ON TORSION OF A 3-WEB

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Abstract. A 3-web on a smooth $2n$-dimensional manifold can be regarded locally as a triple of integrable n-distributions which are pairwise complementary, [5]; that is, we can work on the tangent bundle only. This approach enables us to describe a 3-web and its properties by invariant $(1,1)$-tensor fields P and B where P is a projector and $B^2 = \mathrm{id}$. The canonical Chern connection of a web-manifold can be introduced using this tensor fields, [1]. Our aim is to express the torsion tensor T of the Chern connection through the Nijenhuis $(1,2)$-tensor field $[P,B]$, and to verify that $[P,B]=0$ is a necessary and sufficient conditions for vanishing of the torsion T.

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All objects under considerations will be supposed to be of the class C^∞ (smooth).

1. An (ordered) three-web on a manifold M can be defined as an ordered triple $\mathcal{W} = (D_1, D_2, D_3)$ of integrable distributions of dimension n such that the tangent bundle is a Whitney sum of each couple of them, $TM = D_1 \oplus D_2 = D_2 \oplus D_3 = D_1 \oplus D_3$. Obviously, the web manifold has an even dimension $2n$.

It was proved in [1], [5] that an ordered 3-web on a smooth $2n$-dimensional manifold M_{2n} can be introduced as a couple (P,B) of differentiable (1,1)-tensor fields on M satisfying on M the polynomial equations

$$P^2 - P = 0, \qquad B^2 - I = 0, \tag{1}$$

the identity $B = BP + PB$, and the integrability conditions

$$[P,P] = 0, \qquad [B,B](X,Y) = 0 \qquad \text{for } X,Y \in \ker(B-I) \tag{2}$$

by which the integrability of all the three web distributions is guaranteed. From this viewpoint, a 3-web is an integrable $\{P, B\}$-structure introduced in [1].

Let us denote

$$D_1 = \ker(I - P) = \mathrm{im}\ P, \quad D_2 = \ker P = \mathrm{im}\ (I - P), \quad D_3 = \ker(B - I).$$

Then (D_1, D_2, D_3) satisfies the above definition of a 3-web, and three foliations of integral submanifolds of our distributions form a 3-web in the classical approach.

Let use denote by $\tilde{P} = I - P$ the complementary projector. The following equalities are obvious:

$$P\tilde{P} = \tilde{P}P = 0, \quad PBP = \tilde{P}B\tilde{P} = 0, \quad PB = B\tilde{P}, \quad BP = \tilde{P}B. \tag{3}$$

In [5], all linear connections $\tilde{\nabla}$ were found with respect to which the web distributions D_1, D_2, D_3 are parallel. This property is expressed by the condition saying that both P and B are covariantly constant:

$$\tilde{\nabla} P = 0, \qquad \tilde{\nabla} B = 0. \tag{4}$$

All such connections form a $2n^3$-parameter family, [5]. Among these distributions preserving connections, there exists a unique connection ∇ the torsion tensor of which satisfies

$$T(PX, \tilde{P}Y) = 0, \tag{5}$$

that is, homogeneous vectors $X \in D_{1x}$ and $Y \in D_{2x}$ are conjugated with respect to T; $x \in M$. The covariant derivative of this connection [1] is expressed by tensor fields P, B, $\tilde{P}$ defining the web as follows:

$$\nabla_X Y = PB[PX, BPY] + \tilde{P}B[\tilde{P}X, B\tilde{P}Y] + P[\tilde{P}X, PY] + \tilde{P}[PX, \tilde{P}Y]. \tag{6}$$

Its torsion tensor, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, is given by the formula

$$\begin{aligned} T(X, Y) = PB([PX, BPY] + [BPX, PY]) + \tilde{P}B([\tilde{P}X, B\tilde{P}Y] \\ + [B\tilde{P}X, \tilde{P}Y]) + [\tilde{P}X, PY] + [PX, \tilde{P}Y] - [X, Y]. \end{aligned} \tag{7}$$

Using the above notation, let us recall the proof that the formula (6) defines a covariant derivation with the properties (4), (5), and that any connection $\tilde{\nabla}$ satisfying (4), (5) coincides with ∇ described in (6).

Let ∇ be defined by (6). The additivity in both arguments follows by the additivity of tensor fields and Lie brackets occuring in the formula. We use the identities (1), (3) and

$$[fX, gY] = fg[X,Y] - Yf \cdot X + Xg \cdot Y$$

to obtain

$$\begin{aligned}
&\nabla_X fY \\
&\quad = PB\Big(f[PX, BPY] + (PXf) \cdot BPY\Big) + \tilde{P}B\Big(f[\tilde{P}X, B\tilde{P}Y] + (\tilde{P}Xf) \cdot B\tilde{P}Y\Big) \\
&\qquad + P\Big(f[\tilde{P}X, PY] + (\tilde{P}Xf) \cdot PY\Big) + \tilde{P}\Big(f[\tilde{P}X, \tilde{P}Y] + (PXf) \cdot \tilde{P}Y\Big) \\
&\quad = f\nabla_X Y + (PXf) \cdot PY + (\tilde{P}Xf) \cdot \tilde{P}Y + (\tilde{P}Xf) \cdot PY + (PXf) \cdot \tilde{P}Y \\
&\quad = f\nabla_X Y + Xf \cdot Y,
\end{aligned}$$

$$\begin{aligned}
&\nabla_{fX} Y \\
&\quad = fPB[PX, BPY] - (BPYf) \cdot PBPX + f\tilde{P}B[\tilde{P}X, B\tilde{P}Y] - (B\tilde{P}Xf) \cdot \tilde{P}B\tilde{P}X \\
&\qquad + fP[\tilde{P}X, PY] - (PYf) \cdot P\tilde{P}X + f\tilde{P}[PX, PY] - (\tilde{P}Yf) \cdot \tilde{P}PX \\
&\quad = f\nabla_X Y.
\end{aligned}$$

Further, (5) follows by a direct calculation, and

$$\begin{aligned}
&\nabla P(X;Y) = \nabla_X(PY) - P\nabla_X Y \\
&\quad = PB[PX, BP^2Y] + \tilde{P}B[\tilde{P}X, B\tilde{P}PY] + P[\tilde{P}X, PY] + \tilde{P}[PX, \tilde{P}PY] \\
&\qquad - P^2B[PX, BPY] - P\tilde{P}B[\tilde{P}X, B\tilde{P}Y] - P^2[\tilde{P}X, PY] - P\tilde{P}[PX, \tilde{P}Y] = 0,
\end{aligned}$$

$$\begin{aligned}
\nabla B(X;Y) &= PB[\tilde{P}X, \tilde{P}Y] + \tilde{P}B[\tilde{P}X, PY] + P[\tilde{P}X, PBY] + \tilde{P}[PX, \tilde{P}BY] \\
&\quad - \tilde{P}[PX, BPY] - P[\tilde{P}X, B\tilde{P}Y] - BP[\tilde{P}X, PY] - B\tilde{P}[PX, \tilde{P}Y] = 0.
\end{aligned}$$

On the other hand, let $\tilde{\nabla}$ be a connection satisfying (4) and (5). To prove that ∇ and $\tilde{\nabla}$ coincide, it suffices to calculuate the formula (6) for couples X, Y of homogeneous vector fields belonging to the distribution D_1 or D_2, and to compare it with the identities obtained for $\tilde{\nabla}$, [1].

(a) Let $X \in D_1$, $Y \in D_2$. Then $PY = 0$, $\tilde{P}X = 0$, and $T(X,Y) = 0$. Using $0 = (\tilde{\nabla}P)(X;Y) = \tilde{\nabla}_X(PY) - P(\tilde{\nabla}_X Y)$ we obtain

$$\tilde{\nabla}_X PY = P(\tilde{\nabla}_X Y) = 0,$$

that is $\tilde{\nabla}_X Y \in D_2$. In a similar way, $\tilde{\nabla}\tilde{P} = 0$ yields $\tilde{\nabla}_Y X \in D_1$. By our assumption,

$$[X,Y] = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X.$$

Since the decomposition of the Lie bracket $[X,Y]=P[X,Y]+\tilde{P}[X,Y]$ corresponding to the decomposition of the tangent bundle $TM=D_1\oplus D_2$ is uniquely determined we can write

$$-\tilde{\nabla}_Y X = P[X,Y]\in D_1, \qquad \tilde{\nabla}_X Y=\tilde{P}[X,Y]\in D_2,$$

and we obtain

$$\nabla_X Y=\tilde{P}[PX,\tilde{P}Y]=\tilde{P}[X,Y]=\tilde{\nabla}_X Y.$$

(b) Suppose $X, Y\in D_1$. In this case $\tilde{P}X=\tilde{P}Y=0$, $BY\in D_2$, $\tilde{\nabla}_X Y=B\tilde{\nabla}_X BY$. By (a), $\tilde{\nabla}_X BY=\tilde{P}[X,BY]\in D_2$. We can calculate

$$\begin{aligned}\tilde{\nabla}_X Y&=B\tilde{P}[X,BY]=PB[X,BY],\\ \nabla_X Y&=PB[PX,BPY]=PB[X,BY].\end{aligned}$$

(c) Let X, $Y\in D_2$. Then

$$\begin{aligned}\tilde{\nabla}_X Y&=B\tilde{\nabla}_X(BY)=BP[X,BY]=\tilde{P}B[X,BY],\\ \nabla_X Y&=\tilde{P}B[\tilde{P}X,B\tilde{P}Y]=\tilde{P}B[X,BY].\end{aligned}$$

2. It is well known that vanishing of the torsion tensor of the Chern connection is a necessary (but not sufficient) condition for parallelizability of a given 3-web. We will show now how this condition can be expressed in terms of the tensor fields P, B which determine the web.

Proposition. *Let a 3-web on a manifold M be defined by a couple (P,B) of $(1,1)$-tensor fields satisfying the conditions*

$$\begin{aligned}&P^2=P, \qquad B^2=I, \qquad B=BP+PB,\\ &[P,P]=0, \quad [B,B](X,Y)=0 \ \text{ for } X,Y\in\ker(B-I),\end{aligned}$$

and let T denote the torsion of the Chern connection on a given web manifold. Then

$$\begin{gathered}T\big|D_1\times D_1=B[P,B]\big|D_1\times D_1, \quad T\big|D_2\times D_2=-B[P,B]\big|D_2\times D_2,\\ T\big|D_1\times D_2=B[P,B]\big|D_1\times D_2=0\end{gathered} \tag{8}$$

and consequently,

$$T=0 \qquad \Longleftrightarrow \qquad [P,B]=0. \tag{9}$$

P r o o f. Since $PB + BP = B$ we have

$$[P,B](X,Y) = [PX,BY] + [BX,PY] + B[X,Y] \\ - P[X,BY] - B[X,PY] - P[BX,Y] - B[PX,Y],$$

and

$$B[P,B](X,Y) = B\Big([PX,BY] + [BX,PY]\Big) \\ - BP\Big([X,BY] + [BX,Y]\Big) - [X,PY] - [PX,Y] + [X,Y].$$

(i) Let both X, $Y \in D_1$. A calculation shows that

$$B[P,B](X,Y) = PB\Big([X,BY] + [BX,Y]\Big) - [X,Y],$$

and

$$T(X,Y) = PB\Big([PX,BY] + [BPX,PY]\Big) - [X,Y].$$

We see that on D_1, both tensors coincide:

$$T\big|D_1 \times D_1 = B[P,B]\big|D_1 \times D_1.$$

(ii) Now let X, $Y \in D_2$. In this case

$$\begin{aligned} B[P,B](X,Y) &= -BP[X,BY] - BP[BX,Y] + [X,Y], \\ T(X,Y) &= \tilde{P}B[X,BY] + \tilde{P}B[BX,Y] - [X,Y] \\ &= BP\Big([X,BY] + [BX,Y]\Big) - [X,Y], \end{aligned}$$

which proves that

$$T\big|D_2 \times D_2 = -B[P,B]\big|D_2 \times D_2.$$

(iii) Finally, let $X \in D_1$ and $Y \in D_2$. Then $[P,B](X,Y) = 0$, $T(X,Y) = T(PX,PY) = 0$, and

$$T\big|D_1 \times D_2 \;=\; B[P,B]\big|D_1 \times D_2 = 0.$$

Combining the above results we complete the proof of (8); (9) follows since B is an isomorphism. □

Following Russian authors, either the tensor field T, or the tensor field $[P,B]$ can be called *a torsion* of a given 3-web.

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