EXISTENCE OF NONOSCILLATORY AND OSCILLATORY SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

John R. Graef,[1] Bo Yang,[2] Mississippi State, B. G. Zhang,[2] Qingdao

(Received October 7, 1997)

Abstract. In this paper, we study the existence of oscillatory and nonoscillatory solutions of neutral differential equations of the form

$$\big(x(t)-cx(t-r)\big)' \pm \big(P(t)x(t-\theta)-Q(t)x(t-\delta)\big)=0$$

where $c>0$, $r>0$, $\theta>\delta\geqslant 0$ are constants, and P, $Q\in C(\mathbb{R}^+,\mathbb{R}^+)$. We obtain some sufficient and some necessary conditions for the existence of bounded and unbounded positive solutions, as well as some sufficient conditions for the existence of bounded and unbounded oscillatory solutions.

Keywords: neutral differential equations, nonoscillation, oscillation, positive and negative coefficients

MSC 2000: 34K40, 34K15

1. Introduction

In this paper, we consider the following neutral differential equations with positive and negative coefficients

$$\big(x(t)-cx(t-r)\big)'+P(t)x(t-\theta)-Q(t)x(t-\delta)=0 \tag{1.1}$$

and

$$\big(x(t)-cx(t-r)\big)'=P(t)x(t-\theta)-Q(t)x(t-\delta), \tag{1.2}$$

[1] Research supported by the Mississippi State University Biological and Physical Sciences Research Institute.
[2] Research partially supported by NNSF of China.

where $c > 0$, $r > 0$, $\theta > \delta \geqslant 0$ are constants, and $P,\ Q \in C(\mathbb{R}^+, \mathbb{R}^+)$. Equality (1.1) has been investigated by Ladas and Qian [2, 6], Yu [9], Yu and Wang [8], and Lalli and Zhang [7]. However, results on the existence of positive solutions and the existence of oscillatory solutions of (1.1) and (1.2) are relatively scarce in the literature.

In Section 2, we obtain conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for (1.1) with $c = 1$, and in Section 3, we obtain conditions for the existence of unbounded positive solutions for (1.1) with $c = 1$. Section 4 contains conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for (1.1) with $c \in (0, 1)$, while in Section 5, we obtain conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for (1.2). In Section 6, we consider (1.1) and (1.2) in the case $c > 1$. Obviously, since the equations under consideration are linear, there are corresponding conclusions for negative solutions.

The following hypotheses will often be used in the remainder of this paper:

$$\text{(H1)} \qquad r > 0 \text{ and } \theta > \delta \geqslant 0 \text{ are constants;}$$

$$\text{(H2)} \qquad P,\ Q \in C(\mathbb{R}^+, \mathbb{R}^+);$$

$$\text{(H3)} \qquad \overline{P}(t) = P(t) - Q(t - \theta + \delta) \geqslant 0.$$

The following lemma is taken from Zhang and Yu [10].

Lemma 1.1. *Suppose that* $f \in C\big([t_0, \infty), \mathbb{R}^+\big)$ *and* $r > 0$. *Then*

$$\sum_{j=0}^{\infty} \int_{t_0 + jr}^{\infty} f(t)\,\mathrm{d}t < \infty$$

is equivalent to

$$\int_{t_0}^{\infty} t f(t)\,\mathrm{d}t < \infty.$$

2. Bounded solutions of (1.1) with $c = 1$

In this section, we consider the equation

$$\big(x(t) - x(t - r)\big)' + P(t)x(t - \theta) - Q(t)x(t - \delta) = 0. \tag{2.1}$$

Theorem 2.1. *In addition to* (H1)–(H3), *assume that*

$$\text{(H4)} \qquad \int^{\infty} t\overline{P}(t)\,\mathrm{d}t < \infty,$$

and

(H5) $$\int^{\infty} Q(t)\,\mathrm{d}t < \infty.$$

Then (2.1) *has a bounded positive solution, and for any continuous periodic oscillatory function* $\omega(t)$ *with period* r, *there is a bounded oscillatory solution* $x(t)$ *such that*

(2.2) $$x(t) = \omega(t) + R(t)$$

for $t > T$, *where* $R(t)$ *is a continuous real function,* $|R(t)| < \alpha M$, $M = \min\{\max\omega(t), \max(-\omega(t))\}$, $\alpha \in (0,1)$, *and* T *is sufficiently large.*

To prove the above theorem, we need to establish the following lemma.

Lemma 2.2. *Suppose the hypotheses of Theorem* 2.1 *hold. Then the equations*

(2.3) $$\big(x(t)-x(t-r)\big)' + P(t)\big(x(t-\theta)+2\overline{M}+\omega(t-\theta)\big) - Q(t)\big(x(t-\delta)+2\overline{M}+\omega(t-\delta)\big) = 0$$

and

(2.4) $$\big(x(t)-x(t-r)\big)' + P(t)\big(x(t-\theta)+2\overline{M}\big) - Q(t)\big(x(t-\delta)+2\overline{M}\big) = 0$$

have bounded positive solutions $u_1(t)$ *and* $u(t)$, *respectively, such that*

$$|u(t)| \leqslant \tfrac{1}{2}\alpha M \quad \text{and} \quad |u_1(t)| \leqslant \tfrac{1}{2}\alpha M$$

for $t \geqslant T$, *where* $\overline{M} = \max|\omega(t)|$ *and* T *is sufficiently large.*

Proof. The proof for (2.3) is quite similar to that for (2.4), so we only give the details of the proof for (2.4).

Choose T sufficiently large such that

(2.5) $$\sum_{i=0}^{\infty} \int_{T+ir}^{\infty} \overline{P}(t)\,\mathrm{d}t + n\int_{T-\theta}^{\infty} Q(t)\,\mathrm{d}t < \frac{\alpha M}{16\overline{M}},$$

where $n = [\![\frac{\theta-\delta}{r}]\!] + 2$ and $[\![\cdot]\!]$ denotes the greatest integer function. Set

$$H(t) = \begin{cases} 4\overline{M}\displaystyle\int_t^{\infty} \overline{P}(s)\,\mathrm{d}s + 4\overline{M}\int_{t-\theta+\delta}^{t} Q(s)\,\mathrm{d}s, & t \geqslant T, \\ \big(t-T+r\big)H(T)/r, & T-r \leqslant t \leqslant T, \\ 0, & t \leqslant T-r. \end{cases}$$

Clearly, $H \in C(\mathbb{R}, \mathbb{R}^+)$. Define

$$y(t) = \sum_{i=0}^{\infty} H(t - ir), \ t \geqslant T.$$

It is obvious that $y \in C\big([T, \infty), \mathbb{R}^+\big)$ with $y(t) - y(t-r) = H(t)$ and $0 < y(t) < \frac{1}{4}\alpha M < \overline{M}$, $t \geqslant T$. Define a set X by

$$X = \big\{x \in C\big([T, \infty), \mathbb{R}\big) : \ 0 \leqslant x(t) \leqslant y(t), t \geqslant T\big\}$$

and an operator S on X by

$$(Sx)(t) = \begin{cases} x(t-r) + \displaystyle\int_{t-\theta+\delta}^{t} Q(s)\big(x(s-\delta) + 2\overline{M}\big)\,\mathrm{d}s & \\ \qquad + \displaystyle\int_{t}^{\infty} \overline{P}(s)\big(x(s-\theta) + 2\overline{M}\big)\,\mathrm{d}s, & t \geqslant T + m, \\ (Sx)(T+m)\frac{ty(t)}{(T+m)y(T+m)} + y(t)\big(1 - \frac{t}{T+m}\big), & t \in [T, T+m], \end{cases}$$

where $m = \max\{\theta, r\}$. It is easy to see that

$$(Sx)(t) \leqslant y(t-r) + H(t) = y(t), \quad t \geqslant T + m$$

and

$$(Sx)(t) \leqslant y(t), \quad T \leqslant t \leqslant T + m,$$

for any $x \in X$, i. e., $SX \subset X$. Define a sequence of functions $\big\{x_k(t)\big\}_{k=0}^{\infty}$ as follows:

$$x_0(t) = y(t), \quad t \geqslant T,$$
$$x_k(t) = (Sx_{k-1})(t), \quad t \geqslant T, \quad k = 1, 2, \ldots$$

By induction, we can prove that

$$0 < x_k(t) \leqslant x_{k-1}(t) \leqslant y(t), \quad t \geqslant T, \quad k = 1, 2, \ldots$$

Then there exists a function $u \in X$ such that $\lim\limits_{k\to\infty} x_k(t) = u(t)$ for $t \geqslant T$. Clearly, $u(t) > 0$ on $[T, \infty)$. By the Lebesgue dominated convergence theorem, we have

$$u(t) = u(t-r) + \int_{t-\theta+\delta}^{t} Q(s)\big(u(s-\delta) + 2\overline{M}\big)\,\mathrm{d}s + \int_{t}^{\infty} \overline{P}(s)\big(u(s-\theta) + 2\overline{M}\big)\,\mathrm{d}s$$

for $t \geqslant T + m$. Moreover,

$$\big(u(t) - u(t-r)\big)' = Q(t)\big(u(t-\delta) + 2\overline{M}\big) - P(t)\big(u(t-\theta) + 2\overline{M}\big),$$

i. e., $u(t)$ is a bounded positive solution of (2.4) with $0 < u(t) \leqslant \frac{1}{4}\alpha M$. This completes the proof of the lemma. □

P r o o f of Theorem 2.1. Let

$$U(t) = 2\overline{M} + u(t)$$

and

$$U_1(t) = 2\overline{M} + \omega(t) + u_1(t),$$

where $u(t)$, $u_1(t)$ are defined by Lemma 2.2. It is easy to see that $U(t)$ and $U_1(t)$ are both bounded positive solutions of (2.1). Because (2.1) is linear,

$$x(t) = U_1(t) - U(t) = \omega(t) + \bigl(u_1(t) - u(t)\bigr), \quad t \geqslant T$$

is also a solution of (2.1). It is clear that $x(t)$ is oscillatory and satisfies (2.2), so the proof of the theorem is complete. $\square$

E x a m p l e 2.3. Consider the neutral differential equation

$$\bigl(x(t) - x(t-1)\bigr)' + P_1(t)x(t-1) - Q_1(t)x(t) = 0,\ t \geqslant 5, \tag{2.6}$$

where

$$P_1(t) = \frac{6}{t^2(t-1)(t-2)} \quad \text{and} \quad Q_1(t) = \frac{6t-2}{t(t-1)^4(t+1)}.$$

We have $\overline{P}_1(t) \equiv P_1(t) - Q_1(t-1) \geqslant 0$ for $t \geqslant 5$,

$$\int^{\infty} Q_1(s)\,\mathrm{d}s < \infty, \quad \text{and} \quad \int^{\infty} s\overline{P}_1(s)\,\mathrm{d}s < \infty.$$

By Theorem 2.1, (2.6) has a bounded positive solution. In fact,

$$x(t) = 1 - t^{-2}$$

is such a solution of (2.6).

E x a m p l e 2.4. Consider the neutral differential equation

$$\bigl(x(t) - x(t-2\pi)\bigr)' + P_2(t)x\bigl(t - \tfrac{5}{2}\pi\bigr) - Q_2(t)x(t-\pi) = 0,\ t \geqslant 6\pi, \tag{2.7}$$

where

$$P_2(t) = \frac{4\pi(t-\pi)}{t^2(t-2\pi)^2} \cdot \frac{\bigl(t - \frac{5}{2}\pi\bigr)^2}{(t-\frac{5}{2}\pi)^2 - 1},$$

and

$$Q_2(t) = 4\pi\frac{3t^2 - 6\pi t + 4\pi^2}{(t(t-2\pi))^3} \cdot \frac{(t-\pi)^2}{(t-\pi)^2 - 1}.$$

Now $\overline{P}_2(t) \equiv P_2(t) - Q_2\bigl(t - \frac{3}{2}\pi\bigr) \geqslant 0$ for $t \geqslant 6\pi$,

$$\int^{\infty} Q_2(s)\,\mathrm{d}s < \infty \quad \text{and} \quad \int^{\infty} s\overline{P}_2(s)\,\mathrm{d}s < \infty.$$

By Theorem 2.1, (2.7) has a bounded oscillatory solution, in fact,

$$x(t) = \bigl(1 - t^{-2}\bigr)\sin t$$

is such a solution of (2.7).

Remark 2.5. According to a result of Jaroš and Kusano [5; Theorem 1], if for some $\mu \in (0,1)$,

$$\int_T^{\infty} \mu^{-\frac{s}{r}}\bigl(P(s) + Q(s)\bigr)\,\mathrm{d}s < \infty, \tag{2.8}$$

then (2.1) has oscillatory solutions. Clearly, their condition is much stronger than conditions (H4)–(H5) of Theorem 2.1. For example, (2.8) is not satisfied for (2.7).

The following result gives a necessary condition for the existence of bounded positive solutions of (2.1).

Theorem 2.6. *Assume that* (H1)–(H3) *and* (H5) *hold. If* (2.1) *has a bounded positive solution, then* (H4) *holds.*

Proof. Let $x(t)$ be a bounded positive solution of (2.1). Then there exists $L > 0$ and $t_0 > 0$ such that $0 < x(t) < L$ on $[t_0, \infty)$. Setting

$$y(t) = x(t) - x(t-r) - \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\,\mathrm{d}s,$$

we have

$$y'(t) = -\overline{P}(t)x(t-\theta) \leqslant 0, \quad t \geqslant t_0. \tag{2.9}$$

We claim that $y(t) > 0$ eventually. Assume, to the contrary, that $y(t) < 0$ eventually. Then there exist $t_1 > t_0$ and $\alpha > 0$ such that $y(t) \leqslant -\alpha$ on $[t_1, \infty)$, so

$$x(t) \leqslant -\alpha + x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\,\mathrm{d}s$$

for $t \geqslant t_1$. By induction, we have

$$\begin{aligned} x(t_1 + kr) &\leqslant -k\alpha + x(t_1) + \sum_{i=1}^{k} \int_{t_1+ir-\theta+\delta}^{t_1+ir} Q(s)x(s-\delta)\,\mathrm{d}s \\ &\leqslant -k\alpha + x(t_1) + nL \int_{t_1-\theta}^{\infty} Q(s)\,\mathrm{d}s, \end{aligned}$$

where $n = [\![\frac{\theta-\delta}{r}]\!] + 2$, $k = 1, 2, \ldots$ Then $x(t_1 + kr) < 0$ for sufficiently large k, which is a contradiction.

Hence, we have

$$x(t) > x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\,\mathrm{d}s > x(t-r)$$

eventually. Thus, there exist $J > 0$ and $t_2 > t_1$ such that $x(t) > J$ on $[t_2, \infty)$. From (2.9), we see that

$$y'(t) \leqslant -\overline{P}(t)J, \quad \text{for } t \geqslant t_3 = t_2 + \theta.$$

Integrating, we obtain

$$y(t) \geqslant J \int_{t}^{\infty} \overline{P}(s)\,\mathrm{d}s,$$

and so

$$x(t) \geqslant x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\,\mathrm{d}s + J\int_{t}^{\infty} \overline{P}(s)\,\mathrm{d}s \geqslant x(t-r) + J\int_{t}^{\infty} \overline{P}(s)\,\mathrm{d}s$$

for $t \geqslant t_3$. This implies that

$$L \geqslant x(t_3 + kr) \geqslant x(t_3) + J\sum_{i=1}^{k} \int_{t_3+ir}^{\infty} \overline{P}(s)\,\mathrm{d}s, \tag{2.10}$$

for $k = 1, 2, \ldots$ Letting $k \to \infty$ in (2.10), we obtain

$$\sum_{i=1}^{\infty} \int_{t_3+ir}^{\infty} \overline{P}(s)\,\mathrm{d}s < \infty,$$

which is equivalent to

$$\int^{\infty} s\overline{P}(s)\,\mathrm{d}s < \infty$$

by Lemma 1.1. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 2.7. *Assume that* (H1)–(H3) *and* (H5) *hold. Then* (2.1) *has a bounded positive solution if and only if* (H4) *holds.*

3. Unbounded solutions for (1.1) with $c = 1$

Definition 3.1. A solution $x(t)$ of (2.1) is called a positive (negative) A-type solution if it can be expressed in the form

$$x(t) = \alpha t + \beta(t), \tag{3.1}$$

where $\alpha > 0$ ($\alpha < 0$) is a constant, $\beta\colon [t_x, \infty) \to \mathbb{R}$ is a bounded continuous function, and $t_x > 0$.

Theorem 3.2. *Assume that* (H1)–(H3) *hold,*

$$\int^{\infty} t^2 \overline{P}(t)\,\mathrm{d}t < \infty, \tag{H6}$$

and

$$\int^{\infty} tQ(t)\,\mathrm{d}t < \infty. \tag{H7}$$

Then (2.1) *has a positive A-type solution.*

P r o o f. Choose T sufficiently large such that

$$\sum_{i=0}^{\infty} \int_{T+ir}^{\infty} \overline{P}(t)(t+1)\,\mathrm{d}t + n\int_{T-\theta}^{\infty} Q(t)(t+1)\,\mathrm{d}t < 1,$$

where $n = [\![\frac{\theta-\delta}{r}]\!] + 2$. Set

$$H(t) = \begin{cases} \displaystyle\int_t^{\infty} \overline{P}(s)(1+s)\,\mathrm{d}s + \int_{t-\theta+\delta}^{t} Q(s)(1+s)\,\mathrm{d}s, & t \geqslant T, \\ (t-T+r)H(T)/r, & T-r \leqslant t \leqslant T, \\ 0, & t \leqslant T-r, \end{cases}$$

and observe that $H \in C(\mathbb{R}, \mathbb{R}^+)$. Define

$$y(t) = \sum_{i=0}^{\infty} H(t-ir),\ \ t \geqslant T.$$

It is obvious that $y \in C\big([T,\infty), \mathbb{R}^+\big)$ with $y(t) - y(t-r) = H(t)$ and $0 < y(t) < 1$ for $t \geqslant T$. Define the set X by

$$X = \big\{x \in C\big([T,\infty), \mathbb{R}\big)\colon\ 0 \leqslant x(t) \leqslant y(t), t \geqslant T\big\}$$

and an operator S on X by

$$(Sx)(t) = \begin{cases} x(t-r) + \displaystyle\int_{t-\theta+\delta}^{t} Q(s)\bigl(x(s-\delta)+s-\delta\bigr)\,\mathrm{d}s & \\ \qquad + \displaystyle\int_{t}^{\infty} \overline{P}(s)\bigl(x(s-\theta)+s-\theta\bigr)\,\mathrm{d}s, & t \geqslant T+m, \\ (Sx)(T+m)\frac{ty(t)}{(T+m)y(T+m)} + y(t)\bigl(1-\frac{t}{T+m}\bigr), & t \in [T, T+m], \end{cases}$$

where $m = \max\{\theta, r\}$. Clearly, $SX \subset X$.

Define a sequence of functions $\{x_k(t)\}_{k=0}^{\infty}$ as follows:

$$x_0(t) = y(t), \quad t \geqslant T,$$

$$x_k(t) = (Sx_{k-1})(t), \quad t \geqslant T, \quad k = 1, 2, \ldots$$

By induction, we have

$$0 < x_k(t) \leqslant x_{k-1}(t) \leqslant y(t), \quad t \geqslant T, \quad k = 1, 2, \ldots$$

Then there exists a function $u \in X$ such that $\lim\limits_{k\to\infty} x_k(t) = u(t)$, for $t \geqslant T$. It is obvious that $u(t) > 0$ on $[T, \infty)$. By the Lebesgue dominated convergence theorem, we have $u = Su$. It is easy to see that $x(t) = t + u(t)$ is a positive A-type solution of (2.1), and this completes the proof. □

Similar to Theorem 2.6 and Corollary 2.7, we have the following results.

Theorem 3.3. *Assume that* (H1)–(H3) *and* (H7) *hold. If* (2.1) *has a positive A-type solution, then* (H6) *holds.*

Corollary 3.4. *Assume that* (H1)–(H3) *and* (H7) *hold. Then* (2.1) *has a positive A-type solution if and only if* (H6) *holds.*

4. Bounded solutions of (1.1) with $c \in (0,1)$

In this section, we consider the equation

$$\bigl(x(t) - cx(t-r)\bigr)' + P(t)x(t-\theta) - Q(t)x(t-\delta) = 0, \tag{4.1}$$

where $c \in (0,1)$. Our first result in this section is analogous to Theorem 2.1. Here, condition (H4) gets replaced by (H8) below.

Theorem 4.1. *Suppose that $c \in (0,1)$, conditions* (H1)–(H3) *and* (H5) *hold, and*

$$\text{(H8)}\qquad \sum_{j=0}^{\infty}\int_{T+jr}^{\infty} c^{\frac{s-T-jr}{r}}\,\overline{P}(s)\,\mathrm{d}s < \infty \quad \text{for some } T > 0.$$

Then (4.1) *has a bounded positive solution, and for any continuous periodic oscillatory function $\omega(t)$ with period r,* (4.1) *has a bounded oscillatory solution*

$$x(t) = c^{\frac{t}{r}}\big(\omega(t) + R(t)\big), \tag{4.3}$$

where $\big|R(t)\big| < \alpha M$ and $\alpha \in (0,1)$.

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.2. *Under the hypotheses of Theorem* 4.1*, the equations*

$$\begin{aligned}\big(x(t) - cx(t-r)\big)' &+ P(t)\Big(x(t-\theta) + \big(2\overline{M} + \omega(t-\theta)\big)c^{\frac{t-\theta}{r}}\Big)\\ &- Q(t)\Big(x(t-\delta) + \big(2\overline{M} + \omega(t-\delta)\big)c^{\frac{t-\delta}{r}}\Big) = 0\end{aligned} \tag{4.4}$$

and

$$\big(x(t) - cx(t-r)\big)' + P(t)\Big(x(t-\theta) + 2\overline{M}c^{\frac{t-\theta}{r}}\Big) - Q(t)\Big(x(t-\delta) + 2\overline{M}c^{\frac{t-\delta}{r}}\Big) = 0 \tag{4.5}$$

have bounded positive solutions $u_1(t)$ and $u(t)$, respectively, such that

$$\big|u(t)\big| \leqslant \tfrac{1}{2}\alpha M c^{\frac{t}{r}} \quad \text{and} \quad \big|u_1(t)\big| \leqslant \tfrac{1}{2}\alpha M c^{\frac{t}{r}}.$$

Proof. We give only the outline of the proof for the case of (4.5). Consider the integral equation

$$x(t) = cx(t-r) + \int_{t-\theta}^{t-\delta} Q(s+\delta)\big(x(s) + 2\overline{M}c^{\frac{s}{r}}\big)\,\mathrm{d}s + \int_{t}^{\infty} \overline{P}(s)\big(x(s-\theta) + 2\overline{M}c^{\frac{s-\theta}{r}}\big)\,\mathrm{d}s. \tag{4.6}$$

Letting $z(t) = x(t)c^{-\frac{t}{r}}$, (4.6) becomes

$$z(t) = z(t-r) + \int_{t-\theta}^{t-\delta} Q(s+\delta)\big(z(s) + 2\overline{M}c^{\frac{s-t}{r}}\big)\,\mathrm{d}s + \int_{t}^{\infty} \overline{P}(s)\big(z(s-\theta) + 2\overline{M}\big)c^{\frac{s-t-\theta}{r}}\,\mathrm{d}s. \tag{4.7}$$

To complete the proof of the lemma, it is sufficient to prove that (4.7) has a bounded positive solution $z(t)$ such that $|z(t)| < \frac{\alpha}{2}M$, for $t \geqslant T$, where T is sufficiently large.

If we choose T large enough that

$$\sum_{j=0}^{\infty}\int_{T+jr}^{\infty} c^{\frac{s-T-jr}{r}}\,\overline{P}(s)\,\mathrm{d}s + n\int_{T}^{\infty} Q(s)\,\mathrm{d}s < \frac{\alpha M}{16\overline{M}E},$$

where $E = c^{-\frac{\theta}{r}} > 1$, then the remainder of the proof is similar to the proof of Lemma 2.2 and will be omitted. $\square$

In view of Lemma 4.2, we can prove Theorem 4.1 using a technique similar to that used to prove Theorem 2.1; we omit the details here. Next, we give an explicit condition to guarantee that (H8) holds.

Corollary 4.3. *If, in addition to* (H1)–(H3) *and* (H5), *we have*

$$\int^{\infty} \overline{P}(s)\,\mathrm{d}s < \infty, \tag{H9}$$

then the conclusion of Theorem 4.1 *holds.*

Proof. It suffices to show that (H9) implies (H8). Set $j = [\![\frac{t-T}{r}]\!]$; then $t - r \leqslant T + jr \leqslant t$ and $T + jr \leqslant t \leqslant T + (j+1)r$. Let

$$I = \sum_{j=0}^{\infty} \int_{T+jr}^{\infty} c^{\frac{s-T-jr}{r}} \overline{P}(s)\,\mathrm{d}s.$$

Then

$$\begin{aligned}
I &\leqslant \frac{1}{r} \sum_{j=0}^{\infty} \int_{T+jr}^{T+(j+1)r} \mathrm{d}t \int_{T+jr}^{\infty} c^{\frac{s-T-jr}{r}} \overline{P}(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{cr} \sum_{j=0}^{\infty} \int_{T+jr}^{T+(j+1)r} \mathrm{d}t \int_{t-r}^{\infty} c^{\frac{s-t+r}{r}} \overline{P}(s)\,\mathrm{d}s \;=\; \frac{1}{cr} \int_{T}^{\infty} \mathrm{d}t \int_{t-r}^{\infty} c^{\frac{s-t+r}{r}} \overline{P}(s)\,\mathrm{d}s \\
&= \frac{1}{cr} \int_{T-r}^{\infty} \mathrm{d}t \int_{t}^{\infty} c^{\frac{s-t}{r}} \overline{P}(s)\,\mathrm{d}s \;=\; \frac{1}{cr} \int_{T-r}^{\infty} \overline{P}(s)\,\mathrm{d}s \int_{T-r}^{s} c^{\frac{s-t}{r}}\,\mathrm{d}t \\
&\leqslant \frac{1}{cr} \int_{T-r}^{\infty} \overline{P}(s)\,\mathrm{d}s \int_{0}^{\infty} c^{\frac{u}{r}}\,\mathrm{d}u \;=\; K \int_{T-r}^{\infty} \overline{P}(s)\,\mathrm{d}s,
\end{aligned}$$

where $K = \dfrac{1}{cr} \displaystyle\int_0^{\infty} c^{\frac{u}{r}}\,\mathrm{d}u$. Therefore, (H9) implies (H8), and the proof is complete. □

5. Solutions of (1.2) with $c \in (0,1]$

In this section, we first consider (1.2) with $c = 1$, namely,

$$\big(x(t) - x(t-r)\big)' = P(t)x(t-\theta) - Q(t)x(t-\delta). \tag{5.1}$$

Analogous to Theorem 2.1, we have the following result.

Theorem 5.1. *Suppose conditions* (H1)–(H5) *hold. Then* (5.1) *has a bounded positive solution, and for any continuous periodic oscillatory function* $\omega(t)$ *with period* r*, there is a bounded oscillatory solution* $x(t)$ *such that*

$$x(t) = \omega(t) + R(t) \tag{5.2}$$

for $t > T$*, where* $R(t)$ *is a continuous real function,* $\left|R(t)\right| < \alpha M$*,* $M = \min\left\{\max \omega(t), \max(-\omega(t))\right\}$*,* $\alpha \in (0, 1)$*, and* T *is sufficiently large.*

In order to prove the above theorem, we need the following lemma, which is analogous to Lemma 2.2.

Lemma 5.2. *Under the hypotheses of Theorem* 5.1*, the equations*

$$\big(x(t)-x(t-r)\big)' = P(t)\big(x(t-\theta)+2\overline{M}+\omega(t-\theta)\big)-Q(t)\big(x(t-\delta)+2\overline{M}+\omega(t-\delta)\big) \tag{5.3}$$

and

$$\big(x(t) - x(t - r)\big)' = P(t)\big(x(t - \theta) + 2\overline{M}\big) - Q(t)\big(x(t - \delta) + 2\overline{M}\big) \tag{5.4}$$

have bounded positive solutions $u_1(t)$ *and* $u(t)$*, respectively, such that*

$$\left|u(t)\right| \leqslant \tfrac{1}{2}\alpha M \quad \text{and} \quad \left|u_1(t)\right| \leqslant \tfrac{1}{2}\alpha M$$

for $t \geqslant T$*, where* $\overline{M} = \max\left|\omega(t)\right|$ *and* T *is sufficiently large.*

Proof. We only give a proof for (5.4). Choose T sufficiently large so that (2.5) holds. Define a set X by

$$X = \left\{x \in C\big([T, \infty), \mathbb{R}\big) \colon\ 0 \leqslant x(t) \leqslant \tfrac{1}{4}\alpha M, t \geqslant T\right\}$$

and a sequence of functions $\left\{x_k(t)\right\}_{k=0}^{\infty}$ by

$$x_0(t) = 0, \quad t \geqslant T,$$

$$x_k(t) = \begin{cases} x_{k-1}(t + r) + \displaystyle\int_{t-\theta+\delta+r}^{t+r} Q(s)\big(x_{k-1}(s - \delta) + 2\overline{M}\big)\,\mathrm{d}s & \\ \qquad\qquad + \displaystyle\int_{t+r}^{\infty} \overline{P}(s)\big(x_{k-1}(s - \theta) + 2\overline{M}\big)\,\mathrm{d}s, & t \geqslant T + m, \\ x_k(T + m), & t \in [T, T + m], \end{cases}$$

where $m = \max\left\{0, \theta - r\right\}$, $k = 1, 2, \ldots$ Clearly, $x_1(t) > 0 = x_0(t)$, $t \geqslant T$. By induction, we have

$$x_0(t) < \ldots < x_k(t) < x_{k+1}(t) < \ldots, \quad t \geqslant T, \quad k = 1, 2, \ldots \tag{5.5}$$

It is obvious that $x_0(t) \leqslant \frac{1}{4}\alpha M$ for $t \geqslant T$. Suppose

$$x_k(t) \leqslant \tfrac{1}{4}\alpha M, \quad t \geqslant T, \quad k = 0, 1, \ldots, p-1;$$

we will show that

$$x_p(t) \leqslant \tfrac{1}{4}\alpha M, \quad t \geqslant T.$$

In fact, for $t \geqslant T + m$,

$$\begin{aligned}
x_p(t) &= x_{p-1}(t+r) + \int_{t-\theta+\delta+r}^{t+r} Q(s)\big(x_{p-1}(s-\delta) + 2\overline{M}\big)\,\mathrm{d}s \\
&\quad + \int_{t+r}^{\infty} \overline{P}(s)\big(x_{p-1}(s-\theta) + 2\overline{M}\big)\,\mathrm{d}s \\
&= x_0(t+pr) + \sum_{j=1}^{p} \int_{t-\theta+\delta+jr}^{t+jr} Q(s)\big(x_{p-j}(s-\delta) + 2\overline{M}\big)\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{p} \int_{t+jr}^{\infty} \overline{P}(s)\big(x_{p-j}(s-\theta) + 2\overline{M}\big)\,\mathrm{d}s \\
&\leqslant 4\overline{M}\Big(\sum_{j=1}^{p} \int_{t-\theta+\delta+jr}^{t+jr} Q(s)\,\mathrm{d}s + \sum_{j=1}^{p} \int_{t+jr}^{\infty} \overline{P}(s)\,\mathrm{d}s\Big) \\
&\leqslant \tfrac{1}{4}\alpha M
\end{aligned}$$

by condition (2.5), i.e., $\big\{x_k(t)\big\}_{k=0}^{\infty} \subset X$. In view of (5.5), there exists a function $u \in X$ such that $\lim\limits_{k\to\infty} x_k(t) = u(t)$, for $t \geqslant T$. By the Lebesgue dominated convergence theorem, we have

$$u(t) = \begin{cases} u(t+r) + \displaystyle\int_{t-\theta+\delta+r}^{t+r} Q(s)\big(u(s-\delta) + 2\overline{M}\big)\,\mathrm{d}s & \\ \qquad + \displaystyle\int_{t+r}^{\infty} \overline{P}(s)\big(u(s-\theta) + 2\overline{M}\big)\,\mathrm{d}s, & t \geqslant T + m, \\ u(T+m), & t \in [T, T+m], \end{cases}$$

i.e., $u(t)$ is a solution of (5.4). This completes the proof of the lemma.

In view of Lemma 5.2, we can prove Theorem 5.1 by using an argument similar to the one used to prove Theorem 2.1. We will omit the details. □

Example 5.3. Consider the equation

$$\big(x(t) - x(t-2)\big)' = \frac{\mathrm{e} - 1 + \mathrm{e}^{-t/2}}{2(\mathrm{e}^{t/2} + \mathrm{e})} x(t-2) - \frac{\mathrm{e}^{-t}}{2(1 + \mathrm{e}^{-(t-1)/2})} x(t-1), \quad t \geqslant 0.$$

All the hypotheses of Theorem 5.1 are satisfied, and $x(t) = 1 + \mathrm{e}^{-t/2}$ is a bounded positive solution.

Similar to Theorem 2.6, we have the following result for (5.1). The proof is only slightly different from the proof of Theorem 2.6.

Theorem 5.4. *Assume that* (H1)–(H3) *and* (H5) *hold. If* (5.1) *has a bounded positive solution* $x(t)$ *such that* $\liminf_{t\to\infty} x(t) > 0$, *then* (H4) *holds.*

Corresponding to Theorem 3.2, we have the following result on A-type solutions.

Theorem 5.5. *If* (H1)–(H3), (H6), *and* (H7) *hold, then* (5.1) *has a positive A-type solution.*

Next, we consider the equation

$$\big(x(t) - cx(t-r)\big)' = P(t)x(t-\theta) - Q(t)x(t-\delta). \tag{5.6}$$

For the case where $c \in (0,1)$, we have the following counterpart to Corollary 4.3.

Theorem 5.6. *Suppose that* $c \in (0,1)$ *and* (H1)–(H3), (H5), *and* (H9) *hold. Then* (5.6) *has a bounded positive solution, and for any continuous periodic oscillatory function* $\omega(t)$ *with period* r, (5.6) *has a bounded oscillatory solution*

$$x(t) = c^{\frac{t}{r}}\big(\omega(t) + R(t)\big),$$

where $\big|R(t)\big| < \alpha M$ *and* $\alpha \in (0,1)$.

The proof of Theorem 5.5 is easily modeled after the proofs of Lemma 4.2 and Theorem 4.1 (taking into account the variation in approach used in Lemma 5.2), and then applying the proof of Corollary 4.3 to conclude that (H9) implies (H8).

6. The case $c > 1$

We conclude this paper with results for equations (1.1) and (1.2) in the case $c > 1$. In view of our results in Sections 4 and 5, the proof of the following theorem can easily be constructed.

Theorem 6.1. *Suppose that* $c > 1$ *and conditions* (H1)–(H3), (H5), *and* (H8) *hold. Then* (1.1) *and* (1.2) *each have an unbounded positive solution, and for any continuous periodic oscillatory function* $\omega(t)$ *with period* r, *they have unbounded oscillatory solutions of the form*

$$x(t) = c^{\frac{t}{r}}\big(\omega(t) + R(t)\big),$$

where $\big|R(t)\big| < \alpha M$ *and* $\alpha \in (0,1)$.

Our final result gives an explicit condition to guarantee that (H8) holds in the case $c > 1$.

Corollary 6.2. *Suppose* $c > 1$ *and conditions* (H1)–(H3) *and* (H5) *hold. If*

$$\text{(H10)} \qquad \int^{\infty} c^{\frac{s}{r}} \overline{P}(s) \, \mathrm{d}s < \infty,$$

then the conclusion of Theorem 6.1 *holds.*

Proof. It suffices to prove that (H10) implies (H8). Set $j = [\![\frac{t-T}{r}]\!]$. Then $t - r \leqslant T + jr \leqslant t$ and $T + jr \leqslant t \leqslant T + (j+1)r$. For

$$I = \sum_{j=0}^{\infty} \int_{T+jr}^{\infty} c^{\frac{s-T-jr}{r}} \overline{P}(s) \, \mathrm{d}s,$$

we have

$$\begin{aligned}
I &\leqslant \frac{1}{r} \sum_{j=0}^{\infty} \int_{T+jr}^{T+(j+1)r} \mathrm{d}t \int_{T+jr}^{\infty} c^{\frac{s-t+r}{r}} \overline{P}(s) \, \mathrm{d}s \\
&= \frac{1}{r} \int_{T}^{\infty} \mathrm{d}t \int_{t-r}^{\infty} c^{\frac{s-t+r}{r}} \overline{P}(s) \, \mathrm{d}s = \frac{1}{r} \int_{T-r}^{\infty} \mathrm{d}t \int_{t}^{\infty} c^{\frac{s-t}{r}} \overline{P}(s) \, \mathrm{d}s \\
&= \frac{1}{r} \int_{T-r}^{\infty} \overline{P}(s) \, \mathrm{d}s \int_{T-r}^{s} c^{\frac{s-t}{r}} \, \mathrm{d}t = \frac{1}{r} \int_{T-r}^{\infty} \overline{P}(s) \, \mathrm{d}s \int_{0}^{s-T+r} c^{\frac{u}{r}} \, \mathrm{d}u \\
&\leqslant \frac{1}{\ln c} \int_{T-r}^{\infty} c^{\frac{s-T+r}{r}} \overline{P}(s) \, \mathrm{d}s, \\
&= K \int_{T-r}^{\infty} c^{\frac{s}{r}} \overline{P}(s) \, \mathrm{d}s,
\end{aligned}$$

where $K = \left(\ln c \cdot c^{\frac{T-r}{r}}\right)^{-1}$. Therefore (H10) implies (H8), and the proof is complete. $\square$

References

[1] *D. D. Bainov, D. P. Mishev*: Oscillation Theory for Neutral Differential Equations with Delay. Adam Hilger, Philadelphia, 1991.

[2] *Q. Chuanxi, G. Ladas*: Linearized oscillation for equations with positive, negative coefficients. Hiroshima Math. J. *20* (1990), 331–340.

[3] *L. H. Erbe, Q. Kong, B. G. Zhang*: Oscillation Theory for Functional Differential Equations. Marcel Dekker, New York, 1994.

[4] *I. Györi, G. Ladas*: Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press, Oxford, 1991.

[5] *J. Jaroš, T. Kusano*: Existence of oscillatory solutions for functional differential equations of neutral type. Acta Math. Univ. Comenian. *60* (1991), 185–194.

[6] *G. Ladas, C. Qian*: Oscillation in differential equations with positive, negative coefficients. Canad. Math. Bull. *33* (1990), 442–451.
[7] *B. S. Lalli, B. G. Zhang*: Oscillation of first order neutral differential equations. Appl. Anal. *39* (1990), 265–274.
[8] *Yu Jianshe, Wang Zhicheng*: Some further results on oscillation of neutral differential equations. Bull. Austral. Math. Soc. *46* (1992), 147–154.
[9] *Yu Jianshe*: On the neutral delay differential equation with positive, negative coefficients. Acta. Math. Sinica *34* (1991), 517–523.
[10] *B. G. Zhang, J. S. Yu*: The existence of positive solutions of neutral differential equations. Scientia Sinica *8* (1992), 785–790.

Authors' addresses: *John R. Graef*, *Bo Yang*, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, U. S. A., *B. G. Zhang*, Department of Applied Mathematics, Ocean University of Qingdao, Qingdao 266003, China.