ESTIMATIONS OF COVARIANCE COMPONENTS IN MIXED LINEAR MODELS

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Abstract. An estimation of the linear function of elements of unknown matrices in the covariance components model is presented.

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Introduction

In the mixed linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \tag{1}$$

the covariance matrix of the vector $\mathbf{e}$ with a linear structure

$$\mathbf{e} = \mathbf{U}_1\boldsymbol{\xi}_1 + \ldots + \mathbf{U}_m\boldsymbol{\xi}_m \tag{2}$$

is considered in the form

$$\mathbf{C}(\mathbf{e}) = \mathbf{U}_1\mathbf{C}_1\mathbf{U}_1' + \ldots + \mathbf{U}_m\mathbf{C}_m\mathbf{U}_m' \tag{3}$$

where $\mathbf{U}_i$ is given $n \times \mathbf{c}_i$ matrix ($i = 1, 2, \ldots, m$) and $\boldsymbol{\xi}_i$ is a $\mathbf{c}_i$—vector of random variables with zero mean value and with a covariance matrix $\mathbf{C}_i$ ($i = 1, 2, \ldots, m$).

$$\begin{aligned} E(\boldsymbol{\xi}_i) &= \mathbf{0} \\ \mathbf{C}(\boldsymbol{\xi}_i) &= E(\boldsymbol{\xi}_i\boldsymbol{\xi}_i') = \mathbf{C}_i \\ \mathbf{C}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) &= \mathbf{0}. \end{aligned} \tag{4}$$

If $\mathbf{C}_i = \sigma_i^2 \mathbf{I}_{\mathbf{c}_i}$ $(i = 1, 2, \ldots, m)$, where σ_i^2 are unknown parameters and $\mathbf{I}_{\mathbf{c}_i}$ are $\mathbf{c}_i \times \mathbf{c}_i$ unit matrices, the covariance matrix of the error vector $\mathbf{e}$ in the model (1) is

$$\mathbf{C}(\mathbf{e}) = \sigma_1^2 \mathbf{V}_1 + \ldots + \sigma_m^2 \mathbf{V}_m. \tag{5}$$

The model (1) is called a variance components model ($\mathbf{V}_i = \mathbf{U}_i \mathbf{U}_i'$, $i = 1, \ldots, m$). The problem of estimation of the unknown parameters σ_i^2 or some linear function $\sum_i p_i \sigma_i^2$ of these parameters is solved, for example, in [1], [3], [4], [5]. If $\mathbf{C}_i = \mathbf{C}$ $(i = 1, 2, \ldots, m)$ the covariance matrix of the vector $\mathbf{e}$ in the model (1) is

$$\mathbf{C}(\mathbf{e}) = \mathbf{U}_1 \mathbf{C} \mathbf{U}_1' + \ldots + \mathbf{U}_m \mathbf{C} \mathbf{U}_m' \tag{6}$$

and this model is called a covariance components model. The covariance components are unknown elements of the matrix $\mathbf{C}$. The problem in the covariance components model is to estimate these elements or some linear function $\operatorname{tr} \mathbf{CQ}$ of these elements ($\mathbf{Q}$ is a known matrix of coefficients of linear combinations of the elements; $\operatorname{tr} \mathbf{A}$ is the trace of the matrix $\mathbf{A}$). This problem is solved for example in [2], [3].

In this paper the situation when the covariance matrix in the model (1) is in the form (3) and the matrices $\mathbf{C}_i$ $(i = 1, 2, \ldots, m)$ are unknown matrices is considered. The task is to estimate the unknown elements of these matrices or some linear combinations of the elements. The case (5) and the case (6) are special cases of the situation studied in this paper.

1. Formulation of the problem

Let us consider the model (1) in the form

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi} \tag{7}$$

(the error vector $\mathbf{e} = \mathbf{U}\boldsymbol{\xi}$) where $\mathbf{U} = (\mathbf{U}_1 \vdots \mathbf{U}_2 \vdots \ldots \vdots \mathbf{U}_m)$ and $\boldsymbol{\xi}' = (\boldsymbol{\xi}_1' \vdots \boldsymbol{\xi}_2' \vdots \ldots \vdots \boldsymbol{\xi}_m')$ with the expectation $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ and the covariance matrix

$$D(\mathbf{Y}) = \mathbf{U}_1 \mathbf{C}_1 \mathbf{U}_1' + \ldots + \mathbf{U}_m \mathbf{C}_m \mathbf{U}_m' = \Sigma. \tag{8}$$

Assume that the $\mathbf{C}_i$ $(i = 1, 2, \ldots, m)$ are unknown matrices. The problem is to estimate a linear combination of the unknown elements of the matrices $\mathbf{C}_i$ $(i = 1, 2, \ldots, m)$, which may be written as

$$\sum_{i=1}^{m} \operatorname{tr} \mathbf{C}_i \mathbf{Q}_i \tag{9}$$

where $\mathbf{Q}_i$ are known matrices of coefficients of the linear combination (9).

2. Solution of the problem

We consider a quadratic estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the type of MINQE (minimum norm quadratic estimator; see for example [3] or [4]) of the linear function (9). It is shown in the paper [3] that a quadratic estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is invariant (invariant with respect to the translation of the parameter $\boldsymbol{\beta}$) if $\mathbf{A}\mathbf{X} = \mathbf{0}$. This is the first restriction on the matrix $\mathbf{A}$ in the estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$. The estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of (9) with the condition $\mathbf{A}\mathbf{X} = \mathbf{0}$ is unbiased if

$$
\begin{aligned}
E(\mathbf{Y}'\mathbf{A}\mathbf{Y}) &= \operatorname{tr}\mathbf{A}\mathbf{C}(\mathbf{Y}) = \operatorname{tr}\mathbf{A}(\mathbf{U}_1'\mathbf{C}_1\mathbf{U}_1 + \ldots + \mathbf{U}_m'\mathbf{C}_m\mathbf{U}_m) \\
&= \operatorname{tr}(\mathbf{C}_1\mathbf{U}_1'\mathbf{A}\mathbf{U}_1 + \ldots + \mathbf{C}_m\mathbf{U}_m'\mathbf{A}\mathbf{U}_m) \\
&= \sum_{i=1}^{m} \operatorname{tr}\mathbf{C}_i\mathbf{U}_i'\mathbf{A}\mathbf{U}_i = \sum_{i=1}^{m} \operatorname{tr}\mathbf{C}_i\mathbf{Q}_i
\end{aligned}
$$

for all $(\mathbf{c}_i \times \mathbf{c}_i)$ matrices $\mathbf{C}_i$ $(i = 1, 2, \ldots, m)$. This can be true only if $\mathbf{U}_i'\mathbf{A}\mathbf{U}_i = \mathbf{Q}_i$ $(i = 1, 2, \ldots, m)$. This is the second restriction on the matrix $\mathbf{A}$ in the estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the linear function (9).

Definition 2.1. A quadratic estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the linear function (9) in the model (7) is invariant and unbiased if the matrix $\mathbf{A}$ satisfies the equations

$$
\begin{aligned}
\mathbf{A}\mathbf{X} &= \mathbf{0} \\
\mathbf{U}_i'\mathbf{A}\mathbf{U}_i &= \mathbf{Q}_i
\end{aligned}
\tag{10}
$$

for $(i = 1, 2, \ldots, m)$.

The estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is of the type of MINQE if the difference between $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ and the natural estimator of (9) is minimum. If the variables $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m$ are known the natural estimator of the linear function $\Sigma \operatorname{tr}\mathbf{C}_i\mathbf{Q}_i$ is

$$
\sum_i \boldsymbol{\xi}_i'\mathbf{Q}_i\boldsymbol{\xi}_i = (\boldsymbol{\xi}_1' \vdots \ldots \vdots \boldsymbol{\xi}_m')\boldsymbol{\Delta}\begin{pmatrix} \boldsymbol{\xi}_1 \\ \cdots \\ \vdots \\ \cdots \\ \boldsymbol{\xi}_m \end{pmatrix} = \boldsymbol{\xi}'\boldsymbol{\Delta}\boldsymbol{\xi}
\tag{11}
$$

because the natural estimator of the matrix $\mathbf{C}_i$ is $\boldsymbol{\xi}_i\boldsymbol{\xi}_i'$ $(i = 1, 2, \ldots, m)$. The proposed estimator with respect to the restriction $\mathbf{A}\mathbf{X} = \mathbf{0}$ is

$$
\mathbf{Y}'\mathbf{A}\mathbf{Y} = (\mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi})'\mathbf{A}(\mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi}) = \boldsymbol{\xi}'\mathbf{U}'\mathbf{A}\mathbf{U}\boldsymbol{\xi}.
\tag{12}
$$

Considering (11) and (12) as quadratic expressions in $\boldsymbol{\xi}$, we minimize the norm of their difference by minimizing the norm $\|\mathbf{U}'\mathbf{A}\mathbf{U} - \boldsymbol{\Delta}\| = \operatorname{tr}\mathbf{A}\mathbf{V}\mathbf{A}\mathbf{V}$ $(\mathbf{V} = \mathbf{V}_1 + \ldots + \mathbf{V}_m;\ \mathbf{V}_i = \mathbf{U}_i\mathbf{U}_i',\ i = 1, 2, \ldots, m)$.

Definition 2.2. A quadratic estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the linear function $\Sigma \operatorname{tr} \mathbf{C}_i \mathbf{Q}_i$ is the MINQUE(U, I) (unbiased and invariant) if the matrix $\mathbf{A}$ minimizes the expression $\operatorname{tr} \mathbf{AVAV}$ under the conditions

$$\mathbf{AX} = \mathbf{0}, \quad \mathbf{U}_i'\mathbf{A}\mathbf{U}_i = \mathbf{Q}_i \tag{13}$$

$(i = 1, 2, \ldots, m)$.

We can find the minimum $\operatorname{tr} \mathbf{AVAV}$ under the conditions (13) by the method of Lagrange multipliers:

$$\begin{aligned}
\Phi(\mathbf{A}) &= \operatorname{tr} \mathbf{AVAV} + 2 \operatorname{tr} \mathbf{\Lambda A X} + 2\Sigma \operatorname{tr} \mathbf{\Lambda}_i(\mathbf{U}_i'\mathbf{A}\mathbf{U}_i - \mathbf{Q}_i)\\
\partial\Phi(\mathbf{A})/\partial\mathbf{A} &= 2\mathbf{VAV} + 2\mathbf{\Lambda X}' + 2\Sigma \mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i' = \mathbf{0}\\
\mathbf{VAV} &= -\mathbf{\Lambda X}' - \Sigma \mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i'\\
\mathbf{A} &= \mathbf{V}^{-1}(-\mathbf{\Lambda X}' - \Sigma \mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i')\mathbf{V}^{-1}.
\end{aligned}$$

From $\mathbf{AX} = \mathbf{0}$ we have

$$\mathbf{\Lambda} = -\sum_i \mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i'\mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^- + \mathbf{I} - (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-$$

and then

$$\mathbf{A} = \sum_i \mathbf{Q}_{\mathbf{V}}'\mathbf{V}^{-1}\mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i'\mathbf{V}^{-1}\mathbf{Q}_{\mathbf{V}}$$

where $\mathbf{Q}_{\mathbf{V}} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}$ and the matrices of Lagrange multipliers $\mathbf{\Lambda}_i$ satisfy the equations $\mathbf{U}_i'(\Sigma \mathbf{Q}_{\mathbf{V}}'\mathbf{V}^{-1}\mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i'\mathbf{V}^{-1}\mathbf{Q}_{\mathbf{V}})\mathbf{U}_i = \mathbf{Q}_i$ $(i = 1, 2, \ldots, m)$.

Theorem 2.3. *The* MINQUE(U, I) *of the function* $\Sigma \operatorname{tr} \mathbf{C}_i\mathbf{Q}_i$ *is* $\mathbf{Y}'\mathbf{AY}$, *where*

$$\mathbf{A} = \sum_{i=1}^{m} \mathbf{Q}_{\mathbf{V}}'\mathbf{V}^{-1}\mathbf{U}_i\mathbf{\Lambda}_i\mathbf{U}_i'\mathbf{V}^{-1}\mathbf{Q}_{\mathbf{V}} \tag{14}$$

and the matrices $\mathbf{\Lambda}_i$ *are solutions of the linear system*

$$\sum_{j=1}^{m} \mathbf{U}_i'\mathbf{Q}_{\mathbf{V}}'\mathbf{V}^{-1}\mathbf{U}_j\mathbf{\Lambda}_j\mathbf{U}_j'\mathbf{V}^{-1}\mathbf{Q}_{\mathbf{V}}\mathbf{U}_i = \mathbf{Q}_i \tag{15}$$

$(i = 1, 2, \ldots, m)$. This estimator exists if $\mathbf{MM}^-\mathbf{Q}(\mathbf{M}')^-\mathbf{M}' = \mathbf{Q}$, *where the (i, j)-th element of the hypermatrix* $\mathbf{M}$ *is the matrix* $\mathbf{U}_i'\mathbf{V}^{-1}\mathbf{Q}_{\mathbf{V}}\mathbf{U}_j$ *and* $\mathbf{Q}$ *is a block diagonal hypermatrix* $\mathbf{Q} = \operatorname{diag}(\mathbf{Q}_1', \ldots, \mathbf{Q}_m')$.

P r o o f. It is enough to prove the second part of this theorem because the first part was proved before the theorem. The estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ exists if the linear system (15) has a solution. We can write this system in the form $\mathbf{M}\mathbf{\Lambda}\mathbf{M}' = \mathbf{Q}$, where $\mathbf{M}$, $\mathbf{Q}$ are the hypermatrices defined in Theorem 2.3 and $\mathbf{\Lambda}$ is the diagonal hypermatrix with the unknown matrices $\mathbf{\Lambda}_i$ on the diagonal.

It is not difficult to see that we can get the MINQE estimators in the variance components model [3], [4] from Theorem 2.3 if $\mathbf{C}_i = \sigma_i^2 I_{\mathbf{c}_i}$ $(i = 1, 2, \ldots, m)$ and the MINQUE estimators in the covariance components model [4] if $\mathbf{C}_i = \mathbf{C}$ $(i = 1, 2, \ldots, m)$. $\square$

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