SYMMETRIZED AND CONTINUOUS GENERALIZATION OF TRANSVERSALS

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Abstract. The theorem of Edmonds and Fulkerson states that the partial transversals of a finite family of sets form a matroid. The aim of this paper is to present a symmetrized and continuous generalization of this theorem.

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1. INTRODUCTION

There are two classical results concerning both the transversal theory and the matroid theory. The first is the theorem of Rado [17], who established a necessary and sufficient condition for a finite family of sets to possess a transversal which is independent in a given matroid. The second result, stated by Edmonds and Fulkerson [4] (and also proved independently by Mirsky and Perfect [14]) states that the set of partial transversals of a finite family of sets form a matroid. There are many variations and generalizations of these two theorems. A comprehensive survey of this field can be found in the books of Mirsky [13] and Welsh [20].

In [8] and [9] we introduced $\mathscr{M}$-systems of representatives and $\mathscr{M}$-polytransversals. They present a new concept joining transversals and matroids. An *$\mathscr{M}$-system of representatives* of a finite family $\mathscr{A} = (A_t : t \in T)$ of subsets of a finite set S is a family $(x_t : t \in T)$ of elements of S such that $x_t \in A_t$ for any $t \in T$ and, for any $s \in S$, the set $\{t \in T;\, x_t = s\}$ is independent in a given matroid M_s. Furthermore, the $|S|$ dimensional vector $(u_s : s \in S)$ where $u_s = |\{t \in T;\, x_t = s\}|$ is called an *$\mathscr{M}$-polytransversal* of $\mathscr{A}$. In [9] we proved that the set of $\mathscr{M}$-polytransversals of $\mathscr{A}$

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forms the set of integral independent vectors of a polymatroid. This generalizes the theorem of Edmonds and Fulkerson.

Other generalizations of transversals and the theorem of Edmonds and Fulkerson are presented in [7] and [21].

Now we prove a symmetrized and continuous analogue of the results of [9]. As a motivation let us recall two theorems from transversal theory. We will express them in the language of bipartite graphs. A finite *bipartite graph* $G = (S, T; E)$ consists of two finite disjoint vertex sets S, T and a set E of edges joining the vertex sets S and T. If $X \subseteq S$ and $Y \subseteq T$ we say that $X = \{x_1, \dots x_k\}$ can be *matched into* Y in G if there exists a set of edges joining each x_i to a distinct member of Y (in other words if the subgraph of G determined by $X \cup Y$ has a matching which covers every vertex of X). If $X \subseteq S$ then ∂X is the set of vertices of T which are endpoints of an edge whose other endpoint is in X. The following theorem was proved by Brualdi [2]. Mirsky [13] calls it a symmetrized version of Rado's theorem.

Theorem 1. *Let $G = (S, T; E)$ be a finite bipartite graph. Let M_1, M_2 be matroids on S, T with rank functions ϱ_1, ϱ_2, respectively. Then there exists $X \subseteq S$ with $|X| = k$ such that X is independent in M_1 and X can be matched into an independent set Y of M_2, if and only if for all $X \subseteq S$,*

$$\varrho_1(S \setminus X) + \varrho_2(\partial X) \geqslant k.$$

The next theorem was proved by Perfect [15] (see also [20]) and generalizes the theorem of Edmonds and Fulkerson.

Theorem 2. *Let $G = (S, T; E)$ be a finite bipartite graph. Let M be a matroid on T with rank function ϱ. Then the collection*

$$\{X \colon X \subseteq S,\ X \text{ can be matched in } G \text{ into an independent set of } M\}$$

is the set of independent sets of a matroid M_1 on S with rank function ϱ_1 such that, for any $X \subseteq S$,

$$\varrho_1(X) = \min_{A \subseteq X} (\varrho(\partial A) + |X \setminus A|).$$

The aim of this paper is to show that symmetrized and continuous analogues of $\mathscr{M}$-polytransversals form a polymatroid. Our results generalize Theorems 1 and 2 but also the results from [7], [8], [9] and [21].

We assume familiarity with matroids and transversals. The main literature is the book of Welsh [20] where all basic results regarding matroids, polymatroids and transversals can be found. As other sources let us note [1], [5], [13] and [18].

Let $\mathbb{R}_+$ ($\mathbb{Z}_+$) denote the set of nonnegative real (integer) numbers. If S is a finite set, then denote by $\mathbb{R}_+^S$ ($\mathbb{Z}_+^S$) the space of real (integer) valued nonnegative vectors with coordinates indexed by S. Similarly, if also T is finite, then $\mathbb{R}_+^{S\times T}$ ($\mathbb{Z}_+^{S\times T}$) denotes the space of real (integer) valued nonnegative vectors with coordinates indexed by $S \times T$. For example

$$\begin{aligned} \mathbb{R}_+^S &= \{\mathbf{u} = (u_s \colon s \in S);\, u_s \in \mathbb{R}_+\}\,, \\ \mathbb{Z}_+^S &= \{\mathbf{u} = (u_s \colon s \in S);\, u_s \in \mathbb{Z}_+\}\,, \\ \mathbb{R}_+^{S\times T} &= \{\mathbf{u} = (u_{s,t} \colon s \in S,\ t \in T);\, u_{s,t} \in \mathbb{R}_+\}\,. \end{aligned}$$

For each $\mathbf{x} \in \mathbb{R}_+^S$ and $s \in S$ denote the sth coordinate of $\mathbf{x}$ by $\mathbf{x}(s)$. For $\mathbf{x} \in \mathbb{R}_+^S$ and $A \subseteq S$ we define $\mathbf{x}(A) = \sum_{s\in A} \mathbf{x}(s)$, and $\mathbf{x}\big|A$ denotes the restriction of $\mathbf{x}$ to A. We call the quantity $|\mathbf{x}| = \mathbf{x}(S) = \sum_{s\in S} \mathbf{x}(s)$ the *modulus* of $\mathbf{x}$.

A *polymatroid* $\mathbb{P}$ (on S) is a pair (S, ϱ) where S, the *ground set*, is a non-empty finite set and ϱ, the *ground set rank function*, is a function $\varrho\colon 2^S \to \mathbb{R}_+$ such that

(1) $\quad \varrho(\emptyset) = 0,$

(2) $\quad$ if $A \subseteq B \subseteq S$ then $\varrho A \leqslant \varrho B,$

(3) $\quad$ if $A, B \subseteq S$ then $\varrho A + \varrho B \geqslant \varrho(A \cup B) + \varrho(A \cap B).$

(Items (2) and (3) state that ϱ is *monotone* and *submodular*, respectively.) Then a vector $\mathbf{u} \in \mathbb{R}_+^S$ such that $\mathbf{u}(X) \leqslant \varrho X$ for all $X \subseteq S$ is called an *independent vector* of $\mathbb{P}$.

If $\varrho\colon 2^S \to \mathbb{Z}_+$ then $\mathbb{P} = (S, \varrho)$ is called an *integral* polymatroid. Furthermore, if $\varrho(\{s\}) = 0, 1$ for any $s \in S$ then $\mathbb{P}$ is called a *matroid*. One of the most important properties of polymatroids is expressed in the following theorem (see [3], [12]) known as the polymatroid intersection theorem of Edmonds.

Theorem 3. *Let $\mathbb{P}_1 = (S, \varrho_1)$ and $\mathbb{P}_2 = (S, \varrho_2)$ be polymatroids and let $k \in \mathbb{R}_+$. Then there exists a vector $\mathbf{u} \in \mathbb{R}_+^S$ independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$ and with modulus at least k if and only if for all subsets $X \subseteq S$*

$$\varrho_1(X) + \varrho_2(S \setminus X) \geqslant k.$$

Furthermore, if both $\mathbb{P}_1$, $\mathbb{P}_2$ are integral we may insist that the vector $\mathbf{u}$ be integral.

If $\mathbb{P} = (S, \varrho)$ is a polymatroid and $k \in \mathbb{R}_+$, then it is easy to check that $\mathbb{P}^{(k)} = (S, \varrho^{(k)})$ such that $\varrho^{(k)}(X) = \min\{k, \varrho X\}$ $(X \subseteq S)$ is polymatroid. We call $\mathbb{P}^{(k)}$ the *truncation* of $\mathbb{P}$ at k.

If $\mathbb{P} = (S, \varrho)$ is a polymatroid and $\emptyset \neq X \subseteq S$ then $\mathbb{P}^{(X)} = (X, \varrho^{(X)})$ (where $\varrho^{(X)}$ is the restriction of ϱ to X) is a polymatroid. We call $\mathbb{P}^{(X)}$ the *restriction* of $\mathbb{P}$ to X.

Let I, S_i $(i \in I)$ be finite sets, $S_i \cap S_j = \emptyset$ for any $i \neq j$ and let $\mathbb{P}_i = (S_i, \varrho_i)$ $(i \in I)$ be polymatroids. Let $S = \bigcup_{i \in I} S_i$ and $\varrho\colon 2^S \to \mathbb{R}_+$ be such that for any $X \subseteq S$,

$$\varrho(X) = \sum_{i \in I} \varrho_i(X \cap S_i).$$

Then $\mathbb{P} = (S, \varrho)$ is a polymatroid. We call $\mathbb{P}$ the *product* of the polymatroids $\mathbb{P}_i$ $(i \in I)$ and denote it by $\prod_{i \in I} \mathbb{P}_i$.

Clearly, if $\mathbb{P}$ is integral and $k \in \mathbb{Z}_+$ then $\mathbb{P}^{(k)}$ and $\mathbb{P}^{(X)}$ are integral. If $\mathbb{P}_i$ $(i \in I)$ are integral then $\prod_{i \in I} \mathbb{P}_i$ is integral.

Finally, if $k \in \mathbb{R}_+$, denote by $\mathbb{U}_{k,S}$ the polymatroid (S, ϱ) such that $\varrho X = k|X|$ for any $X \subseteq S$.

Now we introduce the main notions of this paper. Throughout the paper let S, T be two disjoint finite sets. Let $\mathscr{P}_S = (\mathbb{P}_s = (T, \varrho_s)\colon\ s \in S)$, $\mathscr{P}_T = (\mathbb{P}_t = (S, \varrho_t)\colon$ $t \in T)$ be systems of polymatroids, let $\mathbb{P}_2 = (T, \varrho_2)$ be a polymatroid and $X \subseteq S$, $J \subseteq T$.

A vector $\mathbf{a} = (a_{s,t}\colon s \in X, t \in J) \in \mathbb{R}_+^{X \times J}$ is called an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-*system of representatives* (in abbreviation $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-*SR*) if:

- the vector $\mathbf{a}_s = (a_{s,t}\colon t \in J) \in \mathbb{R}_+^J$ is independent in $\mathbb{P}_s$ for any $s \in X$,
- the vector $\mathbf{a}_t = (a_{s,t}\colon s \in X) \in \mathbb{R}_+^X$ is independent in $\mathbb{P}_t$ for any $t \in J$,
- the vector $\mathbf{v} = (v_t = \sum_{s \in X} a_{s,t}\colon t \in J) \in \mathbb{R}_+^J$ is independent in $\mathbb{P}_2$.

Furthermore, the vector $\mathbf{u} = (u_s = \sum_{t \in J} a_{s,t}\colon s \in X) \in \mathbb{R}_+^X$ is called an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-*polytransversal*. In this case $\mathbf{a}$ is called an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-*origin* of $\mathbf{u}$.

Moreover, if $\mathbb{P}_1 = (S, \varrho_1)$ is a polymatroid, then a vector $\mathbf{a}' = (a'_{s,t}\colon s \in X, t \in J) \in \mathbb{R}_+^{X \times J}$ is called an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-*system of representatives* (in abbreviation $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-*SR*) if:

- the vector $\mathbf{a}'$ is an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-SR,
- the vector $\mathbf{u}' = (u'_s = \sum_{t \in J} a'_{s,t}\colon s \in X) \in \mathbb{R}_+^X$ is independent in $\mathbb{P}_1$.

The notions of transversals and systems of distinct representatives and also their generalizations introduced in [7], [8], [9], [10] and [20] are in fact integral $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversals and $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-systems of representatives for special classes of X, J, $\mathscr{P}_S$, $\mathscr{P}_T$, $\mathbb{P}_1$ and $\mathbb{P}_2$. The main distinction introduced here is that we deal with vectors whose coordinates are from $\mathbb{R}_+$ and not only from $\mathbb{Z}_+$. In this way we obtain a "continuous" analogue of transversals and systems of representatives. On the other hand our results presented in the next section (Theorems 4 and 5) remain true if we deal only with integral vectors and integral polymatroids. Thus we generalize the results from [7], [8], [9], [10], [20] and [21].

At the end of this section we introduce another notation. Let $Z \subseteq S \times T$. Then denote

$$Z_{/s} = \{t \in T\colon (s,t) \in Z\} \text{ for any } s \in S, \tag{4}$$

$$Z_{/t} = \{s \in S\colon (s,t) \in Z\} \text{ for any } t \in T. \tag{5}$$

3. The main results

Primarily we generalize the operation product of polymatroids.

Lemma 1. *Let I, S_i $(i \in I)$ be finite sets, $S_i \cap S_j = \emptyset$ for any $i \neq j$. Let $\mathbb{P}_i = (S_i,\, \varrho_i)$ $(i \in I)$ and $\mathbb{P}' = (I, \varrho')$ be (integral) polymatroids. Take $S = \bigcup\limits_{i \in I} S_i$ and $\varrho\colon 2^S \to \mathbb{R}_+$ such that for any $X \subseteq S$,*

$$\varrho(X) = \min_{L \subseteq I} \Big(\varrho'(I \setminus L) + \sum_{i \in L} \varrho_i(X \cap S_i) \Big). \tag{6}$$

Then $\mathbb{P} = (S,\, \varrho)$ is an (integral) polymatroid. Moreover, a vector $\mathbf{a} = (a_s\colon s \in S) \in \mathbb{R}_+^S$ is independent in $\mathbb{P}$ if and only if it is independent in $\prod\limits_{i \in I} \mathbb{P}_i$ and the vector $\mathbf{u} = (u_i = \sum\limits_{s \in S_i} a_s\colon i \in I) \in \mathbb{R}_+^I$ is independent in $\mathbb{P}'$. We will denote $\mathbb{P}$ by $\mathbb{P}' \big| \prod\limits_{i \in I} \mathbb{P}_i$.

P r o o f. It is easy to check that ϱ is monotone and $\varrho(\emptyset) = 0$. Let X (X') be a subset of S and let L (L') be the subset of I for which the minimum occurs in (6). Then using the monotonicity and submodularity of ϱ', ϱ_i $(i \in I)$ we get

$$\begin{aligned}
\varrho X + \varrho X' &= \varrho'(I \setminus L) + \sum_{i \in L} \varrho_i(X \cap S_i) + \varrho'(I \setminus L') + \sum_{i \in L'} \varrho_i(X' \cap S_i) \\
&\geqslant \varrho'(I \setminus (L \cap L')) + \varrho'(I \setminus (L \cup L')) \\
&\quad + \sum_{i \in L \cap L'} \varrho_i((X \cup X') \cap S_i) + \sum_{i \in L \cup L'} \varrho_i((X \cap X') \cap S_i) \\
&\geqslant \varrho(X \cup X') + \varrho(X \cap X').
\end{aligned}$$

Thus ϱ is submodular and $\mathbb{P} = (S,\, \varrho)$ is a polymatroid.

Take $\varphi\colon S \to I$ such that $\varphi(x) = i$ iff $x \in S_i$ $(i \in I)$. Let $\varrho_1\colon 2^S \to \mathbb{R}_+$ be such that $\varrho_1(X) = \varrho'(\varphi(X))$ for any $X \subseteq S$ $(\varphi(X) = \{\varphi(x);\, x \in X\})$. Then it is easy to check that $\mathbb{P}_1 = (S,\ \varrho_1)$ is a polymatroid and that $\mathbf{a} = (a_s\colon s \in S) \in \mathbb{R}_+^S$ is independent in $\mathbb{P}_1$ iff $(\sum\limits_{s\in S_i} a_s\colon i \in I) \in \mathbb{R}_+^I$ is independent in $\mathbb{P}'$. Finally, let $\mathbb{P}_2 = (S,\ \varrho_2)$ denote the polymatroid $\prod\limits_{i\in I} \mathbb{P}_i$. Then for any $X \subseteq S$,

$$\varrho(X) = \min_{A\subseteq X} (\varrho_1(X \setminus A) + \varrho_2(A)). \tag{7}$$

Let $\mathbf{a} \in \mathbb{R}_+^S$ be independent in $\mathbb{P}$. Then $\mathbf{a}(X) \leqslant \varrho X$ and it follows from (7) that $\mathbf{a}$ is independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$. On the other hand let $\mathbf{a}$ be independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$. Then, for any $X \subseteq S$, $\mathbf{a}\big|X$ is independent in both $\mathbb{P}_1^{(X)}$ and $\mathbb{P}_2^{(X)}$ (the restrictions of $\mathbb{P}_1$ and $\mathbb{P}_2$ to X, respectively), and from Theorem 3 and (7) it follows that $\mathbf{a}(X) \leqslant \varrho X$. Thus $\mathbf{a}$ is independent in $\mathbb{P}$.

Finally, if ϱ', ϱ_i $(i \in I)$ are integral then also ϱ is integral. □

Now we generalize Theorem 1 to $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-systems of representatives.

Theorem 4. *Let S, T be finite sets, let $\mathscr{P}_S = (\mathbb{P}_s = (T,\ \varrho_s)\colon\ s \in S)$, $\mathscr{P}_T = (\mathbb{P}_t = (S,\ \varrho_t)\colon\ t \in T)$ be systems of (integral) polymatroids, let $\mathbb{P}_1 = (S,\ \varrho_1)$, $\mathbb{P}_2 = (T,\ \varrho_2)$ be (integral) polymatroids and let $k \in \mathbb{R}_+$. Then there exists an (integral) $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-system of representatives with modulus at least k if and only if*

$$\min_{X\subseteq S, J\subseteq T} \Bigg(\varrho_1(S \setminus X) + \varrho_2(T \setminus J) + \min_{Z\subseteq X\times J} \Big(\sum_{s\in X} \varrho_s(Z_{/s}) + \sum_{t\in J} \varrho_t \left(((X \times J) \setminus Z)_{/t} \right) \Big) \Bigg) \geqslant k.$$

P r o o f. Take $\mathbb{P}'_s = (\{s\} \times T, \varrho'_s)$ such that $\varrho'_s(\{s\} \times J) = \varrho_s(J)$ $(s \in S, J \subseteq T)$ and $\mathbb{P}'_t = (S \times \{t\}, \varrho'_t)$ such that $\varrho'_t(X \times \{t\}) = \varrho_t(X)$ $(t \in T, X \subseteq S)$. Take the polymatroids $\mathbb{P}_S$, $\mathbb{P}_T$ on $S \times T$ such that $\mathbb{P}_S = \mathbb{P}_1 \big| \prod\limits_{s\in S} \mathbb{P}'_s$ and $\mathbb{P}_T = \mathbb{P}_2 \big| \prod\limits_{t\in T} \mathbb{P}'_t$. Then $\mathbf{a} \in \mathbb{R}_+^{S\times T}$ is an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_1, \mathbb{P}_2)$-SR iff $\mathbf{a}$ is independent in both $\mathbb{P}_S$ and $\mathbb{P}_T$, and the theorem follows from Theorem 3. □

The following theorem is a generalization of Theorem 2 but also results from [7] and [9].

Theorem 5. *Let S, T be finite sets, let $\mathscr{P}_S = (\mathbb{P}_s = (T,\ \varrho_s)\colon\ s \in S)$, $\mathscr{P}_T = (\mathbb{P}_t = (S,\ \varrho_t)\colon\ t \in T)$ be systems of polymatroids and let $\mathbb{P}_2 = (S,\ \varrho_2)$ be a polymatroid. Then $\mathbf{u} = (u_s\colon s \in S) \in \mathbb{R}_+^S$ is an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal if and* *only if it is an independent vector of the polymatroid $\mathbb{P} = (S,\, \varrho)$ such that for any $X \subseteq S$,*

$$\varrho(X) = \min_{J \subseteq T} \left(\varrho_2(T \setminus J) \ + \min_{Z \subseteq X \times J} \left(\sum_{s \in X} \varrho_s(Z_{/s}) + \sum_{t \in J} \varrho_t \left(((X \times J) \setminus Z)_{/t} \right) \right) \right).$$

Furthermore, if $\mathbb{P}_2$, $\mathbb{P}_s$, $\mathbb{P}_t$ ($s \in S$, $t \in T$) are integral then also $\mathbb{P}$ is integral. If also $\mathbf{u} \in \mathbb{Z}_+^S$ is an integral $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal, then $\mathbf{u}$ has an integral $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-origin.

Proof. It is easy to check that ϱ is monotone and $\varrho(\emptyset) = 0$. We prove submodularity.

Let $X, Y \subseteq S$. Choose $J \subseteq T$, $K \subseteq T$, $Z \subseteq X \times J$, $V \subseteq Y \times K$ such that

$$\begin{aligned}
\varrho(X) &= \varrho_2(T \setminus J) + \sum_{s \in X} \varrho_s(Z_{/s}) + \sum_{t \in J} \varrho_t(((X \times J) \setminus Z)_{/t}),\\
\varrho(Y) &= \varrho_2(T \setminus K) + \sum_{s \in Y} \varrho_s(V_{/s}) + \sum_{t \in K} \varrho_t(((Y \times K) \setminus V)_{/t}).
\end{aligned}$$

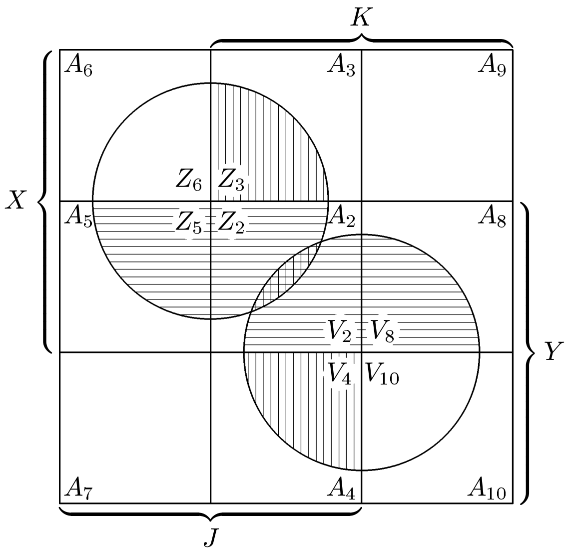
Take the partition of $(X \cup Y) \times (J \cup K)$ into ten sets $A_1, A_2, \ldots, A_{10}$:

$$\begin{aligned}
A_1 &= Z \cap V \subseteq (X \cap Y) \times (J \cap K),\\
A_2 &= ((X \cap Y) \times (J \cap K)) \setminus A_1,\\
A_3 &= (X \setminus Y) \times (J \cap K),\\
A_4 &= (Y \setminus X) \times (J \cap K),\\
A_5 &= (X \cap Y) \times (J \setminus K),\\
A_6 &= (X \setminus Y) \times (J \setminus K),\\
A_7 &= (Y \setminus X) \times (J \setminus K),\\
A_8 &= (X \cap Y) \times (K \setminus J),\\
A_9 &= (X \setminus Y) \times (K \setminus J),\\
A_{10} &= (Y \setminus X) \times (K \setminus J).
\end{aligned}$$

Denote, for any $i \in \{1, 2, \ldots, 10\}$,

$$\begin{aligned}
Z_i &= Z \cap A_i,\\
V_i &= V \cap A_i.
\end{aligned}$$

Clearly $Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_5 \cup Z_6$, $V = V_1 \cup V_2 \cup V_4 \cup V_8 \cup V_{10}$. $V_1 = Z_1 = A_1$, but $V_i \cap Z_i = \emptyset$ for any $i \in \{2, 3, \ldots, 10\}$. (See a symbolic representation of $A_1, A_2, \ldots, A_{10}$ in Fig. 1. Subsets of S and T are expressed as segments of the axis



$A_1 = Z_1 = V_1 \qquad R = \equiv \qquad W = ||||||$

Fig. 1

and subsets of $S \times T$ are depicted as parts of the plane in this figure. For instance Z and V are depicted as circles, A_1 as the intersection of the circles and $A_3, \dots, A_{10}$ as squares.)

Choose R, W and B_i, $i \in \{2, 3, \dots, 10\}$, such that

$$\begin{aligned} R &= Z_1 \cup Z_2 \cup Z_5 \cup V_2 \cup V_8 \quad (\subseteq (X \cap Y) \times (J \cup K)), \\ W &= Z_1 \cup Z_3 \cup V_4 \quad (\subseteq (X \cup Y) \times (J \cap K)), \\ B_i &= A_i \setminus (Z_i \cup V_i). \end{aligned}$$

Then we can check (see e.g. Fig. 1) that

$$\begin{aligned} ((X \cap Y) \times (J \cup K)) \setminus R &= B_2 \cup B_5 \cup B_8, \\ ((X \cup Y) \times (J \cap K)) \setminus W &= B_2 \cup B_3 \cup B_4 \cup Z_2 \cup V_2. \end{aligned}$$

The sets A_1 $(= Z_1 = V_1)$, $Z_2, Z_3, Z_5, Z_6, V_2, V_4, V_8, V_{10}$, $B_2, \dots, B_{10}$ are pairwise disjoint. Using this fact and the submodularity and monotonicity of ϱ_2, ϱ_s, ϱ_t ($s \in S$,

$t \in T$) we get

$$
\begin{aligned}
&\varrho(X) + \varrho(Y) \\
&\quad = \varrho_2(T \setminus J) + \sum_{s\in X} \varrho_s(Z_{/s}) + \sum_{t\in J} \varrho_t(((X \times J) \setminus Z)_{/t}) \\
&\qquad + \varrho_2(T \setminus K) + \sum_{s\in Y} \varrho_s(V_{/s}) + \sum_{t\in K} \varrho_t(((Y \times K) \setminus V)_{/t}) \\
&\quad = \varrho_2(T \setminus J) + \varrho_2(T \setminus K) + \sum_{s\in X\setminus Y} \varrho_s((Z_3 \cup Z_6)_{/s}) + \sum_{s\in Y\setminus X} \varrho_s((V_4 \cup V_{10})_{/s}) \\
&\qquad + \sum_{s\in X\cap Y} \varrho_s((Z_1 \cup Z_2 \cup Z_5)_{/s}) + \sum_{s\in X\cap Y} \varrho_s((V_1 \cup V_2 \cup V_8)_{/s}) \\
&\qquad + \sum_{t\in J\setminus K} \varrho_t((B_5 \cup B_6)_{/t}) + \sum_{t\in K\setminus J} \varrho_t((B_8 \cup B_{10})_{/t}) \\
&\qquad + \sum_{t\in J\cap K} \varrho_t((B_2 \cup B_3 \cup V_2)_{/t}) + \sum_{t\in J\cap K} \varrho_t((B_2 \cup B_4 \cup Z_2)_{/t}) \\
&\quad \geqslant \varrho_2(T \setminus (J \cap K))\varrho_2(T \setminus (J \cup K)) + \sum_{s\in X\setminus Y} \varrho_s((Z_3)_{/s}) + \sum_{s\in Y\setminus X} \varrho_s((V_4)_{/s}) \\
&\qquad + \sum_{s\in X\cap Y} \varrho_s((Z_1)_{/s}) + \sum_{s\in X\cap Y} \varrho_s((Z_1 \cup Z_2 \cup Z_5 \cup V_2 \cup V_8)_{/s}) \\
&\qquad + \sum_{t\in J\setminus K} \varrho_t((B_5)_{/t}) + \sum_{t\in K\setminus J} \varrho_t((B_8)_{/t}) \\
&\qquad + \sum_{t\in J\cap K} \varrho_t((B_2)_{/t}) + \sum_{t\in J\cap K} \varrho_t((B_2 \cup B_3 \cup B_4 \cup Z_2 \cup V_2)_{/t}) \\
&\quad = \varrho_2(T \setminus (J \cup K)) \ + \sum_{s\in X\cap Y} \varrho_s(R_{/s}) + \sum_{t\in J\cup K} \varrho_t\left(((X \cap Y) \times (J \cup K) \setminus R)_{/t}\right) \\
&\qquad + \varrho_2(T \setminus (J \cap K)) \ + \sum_{s\in X\cup Y} \varrho_s(W_{/s}) \\
&\qquad + \sum_{t\in J\cap K} \varrho_t\left(((X \cup Y) \times (J \cap K) \setminus W)_{/t}\right) \\
&\quad \geqslant \varrho(X \cup Y) + \varrho(X \cap Y).
\end{aligned}
$$

Thus ϱ is submodular and $\mathbb{P} = (S, \varrho)$ is a polymatroid.

Replace the polymatroid $\mathbb{P}_1$ in Theorem 4 by $\mathbb{U}_{k,S}$ where k is sufficiently large (e.g., let $k = \sum_{s\in S} \varrho_s(T) + \sum_{t\in T} \varrho_t(S)$). Then it is easy to check that any $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{U}_{k,S}, \mathbb{P}_2)$-SR is just an $(X, J, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-SR and that ϱX is the maximal modulus of an $(X, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-SR (polytransversal).

Therefore, if $\mathbf{u} \in \mathbb{R}^S_+$ is an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal, then $\mathbf{u}|X$ is an $(X, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal, and $\mathbf{u}(X) \leqslant \varrho X$ for any $X \subseteq S$. Thus $\mathbf{u}$ is independent in $\mathbb{P}$.

Let $\mathbf{u} = (u_s \colon s \in S) \in \mathbb{R}_+^S$ be independent in $\mathbb{P}$. Then denote by $\mathscr{P}_S^{(\mathbf{u})}$ the system of polymatroids $(\mathbb{P}_s^{(u_s)} = (T, \varrho_s^{(u_s)}) \colon s \in S)$. Note that $\varrho_s^{(u_s)}(J) = \min\{u_s, \varrho_s(J)\}$ $(J \subseteq T,\ s \in S)$. Let m denote the maximal modulus of an $(S, T, \mathscr{P}_S^{(\mathbf{u})}, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal. Then replacing $\mathbb{P}_1$ by $\mathbb{U}_{k,S}$ (k sufficiently large) and $\mathscr{P}_S$ by $\mathscr{P}_S^{(\mathbf{u})}$ in Theorem 4 and applying the above argument we get

$$
\begin{aligned}
m &= \min_{J \subseteq T} \left(\varrho_2(T \setminus J) + \min_{Z \subseteq S \times J} \left(\sum_{s \in S} \varrho_s^{(u_s)}(Z_{/s}) + \sum_{t \in J} \varrho_t \left(((S \times J) \setminus Z)_{/t} \right) \right) \right) \\
&= \min_{J \subseteq T} \left(\varrho_2(T \setminus J) + \min_{Z \subseteq S \times J} \left(\sum_{s \in S} \min\{u_s, \varrho_s(Z_{/s})\} + \sum_{t \in J} \varrho_t(((S \times J) \setminus Z)_{/t}) \right) \right).
\end{aligned}
$$

Let $X = \{s \in S;\ u_s > \varrho_s(Z_{/s})\}$. Then we can easily check that

$$
\begin{aligned}
m = \mathbf{u}(S \setminus X) + \min_{J \subseteq T} \Bigg(& \varrho_2(T \setminus J) \\
& + \min_{Z \subseteq X \times J} \left(\sum_{s \in X} \varrho_s(Z_{/s}) + \sum_{t \in J} \varrho_t \left(((X \times J) \setminus Z)_{/t} \right) \right) \Bigg) \\
= \mathbf{u}(S \setminus X) + \varrho_1(X) \ \geqslant\ & \mathbf{u}(S \setminus X) + \mathbf{u}(X) = |\mathbf{u}|.
\end{aligned}
$$

Since $\varrho_s^{(u_s)}(T) \leqslant u_s$ $(s \in S)$ then the inequality $m \geqslant |\mathbf{u}|$ is possible iff $\mathbf{u}$ is a $(S, T, \mathscr{P}_S^{(\mathbf{u})}, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal with modulus m. Thus $\mathbf{u}$ is also an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal.

Therefore $\mathbf{u} \in \mathbb{R}_+^S$ is an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversal if and only if $\mathbf{u}$ is independent in $\mathbb{P}$.

Furthermore, if $\mathbb{P}_2$, $\mathbb{P}_s$, $\mathbb{P}_t$ $(s \in S,\ t \in T)$ are integral then $\mathbb{P}$ is integral. If also $\mathbf{u}$ is integer valued then all polymatroids we have dealt with in the proof are integral. Thus, by Theorem 4, we can take an integer valued $(S, T, \mathscr{P}_S^{(\mathbf{u})}, \mathscr{P}_T, \mathbb{P}_2)$-SR $\mathbf{a}$ of modulus m, then it must be an $(S, T, \mathscr{P}_S^{(\mathbf{u})}, \mathscr{P}_T, \mathbb{P}_2)$-origin of $\mathbf{u}$, and also an $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-origin of $\mathbf{u}$, concluding the proof. □

4. Concluding remarks

Note that Theorem 4 is equivalent with the intersection theorem of Edmonds (Theorem 3). But as pointed out by Schrijver [19], Theorem 3 is equivalent to many problems from combinatorial optimization. Thus our results are equivalent to them, too.

As pointed out by Poljak [16], especially interesting is the similarity with the following flow model of Lawler and Martel [11]. By a *polymatroidal flow network* $\mathscr{F}$ we mean a directed multigraph $G = (V, E)$ with a *source* s, a *sink* t and a collection of polymatroids $\mathbb{P}_v^+ = (\Delta_v^+, \varrho_v^+)$, $\mathbb{P}_v^- = (\Delta_v^-, \varrho_v^-)$ where $v \in V$ and Δ_v^+ (Δ_v^-) denotes the set of arcs directed into (out of) v. By a *flow* in $\mathscr{F}$ we mean any vector $f \in \mathbb{R}_+^E$. A flow f is called *feasible* in $\mathscr{F}$ if it satisfies the following conditions:

$$f(\Delta_v^+) = f(\Delta_v^-) \text{ for any } v \in V \setminus \{s,t\},$$
$$f|\Delta_v^+ \text{ is independent in } \mathbb{P}_v^+ \text{ for any } v \in V,$$
$$f|\Delta_v^- \text{ is independent in } \mathbb{P}_v^- \text{ for any } v \in V.$$

By a *value* of a feasible flow f we mean the quantity $v = f(\Delta_s^-) - f(\Delta_s^+) = f(\Delta_t^+) - f(\Delta_t^-)$. A polymatroidal network flow is called *integral* if $\mathbb{P}_v^+$ and $\mathbb{P}_v^-$ are integral for any $v \in V$.

An *arc-partitioned cut* (S,T,L,U) is defined by a partition of vertices into two sets S and T with $s \in S$, $t \in T$ and by a partition of the arcs directed from S to T into two sets L and U. The *capacity* of an arc partitioned cut is defined as

$$c(S,T,L,U) = \sum_{v \in S} \varrho_v^-(U \cap \Delta_v^-) + \sum_{v \in T} \varrho_v^+(L \cap \Delta_v^+).$$

In [11] it is shown that this flow model has the max-flow min-cut property.

Theorem 6. *Let $\mathscr{F}$ be an (integral) polymatroidal flow network. Then the maximal value of an (integral) feasible flow is equal to the minimum capacity of an arc-partitioned cut.*

It is easy to check that Theorem 4 follows from Theorem 6. On the other hand Theorem 6 follows from Theorem 3 (see e.g. [19]) and, thus, also from Theorem 4. Therefore Theorems 3, 4, and 6 are pairwise equivalent.

Note that there exists no analogue of Theorem 5 in flow theory. On the other hand it presents a very natural generalization of results from transversal theory, especially those of Edmonds and Fulkerson [4], Mirsky and Perfect [14] and Perfect [15].

Theorem 5 describes in fact an operation on polymatroids. This "transversal" operation creates the polymatroid $\mathbb{P}$ from a polymatroid $\mathbb{P}_2$ and finite systems of polymatroids $\mathscr{P}_S$ and $\mathscr{P}_T$ (thus we can call $\mathbb{P}$ the *polymatroid of* $(S,T,\mathscr{P}_S,\mathscr{P}_T,\mathbb{P}_2)$-*polytransversals*). It is easy to check that the operation from Lemma 1, the operations truncation and restriction on polymatroids, the polymatroid sum (see [18, pages 351–352]) and thus also the product of polymatroids can be characterized as special cases of the "transversal" operation.

It is well known (see e.g. [20], [1] for more details) that not every matroid can be characterized as a transversal matroid. Nevertheless, every polymatroid $\mathbb{P} = (S, \varrho)$ can be characterized as an $(S,T,\mathscr{P}_S,\mathscr{P}_T,\mathbb{P}_2)$-polytransversal. It suffices to set $T = \{1\}$, $\mathscr{P}_T = (\mathbb{P}_1 = \mathbb{P})$, $\mathscr{P}_S = (\mathbb{P}_s = \mathbb{U}_{k,S} \colon s \in S)$ and $\mathbb{P}_2 = \mathbb{U}_{k,T}$, where k is sufficiently large. Then it is routine to check that the polymatroid of $(S, T, \mathscr{P}_S, \mathscr{P}_T, \mathbb{P}_2)$-polytransversals is equal to $\mathbb{P}$. The situation could change if we required some restrictions for polymatroids from $\mathscr{P}_S$ and $\mathscr{P}_T$. For instance what will happen if $\mathscr{P}_S$ and $\mathscr{P}_T$ are systems of polymatroids of uniform matroids? This could generalize the characterization of transversal matroids (see [20] for more details).

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