm{d}h$$

are mutually inverse functions on $[0,\infty)$, i.e. $\psi = \zeta^{-1}$ and $\zeta = \psi^{-1}$.

The set of all pairs of adjoint Prandtl-Ishlinskii distribution functions (η,ζ) satisfying

$$0 < \alpha \leqslant \eta(r) \leqslant 1/\beta, \qquad 0 < \beta \leqslant \zeta(s) \leqslant 1/\alpha$$

will be denoted by $PI(\alpha,\beta)$.

Besides the input $u(t)$ the values of Prandtl-Ishlinskii operators are determined also by the initial states s_r^0 of the stop or the play operators used in the formulae.

To simplify the notation we have omitted the initial state s_r^0 and we write only $\mathscr{F}[e]$ instead of $\mathscr{F}[e, s_r^0]$ and $\mathscr{G}[\sigma]$ instead of $\mathscr{G}[\sigma, s_r^0]$.

Let us remark that the above introduced operator of stop type $\mathscr{F}$ has concave increasing and convex decreasing branches while the operator of play type $\mathscr{G}$ has convex increasing and concave decreasing branches.

The properties of the Ishlinskii operators are surveyed in the next assertion. For proofs see e.g. [9], [4], [5], [11], (e) is proved in [5], Proposition 2.10.

Proposition 1.3. *Let $(\eta, \zeta) \in PI(\alpha, \beta)$ and let $\mathscr{F}, \mathscr{G}$ be the corresponding operators.*

(a) *Then the operators $\mathscr{F}$ and $\mathscr{G}$ map the spaces*

$$\mathscr{F}, \mathscr{G} \colon W^{1,\infty}(I) \to W^{1,\infty}(I) \qquad \text{and} \qquad \mathscr{F}, \mathscr{G} \colon W^{1,1}(I) \to W^{1,1}(I).$$

(b) *The operators $\mathscr{F}, \mathscr{G}$ are mutually inverse operators satisfying $\sigma = \mathscr{F}[e]$ if and only if $e = \mathscr{G}[\sigma]$.*

(c) *The operators $\mathscr{F}, \mathscr{G}$ are Lipschitz continuous. Moreover, denoting $\sigma_i := \mathscr{F}[e_i]$ or $e_i := \mathscr{G}[\sigma_i]$, $i = 1, 2$, for any $t \in T$ we have*

$$|\sigma_1(t) - \sigma_2(t)| \leqslant \Big(\frac{2}{\beta} - \alpha\Big) \|e_1 - e_2\|_{[0,t]},$$
$$|e_1(t) - e_2(t)| \leqslant \frac{1}{\alpha} \|\sigma_1 - \sigma_2\|_{[0,t]}.$$

(d) *The operators $\mathscr{F}, \mathscr{G}$ are locally monotone in the following sense: denoting $\sigma := \mathscr{F}[e]$ or $e := \mathscr{G}[\sigma]$ we have for almost each $t \in I$ the inequalities*

$$\alpha\, \dot{e}^2(t) \leqslant \dot{e}(t)\, \dot{\sigma}(t) \leqslant \frac{1}{\beta}\, \dot{e}^2(t), \qquad \beta\, \dot{\sigma}^2(t) \leqslant \dot{e}(t)\, \dot{\sigma}(t) \leqslant \frac{1}{\alpha}\, \dot{\sigma}^2(t).$$

(e) *Let $(\eta_i, \zeta_i) \in PI(\alpha, \beta)$ and let $\mathscr{G}_i$ be the corresponding operators of play type. Then for $\sigma_i \in W^{1,1}(I)$ and for every $t \in I$ we have*

$$\|\mathscr{G}_1[\sigma_1] - \mathscr{G}_2[\sigma_2]\|_{[0,t]} \leqslant \zeta_1(\infty)\, \|\sigma_1 - \sigma_2\|_{[0,t]} + \int_0^{\|\sigma_2\|_{[0,t]}} |\zeta_1(r) - \zeta_2(r)|\, \mathrm{d}r.$$

Spatially dependent hysteresis operators.

We will consider the spatially dependent constitutive relation described by the Prandtl-Ishlinskii operators with spatially dependent distribution functions $\eta =$

$\eta(x,r)$ and $\zeta = \zeta(x,r)$. Let $(\eta(x,\cdot),\zeta(x,\cdot)) \in PI(\alpha,\beta)$ for a.e. $x \in J = [0,\ell]$. Given an input $e\colon J \times I \to \mathbb{R}$ we have an output $\sigma\colon J \times I \to \mathbb{R}$ defined by

$$\sigma(x,t) := \mathscr{F}[e](x,t) = \eta(x,\infty)\, e(x,t) - \int_0^\infty \mathscr{S}_r[e(x,\cdot)](t)\,\mathrm{d}_r\eta(x,r).$$

Similarly, for a given input $\sigma\colon J \times I \to \mathbb{R}$ we define an output $e\colon J \times I \to \mathbb{R}$ by

$$e(x,t) := \mathscr{G}[\sigma](x,t) = \zeta(x,0)\,\sigma(x,t) + \int_0^\infty \mathscr{P}_r[\sigma(x,\cdot)](t)\,\mathrm{d}_r\zeta(x,r).$$

To justify passing to the limit in the proof of Theorem 2.2 we need the following result proved in [5]:

Lemma 1.4. *Let $\{\zeta_n\} \subset L^\infty(J \times [0,\infty))$ be a sequence of admissible space dependent distribution functions of $PI(\alpha,\beta)$, i.e. $\zeta_n(x,r)$ are increasing and right-continuous in r and satisfy $\beta \leqslant \zeta_n(x,r) \leqslant 1/\alpha$ for $n \in \mathbb{N}$ and a.e. $x \in J$. Assume that ζ_n converge to ζ in $L^\infty(J \times [0,\infty))$ weakly-star. Let $\mathscr{G}_n$, $\mathscr{G}$ be the operators corresponding to ζ_n, ζ, respectively.*

Let $\{\sigma_n\}$ be a sequence in $L^\infty(I \times J)$ such that $\sigma_n(x,\cdot) \in C^0(I)$ for a.e. $x \in J$ and $\|\sigma_n - \sigma\|_\infty \to 0$ as $n \to \infty$.

Then $\mathscr{G}_n[\sigma_n](\cdot,t)$ converge to $\mathscr{G}[\sigma](\cdot,t)$ for every $t \in I$ in $L^\infty(J)$ weakly-star.

2. Spatially dependent problem

We will start with an analysis of the problem corresponding to a spatially dependent material, i.e. the problem with c and the distribution functions η, ζ in the constitutive operators $\mathscr{F}, \mathscr{G}$ depending, moreover, on x.

Statement of the problem.

Let $I = [0,T]$ be a time interval and $J = [0,\ell]$ a space interval. We will deal with the equation

$$c\,u_t = (\mathscr{F}[u_x])_x + f \qquad \text{in } J \times I, \tag{2.1}$$

completed by initial and boundary conditions

$$\begin{aligned} &u(x,0) = u^0(x) && && x \in J,\\ &u(x,t) = 0 \quad \text{or} && u_x(x,t) = 0 && x = 0,\ell, \quad t \in I, \end{aligned} \tag{2.2}$$

and by an initial state s_r^0 of the Prandtl-Ishlinskii operators. For the sake of simplicity we choose the unperturbed state defined by (1.3) and omit the symbol s_r^0 in the formulae.

Introducing new variables $e = u_x$ and $\sigma = \mathscr{F}[e]$ we rewrite the equation (2.1) into a first order system

$$
\begin{aligned}
c\,u_t &= \sigma_x + f, && e = u_x, \\
\sigma &= \mathscr{F}[e] \quad \text{or equivalently} && e = \mathscr{G}[\sigma]
\end{aligned}
\tag{2.3}
$$

in $J \times I$ for the unknowns u, e, σ completed by initial and boundary conditions (2.2). The condition $u_x(x, \cdot) = 0$ is equivalent to $e(x, \cdot) = \sigma(x, \cdot) = 0$.

The data are assumed to satisfy the following requirements:

Hypothesis 2.1.

(i) $c \in L^\infty(J)$ *and there exist constants* $c_m, c_M > 0$ *such that* $c_m \leqslant c(x) \leqslant c_M$ *for a.e.* $x \in J$,

(ii) $\eta, \zeta \in L^\infty(J \times [0, \infty))$ *are such that* $(\eta, \zeta)(x, \cdot) \in PI(\alpha, \beta)$ *for a.e.* $x \in J$, *where* α, β *are positive constants,*

(iii) $f \in W^{1,1}(I; L^2(J))$,

(iv) $u^0 \in W^{2,2}(J)$ *satisfies the compatibility condition according to the boundary conditions:* $u^0(0) = 0$ *or* $u_x^0(0) = 0$ *and* $u^0(\ell) = 0$ *or* $u_x^0(\ell) = 0$.

We now state the main result of the section.

Theorem 2.2. *Let Hypothesis* 2.1 *hold. Then there exist functions* $u, \sigma \in C(J \times I)$ *and* $e \in L^2(J; C(I))$ *such that* $u_t, \sigma_x \in L^\infty(I, L^2(J))$, $e_t, \sigma_x \in L^\infty(I, L^2(J))$, *and the problem* (2.3)–(2.2) *is satisfied almost everywhere.*

The solution is unique. Moreover, u_t, σ_x *are bounded by* C *in the norm of* $L^\infty(I; L^2(J))$ *and* u_x, σ_t, e_t *are bounded by* C *in the norm of* $L^2(I, L^2(J))$, *where* C *is a constant depending only on* α, β, c_m, c_M, *the norm of* f *in* $W^{1,1}(I; L^2(J))$ *and the norm of* u_0 *in* $W^{2,2}(J)$.

The rest of this section is devoted to the proof. We shall present the proof for the following case of boundary conditions:

$$
u(0, t) = 0 \qquad u_x(\ell, t) \equiv e(\ell, t) = \sigma(\ell, t) = 0.
\tag{2.2a}
$$

Space discretization.

Let $n \in \mathbb{N}$. We discretize the problem in space by dividing the space interval $J = [0, \ell]$ into n subintervals $I_k = (x_{k-1}, x_k)$ of length $h = \ell/n$, where $x_k = kh$,

$k = 0, 1, \dots, n$. The semidiscretized equations read

$$c_k\, \dot{u}_k = \frac{1}{h}\left(\sigma_{k+1} - \sigma_k\right) + f_k, \tag{2.4}$$

$$e_k = \frac{1}{h}\left(u_k - u_{k-1}\right), \tag{2.5}$$

$$e_k = \mathscr{G}_k\left[\sigma_k\right] \quad \text{or equivalently} \quad \sigma_k = \mathscr{F}_k\left[e_k\right], \tag{2.6}$$

$$u_k(0) = u_0(x_k), \tag{2.7}$$

where the unknowns u_k, e_k, σ_k are time dependent. They approximate the values of the corresponding unknowns u, e, σ at $x_k = kh$. According to the above chosen boundary conditions (2.2a) the system (2.4)–(2.7) is considered for $k = 1, 2, \dots, n-1$ for the unknowns u_k, e_k, σ_k with $k = 1, 2, \dots, n-1$. The system is completed by $u_0(t) = 0$ and $\sigma_n(t) = 0$.

The discretized data c_k, f_k are defined by integral averaging

$$c_{k-1} := \frac{1}{h}\int_{I_k} c(x)\,\mathrm{d}x, \qquad f_{k-1}(t) := \frac{1}{h}\int_{I_k} f(x,t)\,\mathrm{d}x. \tag{2.8}$$

The discretized operators $\mathscr{G}_k$,

$$\mathscr{G}_k[\sigma] := \zeta_k(0)\,\sigma + \int_0^\infty \mathscr{P}_r[\sigma]\,\mathrm{d}\zeta_k(r), \tag{2.9}$$

are defined using the averaged distribution functions ζ_k,

$$\zeta_k(r) := \frac{1}{h}\int_{I_k} \zeta(x,r)\,\mathrm{d}x \quad \text{for} \quad r \geqslant 0. \tag{2.10}$$

The functions ζ_k are obviously admissible, the functions η_k are chosen such that (η_k, ζ_k) form pairs of adjoint distribution functions, see Definition 1.2. They satisfy the inequalities of Hypothesis 2.1 (ii), thus $(\eta_k, \zeta_k) \in PI(\alpha, \beta)$.

Let us remark that the distribution functions η_k are defined as adjoint functions to ζ_k, while averaging $\eta(x, r)$ yields, in general, different values.

Lemma 2.3. *The discretized system* (2.4)–(2.7) *admits a unique solution on I such that $u_k, e_k \in W^{2,1}(I)$ and $\sigma_k \in W^{1,\infty}(I)$.*

The proof is analogous to that of Proposition 3.3 in [5]. The existence is proved by the Banach fixed point principle for contractive mappings since we have an initial value problem for a system of ordinary differential equations coupled with equalities with the operators $\mathscr{F}_k$ being causal and Lipschitz continuous.

Starting with continuous $\{u_k, e_k, \sigma_k\}$, equality (2.4) yields $u_k \in C^1(\bar{I})$, (2.5) implies $e_k \in C^1(\bar{I})$. The equation $\sigma_k = \mathscr{F}_k[e_k]$ with Proposition 1.3 (a) yields $\sigma_k \in W^{1,\infty}(I)$. Then due to $f_k \in W^{1,1}(I)$ equation (2.4) yields $u_k \in W^{2,1}(I)$ and (2.5) implies $e_k \in W^{2,1}(I)$.

Estimates.

The following estimates represent the crucial step of the proof. The constants in this subsection are independent of n, h.

Lemma 2.4. *The solutions $\{u_k, e_k, \sigma_k\}$ to system (2.4)–(2.7) satisfy estimates*

$$
\begin{aligned}
&h \sum_k \Big[\dot{u}_k^2 + \Big(\frac{\sigma_{k+1} - \sigma_k}{h}\Big)^2\Big](t) \leqslant C \qquad t \in I,\\
&\int_0^T h \sum_k \Big[\dot{\sigma}_k^2 + \dot{e}_k^2 + \Big(\frac{u_k - u_{k-1}}{h}\Big)^2\Big](\tau)\,\mathrm{d}\tau \leqslant C,
\end{aligned}
\tag{2.11}
$$

where $C > 0$ is a constant depending only on $\alpha, \beta, \varrho_m, \varrho_M$, the norm of f in $W^{1,1}(I; L^2(J))$ and the norm of u^0 in $W^{2,2}(J)$.

Proof. We differentiate equations (2.4) and (2.5) with respect to t. We multiply the first equation by $\dot{u}_k(t)$ and add the second equation multiplied by $\dot{\sigma}_k(t)$. We sum them over $k = 1, \ldots, n-1$. Due to $u_0 = 0$ and $\sigma_n = 0$ we obtain for a.e. $t \in I$ the equality

$$\sum_k \big(c_k\, \ddot{u}_k\, \dot{u}_k + \dot{e}_k\, \dot{\sigma}_k\big)(\tau) = \sum_k \big(\dot{f}_k\, \dot{u}_k\big)(\tau).$$

For a fixed $t \in I$ we integrate the equality from 0 to t and multiply it by h. Estimating the right hand side we obtain

$$
\begin{aligned}
h \sum_k \frac{c_k}{2}\, \dot{u}_k^2(t) + h \sum_k \int_0^t \dot{e}_k(\tau)\, \dot{\sigma}_k(\tau)\,\mathrm{d}\tau \\
\leqslant h \sum_k \frac{c_k}{2}\, \dot{u}_k^2(0) + \Big\| h \sum_k \dot{u}_k^2 \Big\|_{[0,t]}^{1/2} \int_0^t \Big(h \sum_k \dot{f}_k^2(\tau)\Big)^{1/2} \mathrm{d}\tau.
\end{aligned}
$$

To estimate the right hand side we use $h \sum\limits_k \dot{f}_k^2 \leqslant \int_J f_t^2(x,\tau)\mathrm{d}x \leqslant C_1$. The term $\sum\limits_k c_k \dot{u}_k^2(0)$ can be bounded by a constant using the equation (2.4), the properties in Proposition 1.3 and Hypothesis 2.1 (i), (iii), (iv).

Estimating the left hand side using the assumptions of Hypothesis 2.1 and Proposition 1.3 (d), the inequality yields

$$c_m\, h \sum_k \dot{u}_k^2(t) + \int_0^t h \sum_k \left(\dot{e}_k \dot{\sigma}_k\right)(\tau)\mathrm{d}\tau \leqslant C_2 + C_3 \Big\| h \sum_k \dot{u}_k^2 \Big\|_{[0,t]}^{1/2}.$$

Now the desired estimates (2.11) follow by using standard arguments, Lemma 1.3 (d) and equations (2.4), (2.5). □

Approximate solutions.

For each fixed $n \in N$, using solutions u_k, e_k, σ_k and data c_k, f_k, ζ_k we construct piecewise constant (denoted by bar) and continuous piecewise linear (denoted by hat) approximate functions satisfying the system (2.3):

(1) $\bar{c}^{(n)}, \bar{f}^{(n)}, \bar{u}^{(n)}$ are piecewise constant "forward" approximations defined by $\bar{\varphi}^{(n)}(x) := \varphi_{k-1}$ for $x \in I_k$,

(2) $\bar{e}^{(n)}, \bar{\sigma}^{(n)}, \bar{\zeta}^{(n)}$ are piecewise constant "backward" approximations defined by $\bar{\varphi}^{(n)}(x) := \varphi_k$ for $x \in I_k$. Functions $\bar{\zeta}^{(n)}$ determine the operators $\overline{\mathscr{G}}^{(n)}$.

(3) $\hat{u}^{(n)}, \hat{\sigma}^{(n)}$ are continuous approximations piecewise linear in x on each I_k and satisfying e.g. $\hat{u}^{(n)}(x_k) = u_k$. We put $u_n := u_{n-1}$ and $\sigma_0 := \sigma_1 + h\, f_0(t)$.

The above defined functions satisfy the system

$$\bar{c}^{(n)}\, \bar{u}_t^{(n)} = \hat{\sigma}_x^{(n)} + \bar{f}^{(n)}, \qquad \bar{e}_t^{(n)} = \hat{u}_x^{(n)}, \qquad \bar{e}^{(n)} = \overline{\mathscr{G}}^{(n)}[\bar{\sigma}^{(n)}] \tag{2.12}$$

for each $t \in I$ and a.e. $x \in J$, together with boundary conditions (2.2a).

We introduce $\tilde{\sigma}^{(n)}$ which equals to $\hat{\sigma}^{(n)}$, only $\tilde{\sigma}^{(n)} := \sigma_1$ for $x \in I_1$. Then the estimates of Lemma 2.4 yields

Lemma 2.5. *Functions $\bar{u}_t^{(n)}$, $\tilde{\sigma}_x^{(n)}$ are bounded in $L^\infty(I, L^2(J))$ and $\bar{\sigma}_t^{(n)}$, $\bar{e}_t^{(n)}$, $\bar{u}_x^{(n)}$ are bounded in $L^2(I, L^2(J))$.*

Besides $\hat{\sigma}^{(n)}$ satisfying the equations (2.12) we have introduced $\tilde{\sigma}^{(n)}$ which satisfies the estimate. We will see that both $\tilde{\sigma}^{(n)}, \hat{\sigma}^{(n)}$ have the same limit σ.

Passage to the limit.

Due to Lemma 2.5 we can use a compactness argument: the sequence $\{n\}$ contains a subsequence denoted by $\{n_j\}$ such that

$$\begin{aligned} \hat{u}_t^{(n_j)} &\to u_t, & \hat{\sigma}_x^{(n_j)} &\to \sigma_x & &\text{weakly-star in } L^\infty(I; L^2(J)),\\ \hat{u}_x^{(n_j)} &\to u_x, & \bar{\sigma}_t^{(n_j)} &\to \sigma_t & &\text{weakly in } L^2(I; L^2(J)). \end{aligned}$$

A compact embedding yields

$$\hat{u}^{(n_j)} \to u, \qquad \hat{\sigma}^{(n_j)} \to \sigma \qquad \text{uniformly in } C(J \times I).$$

In the system (2.12) we can pass to the limit using Proposition 1.4 and the weak convergence $c^{(n)} \to c$, $\zeta^{(n)} \to \zeta$. If the limit is unique, then the whole sequence converges not only the subsequence.

Uniqueness.

To complete the proof it remains to prove uniqueness of the solution u, e, σ to the problem (2.3), (2.2). Let $u^i, e^i, \sigma^i,\ i = 1, 2$ be two solutions of the system. Then

$$c(u_t^1 - u_t^2) = \sigma_x^1 - \sigma_x^2, \qquad\qquad e^1 - e^2 = u_x^1 - u_x^2$$

with $e^i = \mathscr{G}[\sigma^i]$. We differentiate the second equation with respect to t, multiply the first with $u_t^1 - u_t^2$, the second with $\sigma^1 - \sigma^2$ and sum them up. Integrating the equality in x over J, the right hand side vanishes due to the boundary conditions since $\int_J [(u_t^1 - u_t^2)(\sigma^1 - \sigma^2)]_x \mathrm{d}\, x = 0$. We use the inequality

$$(e_t^1 - e_t^2)(\sigma^1 - \sigma^2) \geqslant \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(\sigma^1 - \sigma^2)^2\right]$$

which is a consequence of Proposition 1.1 (e) and the definition of the operator $\mathscr{G}$. Finally, integrating the inequality up to a $t \in I$ we obtain

$$\int_0^t \int_J c\,(u_t^1 - u_t^2)^2 \mathrm{d}x\, \mathrm{d}\tau + \frac{\beta}{2}\int_J \left[\sigma^1(t) - \sigma^2(t)\right]^2 \mathrm{d}x \leqslant 0.$$

Since both terms are nonnegative the uniqueness follows.

The other boundary conditions.

We have proved the theorem for boundary conditions (2.2a). In the case of other boundary conditions we can proceed similarly. In the case of $u_x(0,t) = e(0,t) = \sigma(0,t) = 0$ we consider the equations in the discretized system from $k = 0$. Instead of $u_0 = 0$ we put $\sigma_0 = e_0 = 0$ and $u_{-1}(t) := 2\,u_0(t) - u_1(t)$. Similarly in the case of $u(\ell, t) = 0$ we consider the equations in the discretized system up to $k = n$ and put $u_n = 0$ and $\sigma_{n+1}(t) := 2\,\sigma_n(t) - \sigma_{n-1}(t)$.

3. Homogenization

We shall consider a spatially dependent problem for a material with fine periodic structure. We consider the so-called "periodic" problem. The constitutive relations contain coefficients periodic in x with period ε—they are denoted by superscript ε, the corresponding solutions are denoted by superscript ε, too.

Let $c(y)$ be a periodic function with period 1 and let $\eta(y,r), \zeta(y,r)$ be admissible distribution functions periodic in y with period 1. We put

$$\text{(3.1)} \qquad c^\varepsilon(x) = c\Big(\frac{x}{\varepsilon}\Big), \qquad \eta^\varepsilon(x,r) = \eta\Big(\frac{x}{\varepsilon}, r\Big), \qquad \zeta^\varepsilon(x,r) = \zeta\Big(\frac{x}{\varepsilon}, r\Big).$$

The distribution functions $\eta^\varepsilon, \zeta^\varepsilon$ define pairs of mutually inverse Prandtl-Ishlinskii operators $\mathscr{F}^\varepsilon, \mathscr{G}^\varepsilon$. For a given small $\varepsilon > 0$ let us consider the equation

$$c^\varepsilon\, u_t^\varepsilon = (\mathscr{F}^\varepsilon[u_x^\varepsilon])_x + f \qquad \text{in} \quad J \times I, \tag{3.2}$$

completed by initial and boundary conditions

$$\begin{aligned} &u^\varepsilon(x,0) = u^0(x) && x \in J, \\ &u^\varepsilon(x,t) = 0 \quad \text{or} \quad u_x^\varepsilon(x,t) = 0 && x = 0, \ell, \quad t \in I, \end{aligned} \tag{3.3}$$

The corresponding "homogenized" problem consists of the equation

$$c^*\, u_t^* = (\mathscr{F}^*[u_x^*])_x + f \qquad x \in J \quad t \in I, \tag{3.4}$$

completed by initial and boundary conditions

$$\begin{aligned} &u^*(x,0) = u^0(x) && x \in J, \\ &u^*(x,t) = 0 \quad \text{or} \quad u_x^*(x,t) = 0 && x = 0, \ell, \quad t \in I, \end{aligned} \tag{3.5}$$

where the homogenized data c^* and ζ^* are the corresponding weak limits defined by

$$c^* = \int_0^1 c(y)\mathrm{d}y, \qquad \zeta^*(r) = \int_0^1 \zeta(y,r)\mathrm{d}y, \tag{3.6}$$

$\eta^*(r)$ is the distribution function adjoint to ζ^*. Finally, $\mathscr{F}^*, \mathscr{G}^*$ denote the corresponding Prandtl-Ishlinskii operators determined by η^*, ζ^*.

We introduce the following assumptions:

Hypothesis 3.1.

(i) *$c \in L^\infty(\mathbb{R})$ is periodic with period 1 and there exist constants $c_m, c_M > 0$ such that $c_m \leqslant c(x) \leqslant c_M$ for a.e. x,*

(ii) *$\eta, \zeta \in L^\infty(\mathbb{R} \times [0,\infty))$ are periodic in y and such that $(\eta,\zeta)(x,\cdot) \in PI(\alpha,\beta)$ for a.e. x, where α, β are positive constants,*

(iii) *$f \in W^{1,1}(I; L^2(J))$,*

(iv) *$u^0 \in W^{2,2}(J)$ and satisfies the compatibility condition according to the boundary conditions considered: $u^0(0) = 0$ or $u_x^0(0) = 0$ and $u^0(\ell) = 0$ or $u_x^0(\ell) = 0$.*

Following Theorem 2.2 both the periodic and homogenized problem admit unique solutions. The homogenization result reads as follows:

Theorem 3.2. *Let Hypothesis* 3.1 *hold and let* u^ε *be the solution to* (3.2)–(3.3) *with* $e^\varepsilon = u^\varepsilon_x$, $\sigma^\varepsilon = \mathscr{F}^\varepsilon[e^\varepsilon]$ *and let* u^* *be the solution to* (3.4)–(3.5) *with* $e^* = u^*_x$, $\sigma^* = \mathscr{F}^*[e^*]$. *Then the following convergences hold:*

$u^\varepsilon \to u^*$, $\sigma^\varepsilon \to \sigma^*$ *uniformly on* $J \times I$,

$u^\varepsilon_t \to u^*_t$, $\sigma^\varepsilon_x \to \sigma^*_x$ *weakly star in* $L^\infty(I, L^2(J))$ *and*

$u^\varepsilon_x \to u^*_x$, $\sigma^\varepsilon_t \to \sigma^*_t$ *weakly in* $L^2(I, L^2(J))$.

P r o o f. Following Theorem 2.2 for any small $\varepsilon > 0$ the solutions to (3.2)–(3.3) are bounded in the following sense:

$u^\varepsilon_t, \sigma^\varepsilon_x$ are bounded by C in the norm of $L^\infty(I; L^2(J))$,

$u^\varepsilon_x, \sigma^\varepsilon_t, e^\varepsilon_t$ are bounded by C in the norm $L^2(I, L^2(J))$.

By a compactness argument the sequence $\{\varepsilon\} \to 0$ contains a subsequence $\{\varepsilon_j\}$ such that the above mentioned functions converge weakly star in $L^\infty(I, L^2(J))$ or weakly in $L^2(I, L^2(J))$, respectively, to functions denoted by star. We need to prove that the limits satisfy the homogenized problem.

Due to the compact imbedding we obtain uniform convergence $\sigma^\varepsilon \to \sigma^*$ and $u^\varepsilon \to u^*$ in $J \times I$. Since $c^\varepsilon \to c^*$ weakly star in $L^\infty(J)$, we obtain $c^\varepsilon\, u^\varepsilon_t \to c^*\, u^*_t$. Finally, due to $\zeta^\varepsilon \to \zeta^*$ and Proposition 1.4, we can pass to the limit even in the equality $e^\varepsilon = \mathscr{G}^\varepsilon[\sigma^\varepsilon]$. Since the homogenized problem admits a unique solution, the whole sequence converges and the proof is complete. □

Generalization.

The homogenization result can be extended to the case of a sequence of the right hand sides f^ε of the form $f^\varepsilon(x,t) = f(x, x/\varepsilon, t)$, where the function $f(x,y,t)$ is periodic in y and continuous in x.

Further, we can weaken the periodicity assumption (3.1) for $c^\varepsilon, \zeta^\varepsilon$ to the weak star convergence

$$c^\varepsilon \to c^*, \qquad \zeta^\varepsilon(\cdot, r) \to \zeta^*(\cdot, r)$$

where c^ε and ζ^ε satisfy the estimates of Hypothesis 2.1 (i), (ii).

Conclusion

We have proved the existence and uniqueness of global solutions to a spatially inhomogeneous diffusion problem with a hysteresis operator. Assuming that the spatial structure is periodic with a period ε tending to 0, we have obtained the form of the homogenized operator in the homogenized constitutive relations and proved that the solutions to the "periodic" problem converge to the solution of the homogenized one as ε tends to 0. In this case, the homogenized operator $\mathscr{F}^*$ is obtained by "averaging" the corresponding inverse operators $\mathscr{G}^\varepsilon$: operator $\mathscr{F}^*$ is the inverse to $\mathscr{G}^*$ which is the "average" of $\mathscr{G}^\varepsilon$.

References

[1] *I. Babuška*: Homogenization approach in engineering. Computing Methods in Applied Sciences and Engineering. Lecture Notes in Econ. and Math. Systems 134, Springer, Berlin, 1976, pp. 137–153.

[2] *A. Bensoussan, J. L. Lions, G. Papanicolaou*: Asymptotic Analysis for Periodic Structure. North Holland, 1978.

[3] *M. Brokate*: Hysteresis operators. Phase Transitions and Hysteresis. Lecture Notes of C.I.M.E., Montecatini Terme, July 13–21, 1993, Springer, Berlin, 1994, pp. 1–38.

[4] *M. Brokate, J. Sprekels*: Hysteresis and Phase Transitions. Appl. Math. Sci., Vol. 121, Springer, New York, 1996.

[5] *J. Franců, P. Krejčí*: Homogenization of scalar wave equation with hysteresis. Continuum Mech. Thermodyn. *11* (1999), 371–390.

[6] *J. Franců*: Homogenization of scalar hysteresis operators. Equadiff 9 CD ROM. Masaryk University, Brno, 1998, pp. 111–122.

[7] *A. Yu. Ishlinskii*: Some applications of statistical methods to describing deformations of bodies. Izv. AN SSSR, Techn. Ser. *9* (1944), 580–590. (In Russian.)

[8] *M. A. Krasnosel'skii, A. V. Pokrovskii*: Systems with Hysteresis. (Rusian edition: Nauka, Moskva, 1983). Springer, Berlin, 1989.

[9] *P. Krejčí*: Hysteresis, Convexity and Dissipation in Hyperbolic Equations. Vol. 8, Gakuto Int. Series Math. Sci. & Appl., Gakkötosho, Tokyo, 1996.

[10] *L. Prandtl*: Ein Gedankenmodell zur kinetischen Theorie der festen Körper. Z. Angew. Math. Mech. *8* (1928), 85–106. (In German.)

[11] *A. Visintin*: Differential Models of Hysteresis. Springer, Berlin, 1994.

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