THE BEST DIOPHANTINE APPROXIMATION FUNCTIONS BY CONTINUED FRACTIONS

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Abstract. Let $\xi = [a_0; a_1, a_2, \ldots, a_i, \ldots]$ be an irrational number in simple continued fraction expansion, $p_i/q_i = [a_0; a_1, a_2, \ldots, a_i]$, $M_i = q_i^2|\xi - p_i/q_i|$. In this note we find a function $G(R, r)$ such that

$$M_{n+1} < R \text{ and } M_{n-1} < r \text{ imply } M_n > G(R, r),$$
$$M_{n+1} > R \text{ and } M_{n-1} > r \text{ imply } M_n < G(R, r).$$

Together with a result the author obtained, this shows that to find two best approximation functions $\tilde{H}(R, r)$ and $\tilde{L}(R, r)$ is a well-posed problem. This problem has not been solved yet.

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1. Introduction

In 1891, Hurwitz [4] proved the following basic fact in the theory of Diophantine approximation:

Theorem 1. *Let ξ be an irrational number. Then there are infinitely many reduced fractions p/q such that $|\xi - p/q| < 1/(\sqrt{5}q^2)$.*

In order to find these fractions p/q, Borel [2] found the theorem below.

Theorem 2. *Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0; a_1, a_2, \ldots, a_i, \ldots]$. If $p_i/q_i = [a_0; a_1, a_2, \ldots, a_i]$, then at least one of the three consecutive p_i/q_i $(i = n-1, n, n+1)$ satisfies $|\xi - p_i/q_i| < 1/(\sqrt{5}q_i^2)$.*

For nearly a century, Borel's theorem has been extensively investigated. There are numerous results about it [1, 3, 6, 7, 10, 11]. The following result of the present author [10] is one of the highlights since it contains asymmetric approximations [5, 9] as a special case.

Theorem 3. *Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0; a_1, a_2, \ldots, a_i, \ldots]$. If $M_i = q_i^2|\xi - p_i/q_i|$ and r is a real number exceeding a_{n+1}, then*

$$M_n < r \text{ implies } \max(M_{n-1}, M_{n+1}) > 4r/(r^2 - a_{n+1}^2),$$
$$M_n > r \text{ implies } \min(M_{n-1}, M_{n+1}) < 4r/(r^2 - a_{n+1}^2).$$

Recently the author generalized Theorem 3 to the inhomogeneous case [12].

Theorem 4. *Let R, r be two real numbers exceeding $1+1/a_{n+1}$, let M_i be defined as in Theorem 3. Then*

$$M_{n+1} < R \text{ and } M_{n-1} < r \text{ imply } M_n > F(R, r),$$
$$M_{n+1} > R \text{ and } M_{n-1} > r \text{ imply } M_n < F(R, r),$$

where $F(R, r) = \left(R^{-1} + r^{-1} + \sqrt{a_{n+1}^2 + 4/(Rr)}\right) \Big/ \left(1 - (R^{-1} - r^{-1})^2/a_{n+1}^2\right)$.

Theorem 3 is a special case of Theorem 4 for $R = r$.

In this paper we give another estimation function $G(R, r)$ to replace $F(R, r)$ in Theorem 4. Using $G(R, r)$, the proof is simpler, it also gives Theorem 3 for $R = r$, the approximation is better than $F(R, r)$ for many values of R, r. These advantages are not essentially important, the significance of introducing $G(R, r)$ is the following consideration.

Suppose $\mathcal{H}$ is the set of all functions $H(R, r)$ such that

$$M_{n+1} < R \text{ and } M_{n-1} < r \text{ imply } M_n > H(R, r),$$

and $\mathcal{L}$ is the set of all functions $L(R, r)$ such that

$$M_{n+1} > R \text{ and } M_{n-1} > r \text{ imply } M_n < L(R, r).$$

Then $\tilde{H}(R, r) = \sup \mathcal{H}$ and $\tilde{L}(R, r) = \inf \mathcal{L}$ are the best approximation functions. Theorem 4 guarantees that both $\mathcal{H}$ and $\mathcal{L}$ are not empty sets. Nevertheless, it could be true that $\tilde{H}(R, r) = F(R, r)$ and $\tilde{L}(R, r) = F(R, r)$. The function $G(R, r)$ which we are going to introduce satisfies $G(R, r) \neq F(R, r)$. This ensures that to find the best approximation functions $\tilde{H}(R, r)$ and $\tilde{L}(R, r)$ is a well-posed problem. These two functions have not been found yet. It is still a challenging Problem.

2. Preliminaries

Let $\xi = [a_0; a_1, a_2, \ldots, a_i, \ldots]$ be an irrational number in simple continued fraction expansion, $p_i/q_i = [a_0; a_1, a_2, \ldots, a_i]$ and $M_i = q_i^2|\xi - p_i/q_i|$. Then it is known [8] that

$$M_i = [a_{i+1}; a_{i+2}, \ldots] + [0; a_i, a_{i-1}, \ldots, a_1]. \tag{1}$$

Let $P = [a_{n+2}; a_{n+3}, \ldots]$, $Q = [a_n; a_{n-1}, \ldots, a_1]$. Then it is easily checked that the following relations are true.

$$M_{n+1} = P + (a_{n+1} + Q^{-1})^{-1}, \tag{2}$$
$$M_n = a_{n+1} + P^{-1} + Q^{-1}, \tag{3}$$
$$M_{n-1} = Q + (a_{n+1} + P^{-1})^{-1}. \tag{4}$$

Now we turn to some inequalities.

If $M_{n+1} < R$, then $P < R - (a_{n+1} + Q^{-1})^{-1}$.

If $M_{n-1} < r$, then $Q < r - (a_{n+1} + P^{-1})^{-1}$.

If both $M_{n+1} < R$ and $M_{n-1} < r$ are true, then by (3) we know that the following two inequalities are correct.

$$M_n > a_{n+1} + P^{-1} + \big(r - (a_{n+1} + P^{-1})^{-1}\big)^{-1}, \tag{5}$$
$$M_n > a_{n+1} + Q^{-1} + \big(r - (a_{n+1} + Q^{-1})^{-1}\big)^{-1}. \tag{6}$$

We will use these inequalities in the next section.

3. The main theorem

We need a useful lemma.

Lemma 1. *The function $f(P) = a + P^{-1} + \big(r - (a + P^{-1})^{-1}\big)^{-1}$ is decreasing in P for $P^{-1} > 2r^{-1} - a$, where $a \geqslant 1$, $r > 1$ are constants.*

Proof. Since $f'(P) = -P^{-2}\big(1 - (ra + rP^{-1} - 1)^{-2}\big)$, if $P^{-1} > 2r^{-1} - a$, we have $ra + rP^{-1} - 1 > 1$, hence $f'(P) < 0$. □

Theorem 5. *Let M_i be defined as in Theorem 3. Then*

$$M_{n+1} < R \text{ and } M_{n-1} < r \text{ imply } M_n > G(R, r),$$
$$M_{n+1} > R \text{ and } M_{n-1} > r \text{ imply } M_n < G(R, r),$$

where

$$G(R,r) = a_{n+1} + 4a_{n+1}\Big/\left((3r-R)a_{n+1} - 4 + \sqrt{\big((R+r)a_{n+1}\big)^2 + 16}\right)$$
$$+ \left(\sqrt{\big((R+r)a_{n+1}\big)^2 + 16} + (R+r)a_{n+1} + 4\right)\Big/\big(2(R+r)\big).$$

P r o o f. If $M_{n+1} < R$ and $M_{n-1} < r$, then $M_{n+1} + M_{n-1} < R + r$. From (2), (4) we have one of the following inequalities:

$$P + (a_{n+1} + P^{-1})^{-1} < \tfrac{1}{2}(R+r), \tag{7}$$
$$Q + (a_{n+1} + Q^{-1})^{-1} < \tfrac{1}{2}(R+r). \tag{8}$$

Suppose (7) holds. Then

$$a_{n+1}P^2 + \big(2 - \tfrac{1}{2}(R+r)a_{n+1}\big)P - \tfrac{1}{2}(R+r) = 0.$$

Hence

$$P < \frac{(R+r)a_{n+1} - 4 + \sqrt{\big((R+r)a_{n+1}\big)^2 + 16}}{4a_{n+1}}. \tag{9}$$

It is obvious that $1 > 2/(a_{n+1} + P^{-1} + 1)$, hence

$$a_{n+1} + P^{-1} > 2/\big(1 + (a_{n+1} + P^{-1})^{-1}\big).$$

Since $1 + (a_{n+1} + P^{-1})^{-1} \leqslant Q + (a_{n+1} + P^{-1})^{-1} = M_{n-1} < r$, we have $P^{-1} > 2r^{-1} - a_{n+1}$. By (5), (9) and Lemma 1 we have

$$M_n > G(R,r).$$

Suppose (8) holds. Noticing (6), we can have $M_n > G(R,r)$ by the similar discussion.

Reversing all the directions of the inequalities above, we can prove that

$$M_{n+1} > R \text{ and } M_{n-1} > r \text{ imply } M_n < G(R,r).$$

□

4. Comparison of the approximation functions $F(R, r)$ and $G(R, r)$

It is easily checked that

$$F(r, r) = G(r, r) = \big(2 + \sqrt{(ra_{n+1})^2 + 4}\big)/r.$$

The function $F(R, r)$ is a symmetric function while $G(R, r)$ is not.

If $a_{n+1} = 1$, then

$$F(4, 2) = 2.1063946,$$
$$G(4, 2) = 2.2018503.$$

In this case $G(R, r)$ gives a better estimation than $F(R, r)$ in the set $\mathcal{H}$, while $F(R, r)$ gives a better estimation than $G(R, r)$ in the set $\mathcal{L}$.

If $a_{n+1} = 3$, then

$$F(2, 6) = 3.7682386,$$
$$G(2, 6) = 3.4463062.$$

In this case the situation reverses.

The function $H(R, r) = \max\big(F(R, r), G(R, r)\big)$ is a better approximation than both $F(R, r)$ and $G(R, r)$ in the set $\mathcal{H}$.

The function $L(R, r) = \min\big(F(R, r), G(R, r)\big)$ is a better approximation than both $F(R, r)$ and $G(R, r)$ in the set $\mathcal{L}$.

The best approximation functions satisfy $\tilde{H}(R, r) \geqslant \tilde{L}(R, r)$ and $\tilde{H}(R, r) \neq \tilde{L}(R, r)$. Now it is an open problem how to find them.

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