MONOTONE APPROXIMATION OF MEASURABLE MULTIFUNCTIONS BY SIMPLE MULTIFUNCTIONS

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Abstract. We investigate the problem of approximation of measurable multifunctions by monotone sequences of measurable simple ones. Our main tool is the Marczewski function, i.e., the characteristic function of a sequence of sets.

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1. Preliminaries

We introduce the notation and basic definitions which will be used throughout the paper. For a topological space Y we set

$$\begin{aligned} CL(Y) &= \{A \in \mathcal{P}(Y)\colon\ A \text{ is a nonempty closed subset of } Y\},\\ K(Y) &= \{A \in CL(Y)\colon\ A \text{ is compact}\},\\ V^- &= \{A \in CL(Y)\colon\ A \cap V \neq \emptyset\},\\ V^+ &= \{A \in CL(Y)\colon\ A \subset V\}, \end{aligned}$$

where V is a subset of Y. Recall that the *Vietoris topology* on $CL(Y)$ is generated by sets V^- and U^+, where $U, V \subset Y$ are open. By a *multifunction* we mean any mapping $\varphi\colon X \to \mathcal{P}(Y)$, where X and Y are arbitrary sets. Let X, Y be two topological spaces. Recall that a multifunction $\varphi\colon X \to CL(Y)$ is *lower semicontinuous*, *l.s.c.* for short (resp. *upper semicontinuous*, *u.s.c.* for short) provided $\varphi^{-1}(V^-) = \{x \in X\colon\ \varphi(x) \cap V \neq \emptyset\}$ $(\varphi^{-1}(V^+) = \{x \in X\colon\ \varphi(x) \subset V\})$ is open in X for every open $V \subset Y$. A multifunction is called *simple* if the set of its values is finite.

Let $(T, \mathfrak{M})$ be a measurable space. A multifunction $\varphi\colon T \to CL(Y)$ is called *measurable* provided $\varphi^{-1}(V^-) \in \mathfrak{M}$ whenever V is open in Y. Let Y be a metrizable space. Then the condition $\varphi^{-1}(V^+) \in \mathfrak{M}$ for each open V implies the measurability of φ. For compact-valued multifunctions the reverse also holds (see Himmelberg [1, Thm. 3.1]).

In [2] we investigated the problem of approximation of measurable multifunctions by sequences of simple ones. It is a consequence of some general results that if Y is separable and metrizable, then each measurable multifunction $\varphi\colon T \to K(Y)$ is the pointwise limit (with respect to the Vietoris topology) of a sequence of simple measurable multifunctions $\varphi_n\colon T \to K(Y)$. Such a theorem is no longer valid for multifunctions with non-compact values (see the counter-example of Spakowski [6]).

In the present paper we look for monotone approximations of measurable multifunctions. We use the Marczewski function and the results of Spakowski [7] on the approximation of semicontinuous multifunctions by simple ones.

Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of T and let χ_n be the characteristic function of A_n. The function $M\colon T \to \{0,1\}^{\mathbb{N}}$ defined by $M(t) = (\chi_n(t))_{n\in\mathbb{N}}$ is called the *Marczewski function of* $\mathcal{A}$ (cf. [5]). We will consider $M(T)$ with the topology induced by the product $\{0,1\}^{\mathbb{N}}$. It is easy to check that the Marczewski function is measurable and $M(A_n)$ is closed-open in $M(T)$ for each $n \in \mathbb{N}$. Some applications of the Marczewski function one can find in [3, 4].

2. The results

We start with an auxiliary technical lemma.

Lemma 1 (cf. [4]). *Let $\mathcal{B}$ be a base of a topological space Y, let X be a set and let $\varphi\colon X \to K(Y)$ be a multifunction. Then for each open $G \subset Y$ we have*

$$\varphi^{-1}(G^+) = \bigcup\{\varphi^{-1}((V_1 \cup \ldots \cup V_k)^+)\colon\ V_i \in \mathcal{B},\ V_i \subset G,\ i = 1, \ldots, k,\ k \in \mathbb{N}\}.$$

We shall need the following versions of two results of Spakowski [7].

Theorem 2 ([7, Thm. 3]). *Let X be a totally bounded metric space, let Y be a metric space and let $F\colon X \to K(Y)$ be an upper semicontinuous multifunction. Then there exists a sequence of simple upper semicontinuous multifunctions $F_n\colon X \to CL(Y)$ pointwise convergent to F with respect to the Vietoris topology and such that $F(x) \subset F_{n+1}(x) \subset F_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$.*

The sequence $(F_n)_{n\in\mathbb{N}}$ is constructed in the following way. Let A_n be a $\frac{1}{n}$-dense subset of X. Put

$$F_n(x) = \bigcap_{k\leqslant n} \bigcap_{s\in A_k} \Theta_{s,k}(x),$$

where

$$\Theta_{s,k}(x) = \begin{cases} \mathrm{cl}(F(B(s,\frac{1}{k}))) & \text{if } x \in B(s,\frac{1}{k}),\\ Y & \text{otherwise.}\end{cases}$$

Note that this is a modification of the definition of Spakowski but the same proof holds. A similar remark applies to the next result.

Theorem 3 ([7, Thm. 4]). *Let X be a totally bounded metric space and let Y be a finite dimensional normed linear space. Assume that $F\colon X \to K(Y)$ is a lower semicontinuous multifunction whose values are convex with nonempty interiors. Then there exists a sequence of simple lower semicontinuous multifunctions $F_n\colon X \to K(Y)\cup\{\emptyset\}$ pointwise convergent to F with respect to the Vietoris topology and such that $F_n(x) \subset F_{n+1}(x) \subset F(x)$ for each $x \in X$ and $n \in \mathbb{N}$.*

Here the sequence $(F_n)_{n\in\mathbb{N}}$ is defined as follows:

$$F_n(x) = \bigcup_{k\leqslant n} \bigcup_{s\in A_k} \Theta_{s,k}(x),$$

where

$$\Theta_{s,k}(x) = \begin{cases} \bigcap\limits_{z\in B(s,\frac{1}{k})} F(z) & \text{if } x \in B(s,\frac{1}{k}),\\ \emptyset & \text{otherwise.}\end{cases}$$

It follows from the proof of [7, Thm. 4] that $F_n(x)$ has nonempty interior for all but finitely many $n \in \mathbb{N}$.

For a multifunction taking the empty set as its value we understand measurability and continuity similarly to the usual case.

Applying Theorems 2, 3 and using the Marczewski function we obtain the following results.

Theorem 4. *Let $(T,\mathfrak{M})$ be a measurable space and let Y be a separable metric space. Then for each measurable multifunction $\varphi\colon T \to K(Y)$ there exists a sequence of simple measurable multifunctions $\varphi_n\colon T \to CL(Y)$ pointwise convergent to φ with respect to the Vietoris topology and such that $\varphi(t) \subset \varphi_{n+1}(t) \subset \varphi_n(t)$ for every $t \in T$ and $n \in \mathbb{N}$.*

P r o o f. Let $\mathcal{B} = \{V_n\colon n \in \mathbb{N}\}$ be a countable base of Y closed under finite unions. Set $A_n = \varphi^{-1}(V_n^+)$. As φ is measurable and compact-valued, each A_n is measurable. Let $M\colon T \to \{0,1\}^{\mathbb{N}}$ be the Marczewski function of $(A_n)_{n\in\mathbb{N}}$. Define a multifunction $\Phi\colon M(T) \to K(Y)$ by setting $\Phi(M(t)) = \varphi(t)$ for every $t \in T$. We need to check that Φ is well-defined. Suppose that $\varphi(t_1) \neq \varphi(t_2)$ and e.g. there is $y_0 \in \varphi(t_1)\setminus\varphi(t_2)$. There exists a $\mathcal{B}_0 \subset \mathcal{B}$ such that $Y\setminus\{y_0\} = \bigcup\mathcal{B}_0$. By compactness we have $\varphi(t_2) \subset B_1\cup\ldots\cup B_k$ for some $B_1,\ldots,B_k \in \mathcal{B}_0$. Now $B_1\cup\ldots\cup B_k = V_m$ for some $m \in \mathbb{N}$. Hence $t_2 \in \varphi^{-1}(V_m^+)$ and $t_1 \notin \varphi^{-1}(V_m^+)$, which means $M(t_1) \neq M(t_2)$.

We now show that Φ is u.s.c. By Lemma 1 it is enough to check that $\Phi^{-1}(V_n^+)$ is open for every $n \in \mathbb{N}$, but we have $\Phi^{-1}(V_n^+) = M(A_n)$ is a closed-open subset of $M(T)$. Observe that $M(T)$ is totally bounded (with a suitable metric). Thus we can apply Theorem 2. There exists a sequence of simple u.s.c. multifunctions $\Phi_n\colon M(T) \to CL(Y)$ pointwise convergent to Φ and such that $\Phi(x) \subset \Phi_{n+1}(x) \subset \Phi_n(x)$ for $x \in M(T), n \in \mathbb{N}$. Define $\varphi_n = \Phi_n\circ M$. Notice that $\varphi_n^{-1}(G^+)$ is measurable for each open $G \subset Y$. Hence φ_n is measurable. Clearly, for the sequence $(\varphi_n)_{n\in\mathbb{N}}$ our statement holds. $\square$

In the above result we cannot require φ_n to be a compact-valued multifunction. Indeed, if we consider a multifunction $\varphi\colon [0,1] \to K(\mathbb{R})$ defined by $\varphi(x) = [0,\frac{1}{x}]$ for $x > 0$ and $\varphi(0) = \{0\}$, then there does not exist a simple compact-valued multifunction ψ with $\varphi(x) \subset \psi(x)$ for all $x \in [0,1]$.

Theorem 5. *Let $(T,\mathfrak{M})$ be a measurable space, let Y be a finite dimensional normed linear space and let $\varphi\colon T \to K(Y)$ be a measurable multifunction whose values are convex with nonempty interiors. Then there exists a sequence of simple measurable multifunctions $\varphi_n\colon T \to K(Y)\cup\{\emptyset\}$ pointwise convergent to φ with respect to the Vietoris topology and such that $\varphi_n(t) \subset \varphi_{n+1}(t) \subset \varphi(t)$.*

Proof. Let $\mathcal{B} = \{V_n\colon n \in \mathbb{N}\}$ be a base of Y. Put $A_n = \varphi^{-1}(V_n^-) \in \mathfrak{M}$. Let M be the Marczewski function of the sequence $(A_n)_{n\in\mathbb{N}}$. Define $\Phi\colon M(T) \to K(Y)$ by setting $\Phi(M(t)) = \varphi(t)$ for $t \in T$. For the proof that Φ is well-defined let us consider $t_1, t_2 \in T$ such that $\varphi(t_1) \neq \varphi(t_2)$ and e.g. $y_0 \in \varphi(t_1)\setminus\varphi(t_2)$. Then there exists $m \in \mathbb{N}$ with $y_0 \in V_m$ and $V_m \cap \varphi(t_2) = \emptyset$. Hence $t_1 \in A_m$ and $t_2 \notin A_m$, which means that $M(t_1) \neq M(t_2)$. Observe that Φ is l.s.c. since $\Phi^{-1}(V_n^-) = M(A_n)$ is closed-open in $M(T)$. Now apply Theorem 3. There exists a sequence of l.s.c. simple multifunctions $\Phi_n\colon M(T) \to K(Y)\cup\{\emptyset\}$ pointwise convergent to Φ and such that $\Phi_n(x) \subset \Phi_{n+1}(x) \subset \Phi(x)$ for $x \in M(T), n \in \mathbb{N}$. Define $\varphi_n = \Phi_n \circ M$. Clearly, $(\varphi_n)_{n\in\mathbb{N}}$ is as desired. $\square$

In the above result, taking $\widehat{\varphi}_n(x) = \operatorname{conv}\varphi_n(x)$ we obtain an increasing sequence of convex-valued simple measurable multifunctions pointwise convergent to φ.

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