THE CROSS-RATIO IN HJELMSLEV PLANES

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Abstract. The cross-ratio in Hjelmslev planes is defined. The cross-ratio in the Hjelmslev plane $H(R)$ is independent of the choice of a coordinate system on a line.

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1. Introduction

A special local ring is a finite commutative local ring R the ideal I of divisors of zero of which is principal. Suppose that R is not a field and that the characteristic of R is odd. Denote the factor ring R/I by the symbol $\overline{R}$. Further denote the set of all regular elements of R by the symbol R^*, thus $R^* = R - I$.

Definition 1.1. A projective Hjelmslev plane (we will denote it by $H(R)$) over R is an incidence structure $H(R) = (\mathcal{B}; \mathcal{P}; \mathcal{I})$ defined in the following way:

– the elements of $\mathcal{B}$—the points of $H(R)$ are classes of ordered triples $(\lambda x_1; \lambda x_2; \lambda x_3)$ where $\lambda \in R^*$, $x_1, x_2, x_3 \in R$ and at least one x_i is regular;
– the elements of $\mathcal{P}$—the lines of $H(R)$ are classes of ordered triples $(\alpha a_1; \alpha a_2; \alpha a_3)$ where $\alpha \in R^*$, $a_1, a_2, a_3 \in R$ and at least one a_i is regular.

A point $X = [x_1; x_2; x_3]$ is incident to the line $a = [a_1; a_2; a_3]$ if and only if

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0. \tag{1.1}$$

Remark 1.1. The canonical homomorphism $\Phi\colon R \to R/I = \overline{R}$ induces a homomorphism of $H(R)$ onto the projective plane $\pi(\overline{R})$.

We will call the points $X, Y \in H(R)$ *neighbouring* if $\overline{X} = \overline{Y}$ where $\Phi(X) = \overline{X}$, $\Phi(Y) = \overline{Y}$. Similarly we will call points $X, Y \in H(R)$ *substantially different* if $\overline{X} \neq \overline{Y}$. Two lines are neighbouring if there are points $A_1, A_2 \in \mathcal{B}$, $A_1 \neq A_2$ such that $A_1 \mathcal{I} a, b$ and $A_2 \mathcal{I} a, b$. Let X be a subset of the R-modul M and let $j\colon X \to M$ be an insertion of the subset X into M. Then $M(R)$ is called the free modul over X if for an arbitrary function $f\colon X \to A$ into the R-modul A there is exactly one linear mapping $t\colon M(R) \to A$ such that $t \circ j = f$.

Remark 1.2. The analytic model of the Hjelmslev plane, introduced by definition 1.1 is really a free modul over R with a factorization defined in the following way: triples $(x_1; x_2; x_3)$ and $(x_1'; x_2'; x_3')$ are identical if there is $\lambda \in R^*$ such that $x_i' = \lambda x_i$ for $i = 1, 2, 3$ and we do not consider the zero triple.

2. The construction and proof of theorem

Definition 2.1. A coordinate system in $H(R)$ is an ordered quadruple of points E_1, E_2, E_3, E_4 such that the points $\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_4$ generate a coordinate system in $\pi(\overline{R})$.

If a point $X = [x_1; x_2; x_3]$ is given by the vector $x = (x_1; x_2; x_3)$, we write $X = \langle x \rangle$.

Lemma 2.1. *Let $M(R)$ be a free modul over R and let e_1, e_2, e_3 be a basis of $M(R)$. Then the points $E_1 = \langle e_1 \rangle$, $E_2 = \langle e_2 \rangle$, $E_3 = \langle e_3 \rangle$, $E_4 = \langle e_1 + e_2 + e_3 \rangle$ generate the coordinate system in the Hjelmslev plane $H(R)$ corresponding to the modul $M(R)$.*

Proof. It is necessary to prove that the points $\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_4$ generate a coordinate system in $\pi(\overline{R})$. Obviously $\overline{e}_1, \overline{e}_2, \overline{e}_3$ form a basis of a vector space over $\overline{R}$ and thus the vectors $\overline{e}_1, \overline{e}_2, \overline{e}_3$ are linearly independent. It follows that the points $\overline{E}_1 = \langle \overline{e}_1 \rangle$, $\overline{E}_2 = \langle \overline{e}_2 \rangle$, $\overline{E}_3 = \langle \overline{e}_3 \rangle$ and $\overline{E}_4 = \langle \overline{e}_1 + \overline{e}_2 + \overline{e}_3 \rangle$ are not on a unique line. $\square$

Conversely, we have

Lemma 2.2. *Let E_1, E_2, E_3, E_4 be a coordinate system in $H(R)$. Then there is a basis of the modul $M(R)$ such that $\langle e_1 \rangle = E_1$, $\langle e_2 \rangle = E_2$, $\langle e_3 \rangle = E_3$, $\langle e_1 + e_2 + e_3 \rangle = E_4$.*

Proof. Let $E_1 = \langle b_1 \rangle$, $E_2 = \langle b_2 \rangle$, $E_3 = \langle b_3 \rangle$ and $E_4 = \langle b_4 \rangle$. Because $\{b_1, b_2, b_3\}$ is a basis of $M(R)$ the vector b_4 can be expressed in the form

$$b_4 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3.$$

If we denote $e_1 = \beta_1 b_1$, $e_2 = \beta_2 b_2$, $e_3 = \beta_3 b_3$ then e_1, e_2, e_3 are the vectors from the statement of the lemma.

Let E_1, E_2, E_3, E_4 and E'_1, E'_2, E'_3, E'_4 be coordinate systems in $H(R)$. If e_1, e_2, e_3 and e'_1, e'_2, e'_3 are the corresponding bases of the modul $M(R)$ then there is a regular matrix $A = [a_{ij}]$ such that

$$e'_i = \sum_j a_{ij} e_j, \quad i = 1, 2, 3.$$

Let $X_E = [x_1; x_2; x_3]$, $X'_E = [x'_1; x'_2; x'_3]$. Then

$$x = \sum_i x'_i e'_i = \sum_i x'_i \sum_j a_{ij} e_j = \sum_j \Big(\sum_i x'_i a_{ij} \Big) e_j = \sum_j x_j e_j.$$

Comparing the two identities, we get

$$x_j = \sum_i x'_i a_{ij}. \tag{2.1}$$

The relation (2.1) can be written also in the form

$$X_E = X'_E A, \quad X'_E = X_E A^{-1}. \tag{2.2}$$

Let an invertible matrix A and a coordinate system E_1, E_2, E_3, E_4 be given, then points E'_1, E'_2, E'_3, E'_4 generate a coordinate system and the corresponding vectors of the point $X \in H(R)$ satisfy

$$X_E = X'_E A.$$

Let the special local ring R be given. We introduce a set Ω by

$$\Omega \cap R = \emptyset, \quad |\Omega| = |I|. \tag{2.3}$$

Thus there is a bijective mapping ω such that

$$\omega\colon I \to \Omega, \quad \omega\colon i \to \omega_i = \omega(i), \quad i \in I \tag{2.4}$$

where ω_i are "inverse" elements of elements $i \in I$, thus $\omega_i \sim 1/i$. Ω is the set of "infinities" corresponding to singular elements. Define an extension of the canonical homomorphism Φ to the set $R \cup \Omega$, let us put

$$\Phi(\Omega) = \infty. \tag{2.5}$$

Let A, B, E be three substantially different points generating a coordinate system on a line. Then every point X of this line can be expressed uniquely (the single-valuedness guarantees the point E) in the form

$$X = sA + tB \tag{2.6}$$

and hence the point $X = [s;t]$ is determined by the pair $(s;t)$.

On the line with the coordinate system A, B, E let us have points P_1, P_2, P_3, P_4 where $P_i = s_iA + t_iB$ thus $P_i[s_i;t_i]$. □

Definition 2.2. The cross-ratio of an ordered quadruple of points P_1, P_2, P_3, P_4 on a line in $H(R)$, of which at least three are substantially different is an element $(P_1P_2, P_3P_4) \in R \cup \Omega$ which is defined by relations

$$(P_1P_2, P_3P_4) = \frac{\begin{vmatrix} s_1 & t_1 \\ s_3 & t_3 \end{vmatrix} \cdot \begin{vmatrix} s_2 & t_2 \\ s_4 & t_4 \end{vmatrix}}{\begin{vmatrix} s_2 & t_2 \\ s_3 & t_3 \end{vmatrix} \cdot \begin{vmatrix} s_1 & t_1 \\ s_4 & t_4 \end{vmatrix}} \tag{2.7}$$

if points P_1P_4 and P_2P_3 are substantially different,

$$(P_1P_2, P_3P_4) = \omega(P_1P_2, P_3P_4) \tag{2.8}$$

if points P_1, P_4 and P_2, P_3 are neighbouring. Suppose that points P_1, P_3 and P_2, P_4 are substantially different.

Remark. If R is a field, $I = \{0\}$ then Definition 2.2 is the same as the definition of the cross-ratio in a projective plane.

Theorem 2.3. *The cross-ratio introduced by relations* 2.7 *and* 2.8 *is independent of the choice of a coordinate system on the line.*

Proof. Let a line $p \in H(R)$ be given and on this line let us have coordinate systems A, B, E and A', B', E'. Let P_1, P_2, P_3, P_4 be points on the given line p whose the cross-ratio we want to investigate. There is obviously a linear transformation which maps the points A, B to the points A', B' on p. We want to verify that the cross-ratio is independent of the choice of the coordinate points on the line. Thus

$$(P_1P_2, P_3P_4)_{AB} = (P_1P_2, P_3P_4)_{A'B'}.$$

We have

$$\begin{aligned} A' &= a_1A + a_2B \\ B' &= b_1A + b_2B \end{aligned}$$

and thus

$$P_i = s'_i A' + t'_i B'$$

and after a substitution we get

$$P_i = (s'_i a_1 + t'_i b_1)A + (s'_i a_2 + t'_i b_2)B = s_i A + t_i B, \quad i = 1, 2, 3, 4.$$

By direct calculation we obtain $(P_1P_2, P_3P_4)_{AB} = (P_1P_2, P_3P_4)_{A'B'}$ which was to be proved. □

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