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NEARLY EQUIVALENT OPERATORS

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Abstract. The properties of the bounded linear operators T on a Hilbert space which satisfy the condition $TT^* = U^*T^*TU$ where U is unitary, are studied in relation to those of normal, hyponormal, quasinormal and subnormal operators.

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1. Introduction

In this paper we introduce the concept of nearly equivalent operators as follows: Two bounded linear operators T and S on a Hilbert space H are said to be *nearly equivalent* if T^*T and S^*S are similar. We first investigate the properties of such operators and later study an interesting class of bounded linear operators T for which T and T^* are nearly equivalent. Such operators are more general than the normal operators and we call them nearly normal operators.

We obtain some properties of these nearly normal operators and relate them to those of other well-known classes of operators such as hyponormal, quasinormal and subnormal. We show that these operators have a special type of polar decomposition and obtain also a necessary and sufficient condition for a nearly normal operator to be normal.

2. Nearly equivalent operators

Let H be a Hilbert space and $B(H)$ the set of bounded linear operators from H into H.

We introduce the following definition:

Definition 2.1. Let T and $S \in B(H)$. Then S is said to be *nearly equivalent* to T, denoted $S ⊜ T$, if and only if there exists an invertible operator $V \in B(H)$ such that $S^*S = V^{-1}T^*TV$. We denote the set of operators S that are nearly equivalent to T by $\xi(T)$.

Remarks.

1. For any T, if $S \in \xi(T)$, then the positive operator $|S| = (S^*S)^{\frac{1}{2}}$ also belongs to $\xi(T)$.
2. For any T and any isometries P and Q (i.e. $P^*P = Q^*Q = I$), $S \in \xi(T)$ if and only if $PS \in \xi(QT) = \xi(T)$.
3. S is nearly equivalent to T if and only if there exists a unitary operator U such that $S^*S = U^*T^*TU$. For, similar normal operators are actually unitarily equivalent.
4. If S is unitarily equivalent to T (i.e. $S = U^*TU$ for a unitary operator U), then S is nearly equivalent to T; but, if S is nearly equivalent to T then S need not even be similar to T. (Recall that S is similar to T if $S = V^{-1}TV$ for an invertible operator V.)

The first part of the remark is easy to prove. For the second part, consider $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$. Then $S^*S = U^*T^*TU$ if $U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Hence, S is nearly equivalent to T, but S is not similar (and hence not unitarily equivalent) to T.

Indeed, if $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|V| = ad - bc \neq 0$ then $V^{-1}TV = \frac{1}{|V|}\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix} \neq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$ for any a and b.

As another example (now in the infinite dimensional space l^2), consider the unilateral shift operator T (see J. Conway [2], p. 28). Then $T^*T = I$ and hence T is nearly equivalent to I but surely T is not unitarily equivalent to I.

5. If T is compact, all operators S in $\xi(T)$ are compact.
6. $S \in \xi(T)$ if and only if for some unitary operator U, $\|Sx\| = \|TUx\|$ for all $x \in H$. Consequently, if $S \in \xi(T)$, then $\|S\| = \|T\|$.

Proposition 2.2. *For an invertible operator $T \in B(H)$, the following are equivalent:*

(1) $T ⊜ T^{-1}$ *and* $\|T\| \leqslant 1$.

(2) T *is unitary.*

Proof. Since $T^{-1} ⊜ T$, $\|T^{-1}\| = \|T\| \leqslant 1$. Hence, for any $x \in H$, $\|x\| = \|T^{-1}Tx\| \leqslant \|Tx\| \leqslant \|x\|$. Thus, $\|Tx\| = \|x\|$ which implies that $T^*T = I$.

Further, since $\|T^*\| = \|T\| \leqslant 1$ and $\|(T^*)^{-1}\| = \|(T^{-1})^*\| = \|T^{-1}\| \leqslant 1$, we have also $TT^* = I$. Hence T is unitary. The converse is trivial. □

3. Nearly normal operators

We start with the following theorem which is mainly a collection of results from Section 7, Chapter 12 of Dunford-Schwartz [3].

Theorem 3.1. *For a densely defined closed operator from H into H, the following are equivalent:*

(i) $T = US$, *U unitary and S positive self-adjoint.*
(ii) $T = UN$, *U unitary and N normal.*
(iii) $TT^* = UT^*TU^*$, *U unitary.*
(iv) *dim* $N(T) =$ *dim* $N(T^*)$.

Proof. (i) ⇒ (ii), evident.

(ii) ⇒ (i), since every normal operator N is of the form $V|N|$ for a unitary operator V and $|N|$ is the positive square root of N^*N.

(ii) ⇒ (iii), evident. (iii) ⇒ (ii), if we set $N = U^*T$, then N is normal. (iii) ⇔ (iv), a known result. □

Now we introduce the following definitions:

Definition 3.2. T is said to be *nearly normal* if and only if $T^* \in \xi(T)$; S is said to be *nearly hyponormal* if there exists a unitary operator U such that $\|Sx\| \geqslant \|S^*Ux\|$ for every $x \in H$.

Remarks.

1. T is nearly normal if and only if there exists a normal operator N such that $T = UN$ for a unitary operator U, (Theorem 3.1).
2. An operator S is hyponormal if and only if $S^*S \geqslant SS^*$. Then, it is easily seen that an operator T is nearly hyponormal if and only if there exists a hyponormal operator S such that $T = US$ for a unitary operator U.

When Paul R. Halmos [4] introduced the concept of subnormal operators in 1950, he also considered a larger class of operators which were termed later by S. K. Berberian as hyponormal operators. Both of these concepts were inspired by the unilateral shift, a very useful example of a non-normal operator.

3. Clearly, every normal operator is nearly normal and every hyponormal operator is nearly hyponormal; also every nearly normal operator is nearly hyponormal.

Example 1. Nearly normal operator that is not normal.

Let H be of dimension 2 and let $T\colon H \to H$ be defined by the corresponding matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $TT^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $T^*T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. T is not normal. But $TT^* = U^*T^*TU$ if we take $U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Proposition 3.3. *A nearly normal operator T is normal if and only if $T^* = VT$ for a unitary operator V.*

Proof. Any nearly normal operator T is of the form $T = UN$ where U is unitary and N normal. Denote by $|T|$ the positive square root of T^*T. Since $N = U_1|N|$ where U_1 is unitary, $T = U_2|T|$ for a unitary U_2.

Since $TT^* = UT^*TU^*$, $|T^*|^2 = U|T|^2U^*$. This implies that $|T^*|^{2n} = U|T|^{2n}U^*$ and consequently for any polynomial f, $f(|T^*|^2) = Uf(|T|^2)U^*$. From this we conclude that $|T^*| = U|T|U^*$ since the positive square root of a positive operator S is the weak limit of a sequence of polynomials in S.

Finally, if T is nearly normal so is T^* and hence $T^* = U_3|T^*|$ for a unitary operator U_3.

a) Suppose now that T is normal. Then $|T| = |T^*|$. Hence $T^* = U_3|T^*| = U_3|T| = U_3U_2^{-1}T = VT$, V unitary.

b) Conversely, suppose $T^* = VT$.

Then $T^* = VU_2|T|$ and hence $TT^* = (VU_2|T|)^*(VU_2|T|) = |T|^2 = T^*T$, i.e. T is normal.

Thus the proof of the proposition is complete. □

Example 2. Nearly normal operator T for which T and T^* are not similar.

Choose a diagonal operator T with diagonal $\{\alpha_n\}$. Then T^* is a diagonal operator with diagonal $\{\bar{\alpha}_n\}$. Since the spectrum of a diagonal operator is the closure of the set of its diagonal terms, it is obvious that $\{\alpha_n\}$ can be chosen so that the spectrum of T is different from the spectrum of T^*.

Hence T and T^* are not similar but, T being normal, T is nearly normal.

Example 3. Hyponormal operator that is not nearly normal.

Let S be the weighted shift operator with weights $(\frac{1}{2}, \frac{3}{4}, 1, 1, \ldots)$. Then if $x = (x_1, x_2, x_3, \ldots)$, $Sx = (0, \frac{1}{2}x_1, \frac{3}{4}x_2, x_3, \ldots)$ and $S^*x = (\frac{1}{2}x_2, \frac{3}{4}x_3, \ldots)$. Hence $S^*Sx = (\frac{1}{4}x_1, \frac{9}{16}x_2, x_3, \ldots)$ and $SS^*x = (0, \frac{1}{2}x_2, \frac{3}{4}x_3, x_4, \ldots)$. Consequently, S is a hyponormal operator.

Suppose now $SS^* = N^{-1}(S^*S)N$ for an invertible operator N. Then, if $e_1 = (1,0,0,\ldots)$, then $SS^*e_1 = 0$ which would imply that $(S^*S)Ne_1 = N(SS^*)e_1 = 0$. This is a contradiction since S^*S and N being one-to-one, $(S^*S)Ne_1 \neq 0$. Hence S cannot be nearly normal.

Example 4. Any invertible operator T is nearly normal. For, $TT^* = (T^{-1})^{-1}T^*T(T^{-1})$.

Polar decomposition and nearly normal decomposition: It is known that a densely-defined closed operator T can be uniquely represented in the form $T = P|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$ is positive self-adjoint, and P is a partial isometry with initial domain $\overline{R(|T|)}$ and final domain $\overline{R(T)}$. This is called the polar decomposition (p.d.) of T.

Now, a nearly normal operator T is, by definition, of the form $T = U|T|$ where U is unitary. We will call this the *nearly-normal decomposition* (n.n.d.) of T. Remark that an operator T has the n.n.d. if and only if T is nearly normal.

First, we remark that for a nearly normal operator T, its n.n.d. need not be the polar decomposition.

Consider, for example, $T = \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1,x_2) = (0,x_1)$. Then $T^*(x_1,x_2) = (x_2,0)$ and $|T|(x_1,x_2) = (x_1,0)$. Let $P(x_1,x_2) = (0,x_1)$ and $U(x_1,x_2) = (-x_2,x_1)$. Then P is a partial isometry and U is unitary.

Since $T = U|T|$, T is nearly normal, but its polar decomposition is $T = P|T|$.

Proposition 3.4. *For an operator $T \in B(H)$, its polar decomposition is a n.n.d. if and only if T^* is injective.*

Proof. Let $T = P|T|$ be the polar decomposition of the operator T. Then,

$$\begin{aligned} T^* \text{ is injective } &\Leftrightarrow R(T) \text{ is dense (since } N(T^*) = R(T)^{\perp}) \\ &\Leftrightarrow \text{ the final domain of } P \text{ is } H \\ &\Leftrightarrow P \text{ is unitary.} \end{aligned}$$

□

Example 5. Nearly normal operator that is not hyponormal.

Choose any non-hyponormal operator A (see p. 270, V. Istrătescu [5]). Let $\lambda \notin \sigma(A)$. Take $T = A - \lambda I$. Then $T^*T - TT^* = A^*A - AA^*$. Hence T is not hyponormal; but T being invertible, T is nearly normal.

Relations between some classes of operators in a Hilbert space. Denote

N—normal operators,

HN—hyponormal operators,

NN—nearly normal operators,

NH—nearly hyponormal operators.

a) If the Hilbert space is *finite dimensional*, we have the relations $N = HN \subsetneqq NN = NH$.

See Example 1 and note that in a finite dimensional space, every hyponormal operator is normal and consequently every nearly hyponormal operator is nearly normal.

b) If the Hilbert space is *infinite dimensional*, we have $N \subsetneqq HN$ and $NN \subsetneqq NH$. For the former, consider the example of a unilateral shift and for the latter, consider Example 3 (noting that every hyponormal operator is nearly hyponormal).

However, the relation between HN and NN is not inclusive. For, the operator T in Example 3 is in $HN \setminus NN$. On the other hand, any operator in Example 5 is in $NN \setminus HN$.

We introduce the following definition:

Definition 3.5. An operator $T \in B(H)$ is said to be *nearly quasinormal* if and only if T^*T commutes with UT for a unitary operator U.

Proposition 3.6. *For an operator $T \in B(H)$ the following are equivalent:*

1. *T is nearly quasinormal.*
2. *UT is quasinormal for a unitary operator U.*
3. *T is of the form $T = VQ$ where V is unitary and Q is quasinormal.*

Proof. 1) $\Rightarrow$ 2): Since T is nearly quasinormal, there exists a unitary operator U such that $(T^*T)UT = UT(T^*T)$.

Let $S = UT$. Then $S^*S = T^*T$ and consequently $(S^*S)S = S(S^*S)$, i.e. S is quasinormal, a notion defined by A. Brown [1].

2) $\Rightarrow$ 3): Suppose $S = UT$ is quasinormal. Then $T = U^*S$ where U^* is unitary.

3) $\Rightarrow$ 1): Suppose $T = VQ$ where Q is quasinormal and hence satisfies the condition $(Q^*Q)Q = Q(Q^*Q)$.

But $Q^*Q = T^*T$ since V is unitary. Hence $(T^*T)V^*T = V^*T(T^*T)$, i.e. T is nearly quasinormal. □

Remark. In Example 5, we have an operator that is nearly quasinormal but not quasinormal.

Proposition 3.7. *The following relations hold: nearly normal $\subsetneqq$ nearly quasinormal $\subset$ nearly hyponormal.*

Proof. a) Let T be nearly normal, i.e. $TT^* = U^*T^*TU$. Then

$$UT(T^*T) = U(TT^*)T = U(U^*T^*TU)T = (T^*T)UT$$

Hence T is nearly quasinormal.

Note, however, that a unilateral shift is nearly quasinormal but not nearly normal.
b) Suppose now T is a nearly quasinormal operator, i.e. $S = UT$ is quasinormal for a unitary U. Then S is hyponormal, i.e. $S^*S \geqslant SS^*$. This means that $T^*T \geqslant UTT^*U^*$, i.e. $\|Tx\| \geqslant \|T^*U^*x\|$ for all $x \in H$. Hence T is nearly hyponormal. $\square$

We introduce now the following definition:

Definition 3.8. An operator $T \in B(H)$ is said to be *nearly subnormal* if $T = US$ where U is unitary and S is subnormal.

Proposition 3.9. *Let $T \in B(H)$ be a nearly subnormal operator. Then there exists a Hilbert space $K \supset H$, H being a closed subspace of K, and a nearly normal operator $A \in B(K)$ such that $Ax = Tx$ for all $x \in H$.*

Proof. Suppose $T = US$ where U is unitary and S is subnormal in H. Since S is subnormal, there exists a Hilbert space $K \supset H$, H being a closed subspace of K, and a normal operator $N \in B(K)$ such that $Nx = Sx$ for all $x \in H$.

Now, define an operator $V \in B(K)$, $K = H \oplus H^{\perp}$, given by $\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Then $VV^* = V^*V = I$ and hence $A = VN$ is nearly normal in K. Moreover, for any $x \in H$, $Ax = VNx = VSx = USx = Tx$. Hence the theorem. $\square$

4. Nearly equivalence relation and partial isometries

Let F be the family of all partial isometries in $B(H)$. We investigate in this section the nature of the equivalence classes in F determined by the equivalence relation in $B(H)$ as defined in Section 2.

Proposition 4.1. *For $T \in B(H)$, the following are equivalent.*
1. $I \in \xi(T)$,
2. *$\xi(T)$ contains an isometric operator,*
3. *$\xi(T)$ is the family of all isometric operators in $B(H)$.*

Proof.
$1 \Rightarrow 2$ and $3 \Rightarrow 1$ are evident.
$2 \Rightarrow 3$: Let $S \in \xi(T)$ be an isometry. Then, $I = S^*S = U^*T^*TU$ and hence $T^*T = I$.

Consequently, since T is isometric, every element in $\xi(T)$ is isometric; moreover, if Q is any isometric operator, then $Q \in \xi(T)$. $\square$

Corollary. *An isometric operator is nearly normal if and only if it is unitary.*

Proof. Let T be unitary. Then it is normal and hence nearly normal. Conversely, let an isometric operator T be nearly normal, i.e. $T^* \in \xi(T)$. Then by the above proposition, T^* also is an isometry. Hence T is unitary. $\square$

Proposition 4.2. *$S \in B(H)$ is a partial isometry if and only if there exists a hermitian projection T such that $S \in \xi(T)$.*

Proof. Let T be a hermitian projection and $S \in \xi(T)$.

Then $S^*S = U^*T^*TU$ and $P = T^*T$ is a projection and consequently, $(S^*S)^2 = (U^*PU)(U^*PU) = U^*PU = S^*S$.

Hence S^*S is a hermitian projection, which implies that S is a partial isometry.

Conversely, for any given operator S and a unitary operator U, there exists a unique hermitian operator R such that $R^2 = U^*S^*SU$, since U^*S^*SU is positive.

Now, if S is moreover a partial isometry, that is, if $S^*S = Q$ is a projection, we have

$$R^4 = (U^*QU)(U^*QU) = U^*QU = R^2.$$

Set $T = R^2$. Then T is a hermitian projection and $T^*T = U^*S^*SU$, i.e. $S \in \xi(T)$. $\square$

Corollary. *If T is a partial isometry, then all operators in $\xi(T)$ are partial isometries.*

Proposition 4.3. *Let c_1 be the class of all partial isometries and $\tilde{c}_1$ the equivalence classes in c_1 determined by the nearly equivalence relation. Let c_2 be the class of all hermitian projections and $\tilde{c}_2$ the equivalence classes in c_2 determined by the unitary equivalence relation. Then $\tilde{c}_1$ is isomorphic to $\tilde{c}_2$.*

Proof. Let $S, T \in c_1$. Then by Proposition 4.2, there exist $P, Q \in c_2$ such that for some unitary operators U and V,

$$S^*S = U^*P^*PU = U^*PU \quad \text{and} \quad T^*T = V^*Q^*QV = V^*QV.$$

a) Suppose S and T are nearly equivalent, i.e. $S^*S = A^*T^*TA$ for a unitary operator A. Then $P = (VAU^*)^*Q(VAU^*)$. Hence P and Q are unitarily equivalent.

b) On the other hand, if P and Q are unitarily equivalent, $P = B^*QB$ for a unitary operator B. Then $S^*S = (V^*BU)^*T^*T(V^*BU)$. Hence S and T are nearly equivalent. Thus, the isomorphic relation between $\tilde{c}_1$ and $\tilde{c}_2$ is established. $\square$

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