PAIRWISE FUZZY CONNECTEDNESS BETWEEN FUZZY SETS

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Abstract. In this paper the concept of fuzzy connectedness between fuzzy sets [6] is generalized to fuzzy bitopological spaces and some of its properties are studied.

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1. Preliminaries

Let X and Y be non-empty sets. A fuzzy set λ in X is a mapping from X to the unit interval [0,1]. The null fuzzy set 0 (resp. the whole fuzzy set 1) is the mapping from X to the unit interval [0,1] which takes the only value 0 (resp. 1) in that interval. The basic operations on fuzzy sets are defined as follows:

$$\bigcup_{\alpha\in\Lambda} \lambda_\alpha(x) = \sup_{\alpha\in\Lambda} \lambda_\alpha(x),\ \forall x \in X,$$
$$\bigcap_{\alpha\in\Lambda} \lambda_\alpha(x) = \inf_{\alpha\in\Lambda} \lambda_\alpha(x),\ \forall x \in X,$$
$$1 \setminus \lambda(x) = 1 - \lambda(x),\ \forall x \in X.$$

A fuzzy topology [2] on X is a family τ of fuzzy sets in X which satisfies the following conditions:
(a) $0, 1 \in \tau$,
(b) $\lambda, \mu \in \tau \Rightarrow \lambda \cap \mu \in \tau$,
(c) for each $\alpha \in \Lambda$, $\lambda_\alpha \in \tau \Rightarrow \bigcup\limits_{\alpha\in\Lambda} \lambda_\alpha \in \tau$.

The pair (X, τ) is called a fuzzy topological space and the members of τ are called fuzzy open sets. The complements of the fuzzy open sets are called fuzzy closed sets. The closure denoted by $\mathrm{cl}(\lambda)$ (interior, denoted by $\mathrm{int}(\lambda)$) of a fuzzy set λ of X is the intersection (union) of all fuzzy closed supersets (fuzzy open subsets, respectively) of λ [2]. For a fuzzy set λ of a fuzzy topological space X, $1-\operatorname{int}(\lambda)=\operatorname{cl}(1-\lambda)$ and 1- $\operatorname{cl}(\lambda)=\operatorname{int}(1-\lambda)$. A fuzzy set λ in X is said to be quasi-coincident [8] with a fuzzy set μ in X denoted by $\lambda \mathrm{~q} \mu$ if there exists a point $x \in X$ such that $\lambda(x)+\mu(x)>1$. If λ and μ are two fuzzy sets of X, then $\lambda \leqslant \mu$ if and only if λ and $1-\mu$ are not quasi-coincident. A fuzzy topological space (X, τ) is said to be fuzzy connected [3] if there is no proper fuzzy set in X which is both fuzzy open and fuzzy closed. A fuzzy topological space (X, τ) is said to be fuzzy connected [6] between its subsets λ and μ if and only if there is no fuzzy closed fuzzy open set δ in X such that $\lambda \leqslant \delta$ and $\neg(\delta \mathrm{~q} \mu)$.

A system (X, τ_1, τ_2) consisting of a set X with two topologies τ_1 and τ_2 on X is called a fuzzy bitopological space [5]. A fuzzy bitopological space (X, τ_1, τ_2) is said to be pairwise fuzzy connected [8] if it has no proper fuzzy set which is both τ_i-fuzzy open and τ_j-fuzzy closed, $i, j=1,2$, $i \neq j$. The purpose of this paper is to introduce and study the concept of pairwise fuzzy connectedness between fuzzy sets in fuzzy bitopological spaces.

Throughout this paper $i, j=1,2$ where $i \neq j$. If P is any fuzzy topological property then τ_i-P and τ_j-P denote the property P with respect to the fuzzy topology τ_i and τ_j, respectively and χ_A denotes the characteristic function of a subset A of X.

2. Pairwise fuzzy connectedness between fuzzy sets

Definition 2.1. A fuzzy bitopological space (X, τ_1, τ_2) is said to be pairwise fuzzy connected between fuzzy sets λ and μ if there is no (i, j)-fuzzy clopen (τ_i-fuzzy closed and τ_j-fuzzy open) set δ in X such that $\lambda \leqslant \delta$ and $\neg(\delta \mathrm{~q} \mu)$.

Remark 2.1. Pairwise fuzzy connectedness between fuzzy sets λ and μ is not equal to the fuzzy connectedness of (X, τ_1) and (X, τ_2) between λ and μ.

Example 2.1. Let $X=\{a, b\}$ and let λ, μ, ν_1 and ν_2 be fuzzy sets on X defined as follows:

$$
\begin{aligned}
\lambda(a)=0.2, &\quad \lambda(b)=0.3, \\
\mu(a)=0.5, &\quad \mu(b)=0.4, \\
\nu_1(a)=0.3, &\quad \nu_1(b)=0.4, \\
\nu_2(a)=0.7, &\quad \nu_2(b)=0.6 .
\end{aligned}
$$

Let $\tau_1=\{0, \nu_1, 1\}$ and $\tau_2=\{0, \nu_2, 1\}$ be fuzzy topologies on X. Then (X, τ_1) and (X, τ_2) are fuzzy connected between the fuzzy sets λ and μ but (X, τ_1, τ_2) is not pairwise fuzzy connected between λ and μ.

Example 2.2. Let $X = \{a, b\}$. Let fuzzy sets ν_1, ν_2, δ_1, δ_2, λ and μ be defined as follows:

$$\begin{aligned} \nu_1(a) &= 0.5, & \nu_1(b) &= 0.6, \\ \nu_2(a) &= 0.5, & \nu_2(b) &= 0.7, \\ \delta_1(a) &= 0.5, & \delta_1(b) &= 0.4, \\ \delta_2(a) &= 0.5, & \delta_2(b) &= 0.3, \\ \lambda(a) &= 0.5, & \lambda(b) &= 0.3, \\ \mu(a) &= 0.5, & \mu(b) &= 0.2. \end{aligned}$$

Let $\tau_1 = \{0, \nu_1, \delta_1, 1\}$ and $\tau_2 = \{0, \nu_2, \delta_2, 1\}$ be fuzzy topologies on X. Then the fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between λ and μ, but neither (X, τ_1) nor (X, τ_2) are fuzzy connected between λ and μ.

Theorem 2.1. *A fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between fuzzy sets λ and μ if and only if there is no (i, j)-fuzzy clopen set δ in X such that $\lambda \leqslant \delta \leqslant 1 - \mu$.*

Proof. Obvious. □

Theorem 2.2. *If a fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between fuzzy sets λ and μ then λ and μ are non-empty.*

Proof. Evident. □

Theorem 2.3. *If a fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between fuzzy sets λ and μ and if $\lambda \leqslant \lambda_1$ and $\mu \leqslant \mu_1$ then (X, τ_1, τ_2) is pairwise fuzzy connected between λ_1 and μ_1.*

Proof. Suppose the fuzzy bitopological space (X, τ_1, τ_2) is not pairwise fuzzy connected between the fuzzy sets λ_1 and μ_1. Then there is an (i, j)-fuzzy clopen set δ in X such that $\lambda_1 \leqslant \delta$ and $\neg(\delta \text{ q } \mu_1)$. Clearly $\lambda \leqslant \delta$. Now we claim that $\neg(\delta \text{ q } \mu)$. If $(\delta \text{ q } \mu)$ then there exists a point $x \in X$ such that $\delta(x) + \mu(x) > 1$. Therefore $\delta(x) + \mu_1(x) > \delta(x) + \mu(x) > 1$ and $\delta \text{ q } \mu_1$, a contradiction. Consequently, (X, τ_1, τ_2) is not pairwise fuzzy connected between λ and μ. □

Theorem 2.4. *A fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between λ and μ if and only if it is pairwise fuzzy connected between τ_i-$\mathrm{cl}(\lambda)$ and τ_j-$\mathrm{cl}(\mu)$.*

Proof. Necessity: It follows by using Theorem (2.3).

Sufficiency: Suppose the fuzzy bitopological space (X, τ_1, τ_2) is not pairwise fuzzy connected between λ and μ. Then there is an (i, j)-fuzzy clopen set δ in X such that $\lambda \leqslant \delta$ and $\neg(\delta \mathrel{q} \mu)$. Since $\lambda \leqslant \delta$, $\tau_i\text{-}\operatorname{cl}(\lambda) \leqslant \tau_i\text{-}\operatorname{cl}(\delta) < \delta$ because δ is τ_i-fuzzy closed. Now,

$$\begin{aligned}\neg(\delta \mathrel{q} \mu) &\Rightarrow \delta \leqslant 1 - \mu \\ &\Rightarrow \delta \leqslant \tau_j\text{-}\operatorname{int}(1 - \mu) \\ &\Rightarrow \delta \leqslant 1 - \tau_j\text{-}\operatorname{cl}(\mu) \\ &\Rightarrow \neg\bigl(\delta \mathrel{q} \tau_j\text{-}\operatorname{cl}(\mu)\bigr).\end{aligned}$$

Hence X is not pairwise fuzzy connected between $\tau_i\text{-}\operatorname{cl}(\lambda)$ and $\tau_j\text{-}\operatorname{cl}(\mu)$, a contradiction. $\square$

Theorem 2.5. *Let (X, τ_1, τ_2) be a fuzzy bitopological space and let λ and μ be two fuzzy sets in X. If $\lambda \mathrel{q} \mu$ then (X, τ_1, τ_2) is pairwise fuzzy connected between λ and μ.*

Proof. If δ is any (i, j)-fuzzy clopen set in X such that $\lambda \leqslant \delta$ then $\lambda \mathrel{q} \mu \Rightarrow \delta \mathrel{q} \mu$. $\square$

Remark 2.2. The converse of Theorem (2.5) may not be true as is shown by the next example.

Example 2.3. Let $X = \{a, b\}$ and let the fuzzy sets λ, μ, δ_1 and δ_2 be defined as follows:

$$\begin{aligned}\lambda(a) &= 0.5, \quad \lambda(b) = 0.4, \\ \mu(a) &= 0.3, \quad \mu(b) = 0.5, \\ \delta_1(a) &= 0.2, \quad \delta_1(b) = 0.9, \\ \delta_2(a) &= 0.8, \quad \delta_2(b) = 0.1.\end{aligned}$$

Let $\tau_i = \{0, \delta_1, 1\}$ and $\tau_2 = \{0, \delta_2, 1\}$ be fuzzy topologies on X. Then the fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between λ and μ but $\neg(\lambda \mathrel{q} \mu)$.

Theorem 2.6. *If a fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected neither between λ and μ, nor between λ and μ_1, then it is not pairwise fuzzy connected between λ and $\mu_0 \cup \mu_1$.*

Proof. Since X is pairwise fuzzy connected neither between λ and μ_0 nor between λ and μ_1, there exists (i, j)-fuzzy clopen fuzzy sets δ_0 and δ_1 in (X, τ_1, τ_2) such that $\lambda \leqslant \delta_0$, $\neg(\delta_0 \mathrel{q} \mu_0)$ and $\lambda \leqslant \delta_1$, $\neg(\delta_1 \mathrel{q} \mu_1)$. Put $\delta = \delta_0 \cap \delta_1$. Then δ is

(i,j)-fuzzy clopen and $\lambda \leqslant \delta$. Now we claim that $\neg\big(\delta \text{ q } (\mu_0 \cup \mu_1)\big)$. If $\delta \text{ q } (\mu_0 \cup \mu_1)$ then there exists a point $x \in X$ such that $\delta(x) + (\mu_0 \cup \mu_1)(x) > 1$. This implies that $\delta \text{ q } \mu_0$ or $\delta \text{ q } \mu_1$, a contradiction. Hence X is not pairwise fuzzy connected between λ and $\mu_0 \cup \mu_1$. $\square$

Theorem 2.7. *A fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected if and only if it is pairwise fuzzy connected between every pair of its non-empty fuzzy subsets.*

Proof. Necessity: Let λ and μ be any pair of non-empty fuzzy subsets of X. Suppose (X, τ_1, τ_2) is not pairwise fuzzy connected between λ and μ. Then there is an (i,j)-fuzzy clopen set δ in X such that $\lambda \leqslant \delta$ and $\neg(\delta \text{ q } \mu)$. Since λ and μ are non-empty, it follows that δ is a non-empty proper (i,j)-fuzzy clopen subset of X. Hence (X, τ_1, τ_2) is not pairwise fuzzy connected.

Sufficiency: Suppose (X, τ_1, τ_2) is not pairwise fuzzy connected. Then there exists a non-empty proper (i,j)-fuzzy clopen subset δ of X. Consequently, (X, τ_1, τ_2) is not pairwise fuzzy connected between δ and $1-\delta$, a contradiction. $\square$

Remark 2.3. If fuzzy bitopological space (X, τ_1, τ_2) is pairwise fuzzy connected between a pair of its subsets then it need not necessarily hold that (X, τ_1, τ_2) is pairwise fuzzy between every pair of its subsets and so it is not necessarily pairwise fuzzy connected as is shown by the next example.

Example 2.4. Let $X = \{a, b\}$ and let δ_1, δ_2, λ_1, λ_2, μ_1 and μ_2 be defined as follows.

$$\begin{aligned} \delta_1(a) &= 0.4, & \delta_1(b) &= 0.6, \\ \delta_2(a) &= 0.6, & \delta_2(b) &= 0.4, \\ \lambda_1(a) &= 0.7, & \lambda_1(b) &= 0.8, \\ \lambda_2(a) &= 0.3, & \lambda_2(b) &= 0.2, \\ \mu_1(a) &= 0.8, & \mu_1(b) &= 0.7, \\ \mu_2(a) &= 0.2, & \mu_2(b) &= 0.3. \end{aligned}$$

Let $\tau_1 = \{0, \delta_1, 1\}$ and $\tau_2 = \{0, \delta_2, 1\}$ be two fuzzy topologies on X. Then (X, τ_1, τ_2) is pairwise fuzzy connected between λ, and μ, but it is not pairwise fuzzy connected between λ_2 and μ_2. Also (X, τ_1, τ_2) is not pairwise fuzzy connected.

Theorem 2.8. *Let $\big(Y, (\tau_1)_Y, (\tau_2)_Y\big)$ be a subspace of a fuzzy bitopological space (X, τ_1, τ_2) and let λ, μ be fuzzy sets of Y. If $\big(Y, (\tau_1)_Y, (\tau_2)_Y\big)$ is pairwise fuzzy connected between λ and μ then (X, τ_1, τ_2) is also pairwise fuzzy connected between λ and μ.*

Proof. Evident. $\square$

Theorem 2.9. *Let $\left(Y,(\tau_1)_Y,(\tau_2)_Y\right)$ be a subspace of a fuzzy bitopological space (X,τ_1,τ_2) and let λ, μ be fuzzy sets of Y. If (X,τ_1,τ_2) is pairwise fuzzy connected between λ and μ and χ_Y is bifuzzy clopen in (X,τ_1,τ_2) then $\left(Y,(\tau_1)_Y,(\tau_2)_Y\right)$ is pairwise fuzzy connected between λ and μ.*

Proof. Suppose $\left(Y,(\tau_1)_Y,(\tau_2)_Y\right)$ is not pairwise fuzzy connected between λ and μ then there exists an (i,j)-fuzzy clopen set δ in X such that $\lambda \leqslant \delta$ and $\neg(\lambda \mathrm{~q~} \delta)$. Since χ_Y is bifuzzy open and bifuzzy closed in (X,τ_1,τ_2), δ is (i,j)-fuzzy clopen in (X,τ_1,τ_2). Therefore (X,τ_1,τ_2) is not pairwise fuzzy connected between λ and μ, which is a contradiction. □

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