ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS TO A ONE-DIMENSIONAL MOTION OF COMPRESSIBLE VISCOUS FLUIDS

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Abstract. We study the one-dimensional motion of the viscous gas represented by the system $v_t - u_x = 0$, $u_t + p(v)_x = \mu(u_x/v)_x + f\left(\int_0^x v\,\mathrm{d}x, t\right)$, with the initial and the boundary conditions $(v(x,0), u(x,0)) = (v_0(x), u_0(x))$, $u(0,t) = u(X,t) = 0$. We are concerned with the external forces, namely the function f, which do not become small for large time t. The main purpose is to show how the solution to this problem behaves around the stationary one, and the proof is based on an elementary L^2-energy method.

Keywords: compressible viscous gas, asymptotic behaviour of the solutions

MSC 1991: 35Q30, 76N10, 76N15

1. Introduction

In this paper we study the asymptotic behavior of solutions to the one-dimensional motion of the viscous gas on a finite interval. In Lagrangian mass coordinate, such a motion is described by the system of equations

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t + p(v)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x v\,\mathrm{d}x, t\right), \tag{1.2}$$

where v, u, p, μ, and f are the specific volume, the velocity, the pressure, the viscosity coefficient, and the external force of the fluid, respectively. We consider these equations in a fixed domain Q_∞ defined by

$$Q_\infty = \{(x,t) \mid 0 < x < X,\ t > 0\}, \tag{1.3}$$

together with the initial and the boundary conditions

$$v(x,0) = v_0(x),\ u(x,0) = u_0(x) \quad \text{on } 0 < x < X, \tag{1.4}$$

$$u(0,t) = u(X,t) = 0 \quad \text{on } t > 0. \tag{1.5}$$

This and related problems have been investigated by a number of authors including Kanel' [5], Itaya [3, 4], Kazhikhov [6], Kazhikhov & Shelukhin [9], Kazhikhov & Nikolaev [7, 8], and so on. For their results and the historical progress, we could refer to the paper of Solonnikov & Kazhikhov [12].

Now we proceed to review this problem in the presence of external forces. Matsumura & Nishida [11] proved the global existence of a solution for any external forces with its derivatives and itself being bounded, assuming that the gas is ideal and isothermal, and obtained the estimate

$$C_0^{-1} \leqslant v(x,t) \leqslant C_0 \qquad \text{for} \quad (x,t) \in Q_\infty, \tag{1.6}$$

where C_0 is a positive constant. Recently, Matsumura [10] improved their results, showing that the solution is exponentially stable if the external force depends only on $\xi = \int_0^x v\,\mathrm{d}x$ and its derivative with respect to ξ is sufficiently small. For a general barotropic gas, Tani obtained in his lecture note [13] the exponential stability of the solution if $f(\xi,t)$ belongs to $L^1(0,\infty;L^\infty(I)) \cap L^2(I \times (0,\infty))$, where $I = [0, \int_0^X v_0\,\mathrm{d}x]$. We shall also mention the papers of Beirão da Veiga. In [2], he proved the global existence of a solution if some norm of the initial date is bounded by constant which is determined by the L^∞-norm of f. We notice that his conclusion is not a result for small data, because the constant mentioned above tends to infinity as the L^∞-norm of f tends to 0. In [1], he also obtained, in his words, the complete characterization of time independent external forces for which the corresponding stationary solutions are known to exist (see also [2]). Finally, we shall show Zlotnik's interesting results. In [15], he proved that if the stationary state of the external force is a decreasing function of ξ, then the solution is exponentially stable.

Our interest in the present paper is to investigate the asymptotic behavior of the solution with external forces depending on time t and not becoming small for large time. We will consider two cases, namely we will investigate an ideal gas in Section 2, and a general barotropic gas in Section 3. In what follows, we assume that the viscosity coefficient is a positive constant, and that the external force $f = f(\xi,t)$, $\xi = \int_0^x v\,\mathrm{d}x$ has a limit function $\hat{f}(\xi)$ in $L^\infty(I)$ satisfying

$$f_0(\xi,t) \equiv f(\xi,t) - \hat{f}(\xi) \in L^2\left(0,\infty;L^\infty(I)\right), \tag{1.7}$$

where $I = [0, \int_0^X v_0 \, \mathrm{d}x]$. To obtain the *strong solution* (see [2], for example), we impose the following assumptions on the initial data and the external force:

$$(v_0, u_0) \in H^1(0, X) \times H_0^1(0, X), \quad \inf_x v_0(x) > 0, \tag{1.8}$$

$$f, \quad f_\xi, \quad \text{and} \quad f_t \in L^\infty\left(I \times (0, \infty)\right), \tag{1.9}$$

where H^k and H_0^k $(k \geqslant 0)$ are the usual Sobolev's spaces with the norm $\|\cdot\|_k$, and we use the notation $\|\cdot\|$ instead of $\|\cdot\|_0$.

2. The case of $p = \dfrac{a}{v}$

2.1. Stationary Problem and Theorem. In this section, we assume that the gas is ideal, i.e.

$$p(v) = \frac{a}{v} \quad (a \text{ being positive constant}). \tag{2.1}$$

Then the equation (1.2) is reduced to

$$u_t + \left(\frac{a}{v}\right)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x v \, \mathrm{d}x, t\right). \tag{2.2}$$

For the global existence of a solution to our system, we have already known the following theorem [11]:

Theorem 2.1. (Matsumura & Nishida) *Assume* (1.8) *and* (1.9). *Then the initial and boundary value problem* (1.1), (1.4), (1.5), (2.2) *has a unique global solution in* $C^0([0,\infty); H^1 \times H_0^1)$ *satisfying* (1.6) *and the estimate*

$$\sup_{t \geqslant 0} \|(v, u)(t)\|_1 \leqslant C(\|(v_0, u_0)\|_1, \ \inf_x v_0, |f|_\infty). \tag{2.3}$$

In order to investigate the asymptotic behavior of the solution, it is necessary to consider the stationary problem. Let $(\eta(x), 0)$ be the stationary solution to (1.1), (1.4), (1.5) and (2.2), then the function $\eta(x)$ must satisfy the system of equations

$$\left(\frac{a}{\eta}\right)_x = \hat{f}\left(\int_0^x \eta \, \mathrm{d}x\right), \tag{2.4}$$

$$\int_0^X \eta(x) \, \mathrm{d}x = \int_0^X v_0(x) \, \mathrm{d}x \quad (\equiv Y). \tag{2.5}$$

We can easily see that this stationary problem has a unique solution in the following way. Let $w(x)$ be defined by $w(x) = \int_0^x \eta \, \mathrm{d}x$. Then (2.4) and (2.5) are reduced to

$$\left(\frac{a}{w_x}\right)_x = \hat{f}(w), \tag{2.6}$$

$$w(0) = 0, \quad w(X) = Y. \tag{2.7}$$

We rewrite (2.6) as

$$-a\frac{w_{xx}}{w_x} = F(w)_x, \tag{2.8}$$

where $F(w)$ is defined by $F(w) = \int_0^w \hat{f}(\xi)\,\mathrm{d}\xi$. Integration of (2.8) with respect to x yields

$$w_x = A\mathrm{e}^{-\frac{1}{a}F(w)}, \tag{2.9}$$

where A is a constant. Since $F(w)$ is a Lipschitz continuous function, the initial value problem (2.9) with $w(0) = 0$ in (2.7) has a unique solution for an arbitrary fixed constant A. We now proceed to show that there is a unique constant A for which the above solution satisfies the relation $w(X) = Y$ in (2.7). As the proof of the existence is trivial, we shall only prove the uniqueness. We note that $A > 0$ because of $Y > 0$. Let A and B satisfy $A > B(> 0)$, and let w_A, w_B be the corresponding unique solutions to (2.9) with $w(0) = 0$. It is enough to show that $w_A(x) > w_B(x)$ for $0 < x \leqslant X$. We shall prove it by reductio ad absurdum. We assume that there exists a point $x_0 \in (0, X]$ such that $w_A(x_0) = w_B(x_0)$ and $w_A(x) > w_B(x)$ for $0 < x < x_0$. Then we must have $w_{Ax}(x_0) \leqslant w_{Bx}(x_0)$. On the other hand, from (2.9), we have $w_{Ax}(x_0) > w_{Bx}(x_0)$. This is a contradiction.

Then our first main theorem is

Theorem 2.2. *Assume* (1.7)–(1.9). *Let* (v, u) *be the unique global solution to* (1.1), (1.4), (1.5), (2.2), *and let* η *be the stationary solution mentioned above. Then there exist constants* $\varepsilon_0 > 0$, $\delta > 0$ *and* $C > 0$ *which depend only on the given data such that if* $|f_\xi|_\infty \leqslant \varepsilon_0$ *then the following estimate is satisfied for all* $t \geqslant 0$:

$$\|(v-\eta)(t)\|_1^2 + \|u(t)\|_1^2 \leqslant C\mathrm{e}^{-\delta t}\left(1 + \int_0^t \mathrm{e}^{\delta s}|f_0(s)|_\infty^2 \,\mathrm{d}s\right). \tag{2.10}$$

The proof of this theorem is done in Section 2.3. In Section 2.2, we will show some energy estimates used in Section 2.3.

2.2. Energy Estimates. In what follows, we shall denote by the letter C a universal constant which depends only on the given data. We first prove the following lemma.

Lemma 2.3. *Let* (v,u) *be the unique solution of* (1.1), (1.4), (1.5), (2.2), *and let* η *be the unique solution of* (2.4), (2.5). *Then the following estimate is valid for all* $t\geqslant 0$:

(2.11)
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X\left\{\frac{1}{2}u^2+P(v,\eta)\right\}\,\mathrm{d}x+\frac{\mu}{2}\int_0^X\frac{u_x^2}{v}\,\mathrm{d}x\leqslant C\biggl(|f_\xi|_\infty\int_0^X vQ_x^2\,\mathrm{d}x+|f_0(t)|_\infty^2\biggr),$$

where P *and* Q *are defined by* $P(v,\eta)=a(\frac{v}{\eta}+\log\frac{\eta}{v}-1)\geqslant 0$ *and* $Q=\frac{a}{v}-\frac{a}{\eta}$, *respectively, and where* $|f_\xi|_\infty$ *denotes the* $L^\infty(I\times(0,\infty))$*-norm of* f_ξ, *while* $|f_0(t)|_\infty$ *denotes the* $L^\infty(I)$*-norm of* f_0.

Proof. We rewrite the equation (2.2) in the form

$$\begin{aligned}(2.12)\qquad u_t+Q_x&=\mu\left(\frac{u_x}{v}\right)_x+f\biggl(\int_0^x v\,\mathrm{d}x,t\biggr)-\hat f\biggl(\int_0^x\eta\,\mathrm{d}x\biggr)\\&=\mu\left(\frac{u_x}{v}\right)_x+f\biggl(\int_0^x v\,\mathrm{d}x,t\biggr)-f\biggl(\int_0^x\eta\,\mathrm{d}x,t\biggr)\\&\quad+f\biggl(\int_0^x\eta\,\mathrm{d}x,t\biggr)-\hat f\biggl(\int_0^x\eta\,\mathrm{d}x\biggr)\\&=\mu\left(\frac{u_x}{v}\right)_x+f_\xi(\cdot,t)\int_0^x(v-\eta)\,\mathrm{d}x+f_0\biggl(\int_0^x\eta\,\mathrm{d}x,t\biggr),\end{aligned}$$

where we have used the relation (2.4). We multiply (1.1) by $-Q$, (2.12) by u and add the results. Integration of this equation over $[0,X]$ yields

(2.13)
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X\left\{\frac{1}{2}u^2+P(v,\eta)\right\}\,\mathrm{d}x+\mu\int_0^X\frac{u_x^2}{v}\,\mathrm{d}x=\int_0^X f_\xi u\,\mathrm{d}x\int_0^x(v-\eta)\,\mathrm{d}x'+\int_0^X f_0u\,\mathrm{d}x.$$

Using (1.6) and the relation $\|u\|\leqslant C\|u_x\|$, each term of the right hand side of (2.13) is estimated as

$$\begin{aligned}(2.14)\qquad\left|\int_0^X f_\xi u\,\mathrm{d}x\int_0^x(v-\eta)\,\mathrm{d}x'\right|&\leqslant|f_\xi|_\infty\int_0^X|u|\,\mathrm{d}x\int_0^x|v-\eta|\,\mathrm{d}x'\\&\leqslant\frac{\mu}{4}\int_0^X\frac{u_x^2}{v}\,\mathrm{d}x+C|f_\xi|_\infty\int_0^X vQ^2\,\mathrm{d}x,\end{aligned}$$

$$\left|\int_0^X f_0 u\,\mathrm{d}x\right| \leqslant \frac{\mu}{4}\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + C|f_0(t)|_\infty^2. \tag{2.15}$$

As discussed in [14], there exists $X_1(t) \in [0, X]$ such that $v(X_1(t), t) = \eta(X_1(t))$, so that Q can be represented by $Q = \int_{X_1(t)}^x Q_x\,\mathrm{d}x$, which gives the relation $\|Q\| \leqslant C\|Q_x\|$. From (2.13)–(2.15) and the above relation, we obtain (2.11). □

In the next lemma, we shall estimate Q_x.

Lemma 2.4. *In the same situation as in Lemma 2.3, the following estimate is satisfied for all $t \geqslant 0$:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X \Big\{\frac{\mu}{2a}(vQ_x)^2 + uvQ_x\Big\}\,\mathrm{d}x + \Big(\frac{1}{2} - C|f_\xi|_\infty\Big)\int_0^X vQ_x^2\,\mathrm{d}x \\ \leqslant C\Big(\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + |f_0(t)|_\infty^2\Big).\end{aligned} \tag{2.16}$$

P r o o f. Owing to the relation $v_t = u_x$, it is easy to see that

$$(vQ_x)_t + \Big(\frac{a}{\eta}\Big)_x u_x = \Big(-\frac{au_x}{v}\Big)_x. \tag{2.17}$$

Thus we can rewrite (2.12) in the form

$$u_t + Q_x + \frac{\mu}{a}(vQ_x)_t + \frac{\mu}{a}\hat{f}\Big(\int_0^x \eta\,\mathrm{d}x\Big)u_x = f_\xi(\cdot,t)\int_0^x (v-\eta)\,\mathrm{d}x + f_0\Big(\int_0^x \eta\,\mathrm{d}x, t\Big). \tag{2.18}$$

Multiplying (2.18) by vQ_x and integrating it over $[0, X]$ gives

$$\begin{aligned}\frac{\mu}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X (vQ_x)^2\,\mathrm{d}x + \int_0^X vQ_x^2\,\mathrm{d}x + \int_0^X u_t vQ_x\,\mathrm{d}x + \frac{\mu}{a}\int_0^X \hat{f}u_x vQ_x\,\mathrm{d}x \\ = \int_0^X f_\xi vQ_x\,\mathrm{d}x\int_0^x (v-\eta)\,\mathrm{d}x' + \int_0^X f_0 vQ_x\,\mathrm{d}x.\end{aligned} \tag{2.19}$$

The third term on the left hand side of (2.19) is calculated as follows:

$$\begin{aligned}\int_0^X u_t vQ_x\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^X uvQ_x\,\mathrm{d}x - \int_0^X u(vQ_x)_t\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^X uvQ_x\,\mathrm{d}x + \int_0^X u\Big\{\hat{f}u_x + \Big(\frac{au_x}{v}\Big)_x\Big\}\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^X uvQ_x\,\mathrm{d}x - \int_0^X \frac{au_x^2}{v}\,\mathrm{d}x + \int_0^X uu_x\hat{f}\,\mathrm{d}x,\end{aligned} \tag{2.20}$$

where we have used (2.17). By using (1.6) and Schwarz's inequality, we conclude from (2.19) and (2.20) that

$$
\begin{aligned}
(2.21)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^X \Big\{\frac{\mu}{2a}(vQ_x)^2 + uvQ_x\Big\}\,\mathrm{d}x + \int_0^X vQ_x^2\,\mathrm{d}x \\
&= \int_0^X \frac{au_x^2}{v}\,\mathrm{d}x - \frac{\mu}{a}\int_0^X \hat{f}u_xvQ_x\,\mathrm{d}x - \int_0^X uu_x\hat{f}\,\mathrm{d}x \\
&\quad + \int_0^X f_\xi vQ_x\,\mathrm{d}x \int_0^x (v-\eta)\,\mathrm{d}x' + \int_0^X f_0vQ_x\,\mathrm{d}x \\
&\leqslant a\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + \frac{1}{4}\int_0^X vQ_x^2\,\mathrm{d}x + C\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x \\
&\quad + C|f_\xi|_\infty \int_0^X vQ_x^2\,\mathrm{d}x + \frac{1}{4}\int_0^X vQ_x^2\,\mathrm{d}x + C|f_0(t)|_\infty^2 \\
&= \Big(\frac{1}{2} + C|f_\xi|_\infty\Big)\int_0^X vQ_x^2\,\mathrm{d}x + C\Big(\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + |f_0(t)|_\infty^2\Big).
\end{aligned}
$$

This completes the proof of Lemma 2.4. □

We finally estimate u_x.

Lemma 2.5. *In the same situation as in Lemma 2.3, we have the following estimate for all $t \geqslant 0$:*

$$
(2.22)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X u_x^2\,\mathrm{d}x + \frac{\mu}{2}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x \leqslant C\Big(\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + \int_0^X vQ_x^2\,\mathrm{d}x + |f_0(t)|_\infty^2\Big).
$$

P r o o f. Multiplying (2.12) by $-u_{xx}$ and integrating it over $[0,X]$ yields

$$
\begin{aligned}
(2.23)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X u_x^2\,\mathrm{d}x + \mu\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x \\
&= \int_0^X Q_xu_{xx}\,\mathrm{d}x + \mu\int_0^X \frac{v_xu_xu_{xx}}{v^2}\,\mathrm{d}x \\
&\qquad - \int_0^X f_\xi u_{xx}\,\mathrm{d}x\int_0^x (v-\eta)\,\mathrm{d}x' - \int_0^X f_0u_{xx}\,\mathrm{d}x.
\end{aligned}
$$

Each term on the right hand side of (2.23) is estimated as follows. First by using Schwarz's inequality,

$$
(2.24)\qquad \Big|\int_0^X Q_xu_{xx}\,\mathrm{d}x\Big| \leqslant \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x + C\int_0^X vQ_x^2\,\mathrm{d}x.
$$

Next,

$$\mu\left|\int_0^X \frac{v_x u_x u_{xx}}{v^2}\,\mathrm{d}x\right| \leqslant \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x + C\int_0^X v_x^2 u_x^2\,\mathrm{d}x. \tag{2.25}$$

Because of $u(0,t) = u(X,t) = 0$, there exists $X_2(t) \in [0, X]$ such that $u_x(X_2(t), t) = 0$, so that

$$u_x^2 = \int_{X_2(t)}^x \frac{\partial}{\partial x} u_x^2\,\mathrm{d}x \leqslant 2\int_0^X |u_x u_{xx}|\,\mathrm{d}x \leqslant \varepsilon\int_0^X u_{xx}^2\,\mathrm{d}x + C\int_0^X u_x^2\,\mathrm{d}x \tag{2.26}$$

for any small $\varepsilon > 0$. Therefore, the last term on the right hand side of (2.25) is estimated as follows:

$$C\int_0^X v_x^2 u_x^2\,\mathrm{d}x \leqslant \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x + C\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x. \tag{2.27}$$

Here we have used (2.3). Next,

$$\left|\int_0^X f_\xi u_{xx}\,\mathrm{d}x \int_0^x (v-\eta)\,\mathrm{d}x'\right| \leqslant \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x + C\int_0^X vQ_x^2\,\mathrm{d}x. \tag{2.28}$$

Finally,

$$\left|\int_0^X f_0 u_{xx}\,\mathrm{d}x\right| \leqslant \frac{\mu}{10}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x + C|f_0(t)|_\infty^2. \tag{2.29}$$

Inserting the above inequalities (2.24)–(2.29) into (2.23), we immediately obtain (2.22). □

2.3. Proof of Theorem 2.2. We are now in position to prove the Theorem 2.2. Multiplying (2.16) by θ_1, (2.22) by θ_2 and adding the results together with (2.11) implies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E^2(t) + \Big(\frac{\mu}{2} - C\theta_1 - C\theta_2\Big)\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x \\ &\quad + \left(\frac{\theta_1}{2} - C\,(1+\theta_1)\,|f_\xi|_\infty - C\theta_2\right)\int_0^X vQ_x^2\,\mathrm{d}x + \frac{\mu\theta_2}{2}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x \\ &\leqslant C(1+\theta_1+\theta_2)|f_0(t)|_\infty^2, \end{aligned} \tag{2.30}$$

where $E^2(t)$ is defined by

$$E^2(t) = \int_0^X \left\{\frac{1}{2}u^2 + P(v,\eta) + \frac{\mu\theta_1}{2a}(vQ_x)^2 + \theta_1 uvQ_x + \frac{\theta_2}{2}u_x^2\right\}\mathrm{d}x. \tag{2.31}$$

Using Schwarz's inequality, we can estimate the term $\theta_1 uvQ_x$ as

$$|\theta_1 uvQ_x| \leqslant \frac{\mu\theta_1}{4a}(vQ_x)^2 + \frac{a\theta_1}{\mu}u^2. \tag{2.32}$$

Thus if $|f_\xi|_\infty$ is sufficiently small, we can choose the positive constants θ_1 and θ_2 to satisfy

$$\frac{\mu}{2} - C\theta_1 - C\theta_2 > 0, \quad \frac{\theta_1}{2} - C(1+\theta_1)|f_\xi|_\infty - C\theta_2 > 0, \quad \frac{1}{2} - \frac{a\theta_1}{\mu} > 0, \tag{2.33}$$

so that $E^2(t) \geqslant 0$, and the coefficient on the second and the third term of the left hand side of (2.30) is positive. We note that because of (1.6), P and Q^2 are equivalent. Furthermore, as stated in Section 2.2, we have the relation $\|Q\| \leqslant C\|Q_x\|$. Thus it follows from these remarks and (2.30) that there exists a positive constant δ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^2(t) + \delta E^2(t) \leqslant C|f_0(t)|_\infty^2 \tag{2.34}$$

holds for all $t \geqslant 0$, from which we obtain

$$E^2(t) \leqslant E^2(0)\mathrm{e}^{-\delta t} + C\mathrm{e}^{-\delta t}\int_0^t \mathrm{e}^{\delta s}|f_0(s)|_\infty^2 \,\mathrm{d}s. \tag{2.35}$$

It is easy to see that (2.35) implies (2.10), and the proof of Theorem 2.2 is complete. $\square$

3. The case of $p = av^{-\gamma}$, $\gamma > 1$

3.1. Stationary Problem and Theorem. In this section we consider the general barotropic gas represented by

$$p(v) = av^{-\gamma} \quad (a > 0, \gamma > 1 \ \text{ constants}). \tag{3.1}$$

Then the equation (1.2) is reduced to

$$u_t + \left(\frac{a}{v^\gamma}\right)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x v\,\mathrm{d}x, t\right). \tag{3.2}$$

As mentioned in Section 1, we have already known the following global existence theorem [2]:

Theorem 3.1. (H. Beirão da Veiga) *Assume* (1.8) *and* (1.9). *Then there exists a decreasing function* $R(\cdot)$ *satisfying* $R(0) = \infty$ *such that if* $\|(v_0, u_0)\|_1 \leqslant R(|f|_\infty)$ *then the initial and boundary value problem* (1.1), (1.4), (1.5), (2.2) *has a unique global solution in* $C^0([0,\infty); H^1 \times H_0^1)$ *satisfying* (1.6) *and* (2.3).

The stationary problem considered in this section is

$$\left(\frac{a}{\eta^\gamma}\right)_x = \hat{f}\left(\int_0^x \eta \,\mathrm{d}x\right), \tag{3.3}$$

$$\int_0^X \eta(x)\,\mathrm{d}x = \int_0^X v_0(x)\,\mathrm{d}x \quad (\equiv Y). \tag{3.4}$$

Performing the same calculation as in Section 2.1, (3.3) and (3.4) are rewritten as

$$\Phi(w_x)_x = F(w)_x, \tag{3.5}$$

$$w(0) = 0, \quad w(X) = Y, \tag{3.6}$$

where $w(x)$, $\Phi(w)$, and $F(w)$ are defined by $w(x) = \int_0^x \eta\,\mathrm{d}x$, $\Phi(w) = \frac{a\gamma}{\gamma-1}(w^{1-\gamma} - 1)$, and $F(w) = \int_0^w \hat{f}(\xi)\,\mathrm{d}\xi$.

From (3.5) we have

$$\Phi(w_x) = F(w) + c, \tag{3.7}$$

where c is a constant. Let M and m be defined by $M = \max\limits_{0\leqslant w\leqslant Y} F(w)$ and $m = \min\limits_{0\leqslant w\leqslant Y} F(w)$. Then we must have

$$m + c > -\frac{a\gamma}{\gamma - 1} \quad \left(= \inf_{0\leqslant w\leqslant Y} \Phi(w)\right), \tag{3.8}$$

because we are looking for a solution that satisfies $\inf\limits_{0\leqslant x\leqslant X} \eta(x) > 0$. Now let us fix a constant c that satisfies (3.8). Since $\Phi(w)$ is a decreasing function of w, we can solve (3.7) obtaining

$$w_x = \Phi^{-1}\left(F(w) + c\right). \tag{3.9}$$

It is easy to see that the initial value problem (3.9) with $w(0) = 0$ in (3.6) has a unique solution for an arbitrary fixed constant c satisfying (3.8), and we denote this solution by $w_c(x)$. The unique existence of a constant c satisfying $w_c(X) = Y$ is our problem. As the proof of the uniqueness is easily verified by using the comparison theorem, we shall only consider the existence. Integration of (3.9) over $[0, X]$ yields

$$Y = \int_0^X \Phi^{-1}\left(F(w) + c\right)\,\mathrm{d}x. \tag{3.10}$$

Thus the necessary and sufficient condition for the existence is given by

$$\lim_{c\to -m-\frac{a\gamma}{\gamma-1}} \int_0^X \Phi^{-1}\left(F(w)+c\right)\,\mathrm{d}x > Y, \tag{3.11}$$

from which we obtain one of the sufficient condition as follows:

$$\Phi\left(\frac{Y}{X}\right) > M - m - \frac{a\gamma}{\gamma-1}. \tag{3.12}$$

Then our final main theorem is

Theorem 3.2. *Assume the hypotheses as in Theorem* 3.1 *and the existence of the stationary solution. Then there exist constants* $\varepsilon_0 > 0$, $\delta > 0$ *and* $C > 0$ *which depend only on the given data such that if* $|f_\xi|_\infty \leqslant \varepsilon_0$ *then the following estimate is satisfied for all* $t \geqslant 0$*:*

$$\|(v-\eta)(t)\|_1^2 + \|u(t)\|_1^2 \leqslant C\mathrm{e}^{-\delta t}\left(1+\int_0^t \mathrm{e}^{\delta s}|f_0(s)|_\infty^2\,\mathrm{d}s\right). \tag{3.13}$$

The proof of this theorem is similar to that of Theorem 2.2, so we will only show the sketch of proof in the next subsection.

3.2. Sketch of Proof of Theorem 3.2. As in Section 2.2, we derive the following three energy estimates.

Lemma 3.3. *Let* (v,u) *be the unique solution of* (1.1), (1.4), (1.5), (3.2), *and let* η *be the unique solution of* (3.3), (3.4)*. Then the following estimate is valid for all* $t \geqslant 0$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X \left\{\frac{1}{2}u^2 + P(v,\eta)\right\}\mathrm{d}x + \frac{\mu}{2}\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x \leqslant C\Big(|f_\xi|_\infty \int_0^X v^\gamma Q_x^2\,\mathrm{d}x + |f_0(t)|_\infty^2\Big), \tag{3.14}$$

where P *and* Q *are defined by* $P(v,\eta) = a\left(\frac{1}{\gamma-1}v^{-\gamma+1} + v\eta^{-\gamma} - \frac{\gamma}{\gamma-1}\eta^{-\gamma+1}\right) \geqslant 0$ *and* $Q = \frac{a}{v^\gamma} - \frac{a}{\eta^\gamma}$*, respectively.*

Lemma 3.4. *In the same situation as in Lemma* 3.3*, the following estimate is satisfied for all* $t \geqslant 0$*:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X &\left\{\frac{\mu}{2a\gamma}(v^\gamma Q_x)^2 + uv^\gamma Q_x\right\}\mathrm{d}x + \left(\frac{1}{2} - C|f_\xi|_\infty\right)\int_0^X v^\gamma Q_x^2\,\mathrm{d}x \\ &\leqslant C\Big(\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + |f_0(t)|_\infty^2\Big).\end{aligned} \tag{3.15}$$

Lemma 3.5. *In the same situation as in Lemma* 3.3, *we have the following estimate for all* $t \geqslant 0$*:*

$$\text{(3.16)}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^X u_x^2\,\mathrm{d}x + \frac{\mu}{2}\int_0^X \frac{u_{xx}^2}{v}\,\mathrm{d}x \leqslant C\left(\int_0^X \frac{u_x^2}{v}\,\mathrm{d}x + \int_0^X v^\gamma Q_x^2\,\mathrm{d}x + |f_0(t)|_\infty^2\right).$$

The proof of these lemmas is done by the same procedure as in Lemma 2.3–Lemma 2.5, and we ommit it only noting that we use the following relation in Lemma 3.4 instead of (2.17):

$$\text{(3.17)}\qquad (v^\gamma Q_x)_t + \left(\frac{a}{\eta^\gamma}\right)_x \gamma v^{\gamma-1} u_x = -\gamma a \left(\frac{u_x}{v}\right)_x.$$

Now the proof of Theorem 3.2 is easy; with these three inequalities, the same consideration as in Section 2.3 leads to Theorem 3.2.

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