ON ITERATED LIMITS OF SUBSETS OF A CONVERGENCE ℓ-GROUP

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Abstract. In this paper we deal with the relation

$$\lim_{\alpha}\lim_{\alpha} X = \lim_{\alpha} X$$

for a subset X of G, where G is an ℓ-group and α is a sequential convergence on G.

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For a convergence ℓ-group (shorter: cl-group) we apply the same notation and definitions as in [4] with the distinction that now we do not assume the commutativity of the group operation.

Let (G,α) be a cl-group (where G is an ℓ-group and α is a convergence on G). For $X \subseteq G$ the symbol $\lim_{\alpha} X$ has the usual meaning. X will be said to be regular with respect to (G,α) if the relation

$$\lim_{\alpha}\lim_{\alpha} X = \lim_{\alpha} X$$

is valid.

An ℓ-group G will be called absolutely regular, if whenever (G,α) is a convergence ℓ-group and H is an ℓ-subgroup of G, then H is regular with respect to (G,α).

We denote by F the class of all ℓ-groups K such that each disjoint subset of K is finite; such ℓ-groups were studied in [1] (cf. also [2] and [6]).

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In the present paper we prove that each ℓ-group belonging to F is absolutely regular.

This generalizes a result from [5] concerning ℓ-groups which can be represented as direct products of a finite number of linearly ordered groups.

1. Preliminaries

In the whole paper G is an ℓ-group; the group operation is written additively, but we do not assume commutativity of this operation.

For the notion of convergence $\alpha \in \operatorname{conv} G$ we apply the same definition as in [4] with the distinction that to the conditions for α used in [4] we add the following one:

($*$) α is a normal subset of $(G^N)^+$ (i.e., if $s \in (G^N)^+$, then $s + \alpha = \alpha + s$).

The corresponding convergence ℓ-group will be denoted by (G, α).

If X is a nonempty subset of G, then by $\lim\limits_{\alpha} X$ we denote the set of all $g \in G$ such that there exists a sequence $(x_n) \in X$ with $x_n \to_\alpha g$.

It is easy to verify that

(i) if X is an ℓ-subgroup of G, then $\lim\limits_{\alpha} X$ is an ℓ-subgroup of G as well;

(ii) if X is convex in G, then the same holds for $\lim\limits_{\alpha} X$.

We shall often apply the following rule:

If $x_n \to_\alpha g$ and $x_n \leqslant g$ for each $n \in N$, then $\bigvee\limits_{n\in N} x_n = g$ (and dually).

A subset Y of G is called disjoint if $Y \subseteq G^+$ and $y_1 \wedge y_2 = 0$ whenever y_1 and y_2 are distinct elements of G.

The direct product of ℓ-groups $G_1, G_2, \ldots, G_k$ is defined in the usual way; it will be denoted by $G_1 \times G_2 \times \ldots \times G_n$.

If H is a convex ℓ-subgroup of G such that $g > h$ for each $g \in G^+ \setminus H$ and each $h \in H$, then G is said to be a lexico extension of H; we express this fact by writing $G = \langle H \rangle$. For the properties of the lexico extension cf., e.g., [2].

2. Auxiliary results

Let (G, α) be a cl-group.

2.1. Lemma. *Let (x_n) be a sequence in G, $x_n \leqslant x_{n+1}$ for each $n \in N$, $g \in G$, $x_n \to_\alpha g$. Then $\bigvee\limits_{n\in N} x_n = g$.*

P r o o f. If there exists a subsequence (x_n^1) of (x_n) such that $x_n^1 \leqslant g$ for each $n \in N$, then $\bigvee\limits_{n\in N} x_n^1 = g$, and hence we have also $\bigvee\limits_{n\in N} x_n = g$. If such a subsequence

(x_n^1) does not exist, then there is a subsequence (x_n^2) of (x_n) such that for each $n \in N$, either $x_n^2 > g$ or x_n^2 is incomparable with g. Hence $x_n^2 \vee g > g$ for each $n \in N$. Thus we obtain

$$(*_1) \qquad x_n^2 \vee g \to_\alpha g$$

and

$$g < x_1^2 \vee g \leqslant x_n^2 \vee g \quad \text{for each } n \in N,$$

so that the relation $(*_1)$ cannot be valid. $\square$

2.2. Lemma. *Let H be an ℓ-subgroup of the ℓ-group G. Suppose that H can be represented as a lexico extension $H = \langle A \rangle$ with $A \neq \{0\}$. Then*

$$\lim_\alpha H = \bigcup_{h \in H} \lim_\alpha (h + A).$$

Moreover, if $h_1, h_2 \in H$ and $h_1 \notin h_2 + A$, then

$$\lim_\alpha (h_1 + A) \cap \lim_\alpha (h_2 + A) = \emptyset.$$

Proof. For $h \in H$ we put $\overline{h} = h + A$. If $h_1, h_2 \in H$ and $h_1 \notin h_2 + A$, then from the properties of the lexico extension we infer that either

(i) $h_1' < h_2'$ for each $h_1' \in h_1 + A$ and each $h_2' \in h_2 + A$,

or

(ii) $h_2' < h_1'$ for each $h_1' \in h_1 + A$ and each $h_2' \in h_2 + A$.

Let $g \in G$ and suppose that there exists a sequence (h_n) in H such that $h_n \to_\alpha g$.

a) First suppose that there exist $h_1 \in H$ and a subsequence (h_n') of (h_n) such that $h_n' \in h_1 + A$ for each $n \in N$. Then $h_n' \to_\alpha g$, whence $g \in \lim_\alpha (h_1 + A)$.

b) Now suppose that the assumption from a) is not valid. Then there exists a subsequence (h_n') of (h_n) such that, whenever $n(1)$ and $n(2)$ are distinct positive integers, then

$$h_{n(1)}' + A \neq h_{n(2)}' + A.$$

Thus in view of the relations (i) and (ii) above, if $n(1)$ and $n(2)$ are distinct, then either $h_{n(1)}' < h_{n(2)}'$ or $h_{n(1)}' > h_{n(2)}'$. This implies that there exists a subsequence (h_n'') of (h_n') such that either

$$h_n'' < h_{n+1}'' \quad \text{for each } n \in N,$$

or

$$h''_n > h''_{n+1} \quad \text{for each } n \in N.$$

Suppose that the first case occurs (in the second case we apply a dual argument). We have $h''_n \to_\alpha g$ and thus according to 2.1 the relation

$$\bigvee_{n\in N} h''_n = g$$

is valid.

If there exists $n(1) \in N$ such that $h''_{n(1)} + A = g + A$, then $h''_{n(1)+1} > g$, which is a contradiction. Hence

$$h''_{n(1)} + A \neq g + A \quad \text{for each } n(1) \in N.$$

Since $A \neq \{0\}$, there exists $a \in A$ with $a > 0$. Then

$$h''_n < g - a \quad \text{for each } n \in N,$$

which is imposible. Thus we have verified that the condition from a) must be valid. Therefore

$$\bigcup_{h\in H} \lim_\alpha (h + A) \subseteq \lim_\alpha H \subseteq \bigcup_{h\in H} \lim_\alpha (h + A),$$

which proves the first assertion of the lemma.

c) Let g be as above; we have shown that there is $h_1 \in H$ such that $g \in \lim\limits_\alpha (h_1+A)$. Let $h_2 \in H$, $h_1 \notin h_2 + A$. By way of contradiction, suppose that $g \in \lim\limits_\alpha (h_2 + A)$. Hence there exists a sequence (h_n^2) in $h_2 + A$ such that $h_n^2 \to_\alpha g$. At the same time, there exists a sequence (h_n^1) in $h_1 + A$ such that $h_n^1 \to_\alpha g$. Let a be as above. If (i) is valid, then

$$h_n^1 + a < h_n^2 \quad \text{for each } n \in N,$$

thus $g + a \leqslant g$, which is a contradiction. In the case when (ii) is valid we proceed dually. □

2.3. Lemma. *Let H be as in 2.2. Then $\lim\limits_\alpha H = \langle \lim\limits_\alpha A \rangle$.*

P r o o f. We obviously have $\lim\limits_\alpha A \subseteq \lim\limits_\alpha H$ and thus $\lim\limits_\alpha A$ is an ℓ-subgroup of $\lim\limits_\alpha H$. Let $h_1, h_2 \in \lim\limits_\alpha A$, $h \in \lim\limits_\alpha H$, $h_1 \leqslant h \leqslant h_2$. Then there exist sequences $(h_n^1), (h_n^2)$ in A and (h'_n) in H such that

$$h_n^1 \to_\alpha h_1, \quad h_n^2 \to_\alpha h_2, \quad h'_n \to_\alpha h.$$

Put $(h'_n \vee h^1_n) \wedge h^2_n = h''_n$. Then $h''_n \in A$ for each $n \in N$ and

$$h''_n \to_\alpha (h \vee h_1) \wedge h_2 = h,$$

whence $h \in \lim_\alpha A$. Thus $\lim_\alpha A$ is a convex subset of $\lim_\alpha H$.

Let $h \in (\lim_\alpha H)^+ \setminus \lim_\alpha A$. In view of 2.2 there exist $h^1 \in H$ and a sequence (h_n) in $h^1 + A$ such that $h_n \to_\alpha h$. Moreover, h^1 does not belong to A. Since $h \in G^+$, without loss of generality we can suppose that all h_n belong to G^+. Further, 2.2 yields that there is a subsequence (h^1_n) of (h_n) such that for each $n \in N$ the relation $h^1_n \notin A$ is valid. Thus $h^1_n > a$ for each $a \in A$. Therefore $h \geqslant a$; since $h \notin A$ we obtain that $h > a$ for each $a \in A$.

If $a' \in \lim_\alpha A$, then there exists a sequence (a_n) in A with $a_n \to_\alpha a'$. Thus $h > a_n$ for each $n \in N$, hence $h \geqslant a'$. Since $h \notin \lim_\alpha A$ we get $h > a'$ for each $a' \in \lim_\alpha A$. Therefore $\lim_\alpha H = \langle \lim_\alpha A \rangle$. □

2.4. Corollary. *If H is as in 2.2 and if A is regular with respect to (G, α), then H is regular with respect to (G, α).*

2.5. Corollary. *Let H be and ℓ-group, $H = \langle A \rangle$, $A \neq \{0\}$ and suppose that A is absolutely regular. Then H is absolutely regular.*

2.6. Proposition. *Let A be an ℓ-group which can be represented as a direct product of a finite number of linearly ordered groups. Suppose that $A \neq \{0\}$ and $H = \langle A \rangle$. Then H is absolutely regular.*

P r o o f. This is a consequence of 2.6 and of Theorem 3.6, [3]. □

2.7. Lemma. *Let H be an ℓ-subgroup of G such that*

(i) *H can be represented as a direct product $H_1 \times H_2 \times \ldots \times H_k$;*

(ii) *there are ℓ-subgroups A_i of H_i such that $H_i = \langle A_i \rangle$, $H_i \neq A_i \neq \{0\}$ $(i = 1, 2, \ldots, k)$.*

Then $\lim_\alpha H = \lim_\alpha H_1 \times \ldots \times \lim_\alpha H_k$.

P r o o f. Let $i \in \{1, 2, \ldots, k\}$. In view of 2.3,

$$\lim_\alpha H_i = \langle \lim_\alpha A_i \rangle.$$

Now we proceed by induction with respect to k. For $k = 1$ the assertion is trivial. Let $k > 1$. Consider an element $g \in \lim_\alpha H$ with $g > 0$. Then there exists a sequence (z_n) in H such that $z_n \to_\alpha g$ and $z_n > 0$ for each $n \in N$.

a) First we prove that g cannot be an upper bound of the set H. In fact, if $g \geqslant h$ for each $h \in H$, then $g \geqslant z_n$ for each $n \in N$, whence $g = \bigvee_{n \in N} z_n$ and thus $g = \sup H$. There exists $h_0 \in H$ with $h_0 > 0$. Then $h + h_0 \in H$ for each $h \in H$, yielding that $h + h_0 \leqslant g$. Hence $h \leqslant g - h_0 < g$ for each $h \in H$, which is a contradiction.

b) For $h \in H$ and $i \in I$ we denote by $h(H_i)$ the component of h in H_i. If $h \geqslant 0$, then

$$h = h(H_1) + h(H_2) + \ldots + h(H_n) = h(H_1) \vee h(H_2) \vee \ldots \vee h(H_n).$$

Thus in view of a) there exists $i_0 \in \{1, 2, \ldots, k\}$ such that g fails to be an upper bound of the set H_{i_0}. Without loss of generality we can suppose that $i_0 = k$. Therefore there exists $x_0 \in H_k^+$ such that $x_0 \nleqq g$.

We have

$$z_n \wedge x_0 = (z_n(H_1) \vee z_n(H_2) \vee \ldots \vee z_n(H_k)) \wedge x_0 = z_n(H_k) \wedge x_0 \in H_k$$

(since $z_n(H_i) \wedge x_0 = 0$ for $i = 1, 2, \ldots, k-1$). Then

$$z_n(H_k) \wedge x_0 \to g \wedge x_0,$$

whence $g \wedge x_0 \in \lim_\alpha H_k \subseteq \lim_\alpha H$.

For each $h^k \in H_k$ we denote $\overline{h^k} = h^k + A_k$. Further we put

$$\overline{H}_k = \{\overline{h^k} \colon \ h^k \in H_k\}.$$

If $\overline{h_1^k}$ and $\overline{h_2^k}$ are distinct elements of $\overline{H}_k$ and $h_1^k < h_2^k$, then we put $\overline{h_1^k} < \overline{h_2^k}$. In this way $\overline{H}_k$ turns out to be a linearly ordered set.

Consider the sequence $(\overline{z_n(H_k)})$. If there existed a subsequence $(\overline{y}_n)$ of $(\overline{z_n(H_k)})$ such that $\overline{y}_n > \overline{x}_0$ for each $n \in N$, then we would have $g \geqslant x_0$, which is a contradiction. Hence there is a subsequence $(\overline{y}_n)$ of $(\overline{z_n(H_k)})$ such that $\overline{y}_n \leqslant \overline{x}_0$ for each $n \in N$.

Since $H_k \neq A_k$ there exists $x_0' \in H_k$ such that $\overline{x}_0 < \overline{x_0'}$. We can replace $\overline{x}_0$ by $\overline{x_0'}$ and then the previous considerations remain valid. Moreover, $\overline{y}_n < \overline{x_n'}$ for each $n \in N$. We have $y_n = z_n^1(H_k)$, where (z_n^1) is a subsequence of (z_n). Thus

$$z_n^1(H_k) < x_0' \quad \text{for each } n \in N,$$

and $z_n^1(H_k) \wedge x_0' \to_\alpha g \wedge x_0'$. Hence $z_n^1(H_k) \to_\alpha g \wedge x_0'$. This yields that

$$z_n' - z_n'(H_k) \to_\alpha g - (g \wedge x_0').$$

Since

$$z_n' - z_n'(H_k) = z_n'(H_1) + z_n'(H_2) + \ldots + z_n'(H_{k-1}) \in H_1 \times \ldots \times H_{k-1},$$

in view of the induction hypothesis we obtain

$$g - (g \wedge x_0') \in \lim_\alpha H_1 \times \lim_\alpha H_2 \times \ldots \times \lim_\alpha H_{k-1}.$$

Denote

$$\lim_\alpha H_1 \times \lim_\alpha H_2 \times \ldots \times \lim_\alpha H_{k-1} = Y_{k-1}.$$

It is easy to verify that if $y_{k-1} \in (Y_{k-1})^+$ and $y_k \in (\lim_\alpha H_k)^+$, then

$$y_{k-1} \wedge y_k = 0.$$

Further, we obviously have

$$0 \in (Y_{k-1})^+ \cap (\lim H_k)^+.$$

Let Y be the sublattice of the lattice G^+ generated by the set

$$(Y_{k-1})^+ \cup (\lim H_k)^+.$$

Since the lattice G^+ is distributive, we obtain

$$Y = \{y_{k-1} \vee y_k \colon\ y_{k-1} \in (Y_{k-1})^+ \text{ and } y_k \in (\lim H_k)^+\}.$$

Thus in view of Lemma 3.4 in [5] we get

$$Y = (Y_{k-1})^+ \times Y_k^+, \tag{1}$$

where Y_k^+ is the underlying lattice of the lattice ordered semigroup $(\lim_\alpha H_k)^+$.

For $A, B \subseteq G$ we put

$$A - B = \{a - b \colon\ a \in A \text{ and } b \in B\}.$$

Clearly

$$\lim_\alpha H_k = Y_k^+ - Y_k^+.$$

Therefore according to (1) and by applying Theorem 2.9 in [3] we obtain

$$\begin{aligned} \lim_\alpha H = Y - Y = ((Y_{k-1})^+ - (Y_{k-1})^+) \times (Y_k^+ - Y_k^+) = Y_{k-1} \times \lim_\alpha H_k \\ = \lim_\alpha H_1 \times \lim_\alpha H_2 \times \ldots \times \lim_\alpha H_{k-1} \times \lim_\alpha H_k. \end{aligned}$$

□

2.8. Lemma. *Let H and $H_1, H_2, \dots, H_k$ be as in 2.7. Further suppose that all A_i $(i = 1, 2, \dots, k)$ are regular with respect to (G, α). Then $\lim_\alpha H$ can be represented in the form*

$$\lim_\alpha H = \langle \lim_\alpha A_1 \rangle \times \langle \lim_\alpha A_2 \rangle \times \dots \times \langle \lim_\alpha A_k \rangle$$

and all $\lim_\alpha A_i$ $(i = 1, 2, \dots, k)$ are regular with respect to (G, α).

P r o o f. The first assertion is a consequence of 2.7 and 2.3; the latter is obvious. □

2.9. Lemma. *Let H and $H_1, H_2, \dots, H_k$ be as in 2.8. Then H is regular with respect to (G, α).*

P r o o f. In view of 2.3, 2.7 and 2.8 we have

$$\begin{aligned} \lim_\alpha \lim_\alpha H &= \lim_\alpha \langle \lim_\alpha A_1 \rangle \times \dots \times \lim_\alpha \langle \lim_\alpha A_k \rangle \\ &= \langle \lim_\alpha \lim_\alpha A_1 \rangle \times \dots \times \langle \lim_\alpha \lim_\alpha A_k \rangle \\ &= \langle \lim_\alpha A_1 \rangle \times \dots \times \langle \lim_\alpha A_k \rangle = \lim_\alpha H. \end{aligned}$$

□

2.10. Corollary. *Let H and H_i $(i = 1, 2, \dots, k)$ be ℓ-groups such that the conditions* (i) *and* (ii) *from 2.7 are valid. Further suppose that all A_i $(i = 1, 2, \dots, k)$ are absolutely regular. Then H is absolutely regular.*

3. On ℓ-groups belonging to F

In this section we assume that H is an ℓ-group belonging to the class F and that $H \neq \{0\}$.

It follows from the results of [1] concerning the structure of ℓ-groups belonging to the class F that there exist a positive integer n and finite systems $F_1, F_2, \dots, F_n$ of convex nonzero subgroups of H such that

(i) $F_1 = \{A_1^1, A_2^1, \dots, A_{n(1)}^1\}$, all ℓ-groups A_i^1 $(i = 1, \dots, n(1))$ are linearly ordered and $A_{i(1)}^1 \cap A_{i(2)}^1 = \{0\}$ whenever $i(1), i(2)$ are distinct elements of the set $\{1, 2, \dots, n(1)\}$.

(ii) If $k > 1$, then $F_k = \{A_1^k, A_2^k, \dots, A_{n(k)}^k\}$ such that

 (ii$_1$) $A_{i(1)}^k \cap A_{i(2)}^k = \{0\}$ whenever $i(1), i(2)$ are distinct elements of the set $\{1, 2, \dots, n(k)\}$;

(ii$_2$) if $i \in \{1, 2, \ldots, n(k)\}$, then either A_i^k is equal to an element of F_{k-1}, or there are $B_1, B_2, \ldots, B_{t(i)} \in F_{k-1}$ such that $t(i) \geqslant 2$ and $A_i^k = \langle B_1 \times B_2 \times \ldots \times B_{t(i)} \rangle$.

(iii) $F_n = \{H\}$.

3.1. Lemma. *Let us apply the above notation and let $k \in \{1, 2, \ldots, n\}$. Then all ℓ-groups of the system F_k are absolutely regular.*

Proof. We proceed by induction with respect to k. For $k = 1$, this is a consequence of Theorem 3.6 in [5]. Suppose that $k > 1$ and that the assertion is valid for $k - 1$. Then 2.10 yields that the elements of F_k are absolutely regular. $\square$

As a corollary we obtain

3.2. Theorem. *Each ℓ-group belonging to F is absolutely regular.*

If an ℓ-group H is a direct product of a finite number of linearly ordered groups, then H belongs to F. Hence 3.2 generalizes Theorem 3.6 from [5].

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