RELAXATION OF VECTORIAL VARIATIONAL PROBLEMS

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Abstract. Multidimensional vectorial non-quasiconvex variational problems are relaxed by means of a generalized-Young-functional technique. Selective first-order optimality conditions, having the form of an Euler-Weiestrass condition involving minors, are formulated in a special, rather a model case when the potential has a polyconvex quasiconvexification.

Keywords: relaxed variational problems, Young measures, minors of gradients, optimality conditions, Weierstrass-type maximum principle

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0. INTRODUCTION

We will deal with a general multidimensional vectorial variational problem (see [2, 3, 5–7, 10–12, 17, 20, 22] and the references therein):

$$\text{(VP)} \qquad \text{minimize} \quad \Phi(u) = \int_\Omega \varphi(x, u(x), \nabla u(x))\,\mathrm{d}x \qquad \text{for} \quad u \in W^{1,p}(\Omega;\mathbb{R}^m),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $W^{1,p}(\Omega;\mathbb{R}^m)$ the Sobolev space of functions $u\colon \Omega \to \mathbb{R}^m$ with the norm $\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \|\nabla u\|_{L^p(\Omega;\mathbb{R}^{nm})}$, and $\varphi\colon \Omega \times (\mathbb{R}^m \times \mathbb{R}^{nm}) \to \mathbb{R}\colon (x,u,A) \mapsto \varphi(x,u,A)$ a Carathéodory function; $1 < p < +\infty$, $m, n \geqslant 1$. In accord with Morrey [22], we will call $v\colon \mathbb{R}^{nm} \to \mathbb{R}$ quasiconvex if $\int_\Omega v(A + \nabla u(x))\,\mathrm{d}x \geqslant \operatorname{meas}(\Omega) v(A)$ for every matrix $A \in \mathbb{R}^{nm}$ and every $u \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. Supposing coercivity of (VP), we are especially interested in the case when $\varphi(x,u,\cdot)$ is not quasiconvex. Then the minimum in (VP) is generally not achieved. In other words, the problem (VP) may have no solution in $W^{1,p}(\Omega;\mathbb{R}^m)$ and the need of its natural extension (= relaxation) immediately appears.

The aim, pursued in Sec. 2, is to build up a general relaxation theory for (VP), covering simultaneously both the coarsest case (i.e. the relaxation by quasiconvexification of $\varphi(x,u,\cdot)$ as in [12]) and the relaxation by means of Young measures [35–37], or rather a generalization of them. An interesting result in this part says that, if the quasiconvexified problem fails to be convex, then (VP) does not admit any Hausdorff relaxation which would have a convex structure; for a precise formulation see Corollary 2.1 below.

The further aim, achieved in Sec. 3, is to formulate selective optimality conditions for the generalized solution of (VP), i.e. for the solution of the relaxed problem. This is, however, a very delicate problem related with the question of an effective characterization of (generalized) Young measures, which was investigated in [31]. We will choose here the simplest possibility, requiring a commutation with all minors; cf. (3.3). This forms only a finite number of equality constraints, but imposes quite a severe restriction on the potential density φ, namely that the quasiconvexification (= the highest quasiconvex minorant) of $\varphi(x,u,\cdot)$ is polyconvex; recall that $v\colon \mathbb{R}^{nm}\to\mathbb{R}$ is called polyconvex if $v(A)=\omega(A,\mathrm{adj}_2A,\ldots,\mathrm{adj}_{\min(n,m)}A)$ for some convex function $\omega\colon \prod_{s=1}^{\min(n,m)}\mathbb{R}^{\sigma(s)}\to\mathbb{R}$; see Ball [3]. Without this restriction, we could not guarantee sufficiency of the conditions on minors (cf. (3.3)) for the relaxed problem to be a proper relaxation of (VP) (cf. Definition 1.1), as follows from the counterexample by Ball and James [6].

The main result can be (with help of Remark 3.1 below) summarized as follows. If a couple (u,ν), where $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\nu\colon x\mapsto\nu_x$ is a Young measure representing the probability-measure-valued gradient of u, is a generalized solution of (VP) in the sense that it minimizes $j(u,\nu)=\int_\Omega\int_{\mathbb{R}^{nm}}\varphi(x,u(x),A)\nu_x(\,\mathrm{d}A)\,\mathrm{d}x$, cf. also [5, 6, 10, 20], then, for some $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\min(n,m)})\in\prod_{s=1}^{\min(n,m)}L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$,

$$\sum_{s=1}^{\min(n,m)}\mathrm{div}\left(\lambda_s\frac{\partial\mathrm{adj}_s}{\partial A}(\nabla u)\right) \;=\; \int_{\mathbb{R}^{nm}}\frac{\partial\varphi}{\partial u}(x,u(x),A)\nu_x(\,\mathrm{d}A) \tag{0.1}$$

in the sense of $W^{-1,p/(p-1)}(\Omega;\mathbb{R}^m)$ and with appropriate boundary conditions (see (3.8b)), and the following "Weierstrass-type" maximum principle is satisfied:

$$\int_{\mathbb{R}^{nm}}\mathcal{H}_{u,\lambda}(x,A)\nu_x(\,\mathrm{d}A) \;=\; \max_{A\in\mathbb{R}^{nm}}\mathcal{H}_{u,\lambda}(x,A)\qquad\text{for a.a. }x\in\Omega, \tag{0.2}$$

where the "Hamiltonian" $\mathcal{H}_{u,\lambda}\colon\Omega\times\mathbb{R}^{nm}\to\mathbb{R}$ is given by the formula

$$\mathcal{H}_{u,\lambda}(x,A) \;=\; -\varphi(x,u(x),A) \;+\; \sum_{s=1}^{\min(n,m)}\lambda_s(x)\mathrm{adj}_sA \tag{0.3}$$

with $\mathrm{adj}_s\colon \mathbb{R}^{nm} \to \mathbb{R}^{\sigma(s)}$ assigning each matrix $A \in \mathbb{R}^{nm}$ its minors of the order s, i.e. the determinants of all $s \times s$-submatrices; apparently $\sigma(s) = \binom{m}{s}\binom{n}{s} = \frac{m!}{s!(m-s)!}\frac{n!}{s!(n-s)!}$. This notation is due to [11, 12]. For the scalar case (i.e. $m = 1$), (0.1) is just one half of the Euler-Lagrange equation while (0.2)–(0.3) is the Weierstrass condition, extended in terms of Young measures, which has been already obtained in [30] but under much stronger assumptions, generalizing also the original result by Young [37] derived only for $\varphi(x,\cdot,A)$ constant. (Note that $\mathrm{adj}_1 A = A$.) In the one-dimensional case (i.e. $n = 1$) such optimality conditions were investigated already by Young [36] and McShane [21].

These conditions are well selective because, if u solves the relaxed problem obtained by quasiconvexification and ν satisfies (0.2)–(0.3) with some λ, then (u,ν) is a generalized solution of (VP). No better selectivity can probably be expected because, as outlined above, in the general case the relaxed problem cannot be made convex. Let us emphasize that if one were tempted to extend the usual Euler-Lagrange equation in terms of Young measures, such extended equation would admit enormously many "false" solutions which do not minimize the energy. Such dramatical loss of selectivity was shown in [29] even for the case $n = m = 1$.

Our necessary conditions give a rigorous method to estimate the support of (generalized) Young measure minimizers and, for $m = 1$, also to establish uniqueness on some very special occasions, cf. [18]. Such kind of investigations has been so far performed only in special cases basically always with $\lambda \equiv 0$, cf. [5, 6, 8–10], which can hardly be expected in general situations, however. For $m = 1$, see also [16].

On the other hand, due to strict assumptions, Sec. 3 is to be considered rather as a first attempt to attack the very difficult problem. Further directions should probably involve a more sophisticated set of constraints including also inequality constraints of the type of the Jensen inequalities with some quasiconvex functions (cf. [31]) and/or an extension of the potential Φ only by lower semi-continuity, and not by continuity used here.

1. An abstract relaxation pattern

To provide a straightforward insight, we first treat an abstract formulation using only classical tools. Roughly speaking, we want to construct an envelope of (VP), called an abstract relaxed problem (AP), which will always have the following structure:

$$\text{(AP)}\qquad \begin{cases} \text{minimize} & \bar{\Phi}(u,\eta) \quad \text{for} \quad (u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times H^*, \\ \text{subject to} & N(u) = L\eta,\ \ \eta \in K, \end{cases}$$

where H^* is a Banach space which is dual to a normed linear space H, and Z is a Banach space, K a closed subset of H^*, $\bar{\Phi}\colon W^{1,p}(\Omega;\mathbb{R}^m) \times H^* \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous (l.s.c.), $N\colon W^{1,p}(\Omega;\mathbb{R}^m) \to Z$ is continuous (generally nonlinear), and $L\colon H^* \to Z$ is a linear continuous mapping. (For examples of concrete Z, K, N, and L, we refer to (RVP$_H$) and (RVP$'_H$) in Sections 2 and 3, respectively.) Besides, $W^{1,p}(\Omega;\mathbb{R}^m)$ is imbedded into $W^{1,p}(\Omega;\mathbb{R}^m)\times K$ via a continuous injection i. A sequence (or a net) $\{u_\alpha\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ will be called minimizing for (VP) if $\lim \Phi(u_\alpha) = \inf(\mathrm{VP}) \equiv \inf_{u\in W^{1,p}(\Omega;\mathbb{R}^m)} \Phi(u)$. Speaking about its cluster (or limit) point in $W^{1,p}(\Omega;\mathbb{R}^m) \times H^*$, we will naturally understand the net $\{u_\alpha\}$ imbedded into $W^{1,p}(\Omega;\mathbb{R}^m) \times H^*$ via i.

Definition 1.1. (AP) will be called a proper relaxation of (VP) if, referring to the (weak×weak*)-topology on $W^{1,p}(\Omega;\mathbb{R}^m) \times H^*$, the following four properties are satisfied:
a) (AP) has a solution,
b) $\min(\mathrm{AP}) = \inf(\mathrm{VP})$,
c) every cluster point of every minimizing sequence of (VP) solves (AP),
d) every solution of (AP) can be attained by a net minimizing (VP).

Let us denote by $\mathcal{D}_{\mathrm{ad}}(\mathrm{AP}) = \{(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times K\,;\, N(u) = L\eta\}$ the admissible domain for (AP). We suppose, again referring to the (weak×weak*)-topology on $W^{1,p}(\Omega;\mathbb{R}^m) \times H^*$, that

(1.1a) $\quad i(W^{1,p}(\Omega;\mathbb{R}^m))$ is dense in $\mathcal{D}_{\mathrm{ad}}(\mathrm{AP})$,

(1.1b) $\quad \forall (u,\eta) \in \mathcal{D}_{\mathrm{ad}}(\mathrm{AP})\colon\ \bar{\Phi}(u,\eta) = \liminf_{i(u_\alpha)\to(u,\eta)} \Phi(u_\alpha)$,

(1.1c) $\quad \exists\, c > \inf(\mathrm{AP})\colon\ \{(u,\eta) \in \mathcal{D}_{\mathrm{ad}}(\mathrm{AP})\,;\, \bar{\Phi}(u,\eta) \leqslant c\}$ is compact.

Proposition 1.1. *If* (1.1) *is valid, then* (AP) *is a proper relaxation of* (VP).

The proof, being obvious, is omitted. Let us only remark that (1.1) can be used only in rather simple cases, and generally more sophisticated techniques must be employed to guarantee (AP) to be a proper relaxation of (VP), cf. Proposition 3.2 below.

Furthermore, we want to study first-order optimality conditions for the solution of (AP). We suppose, now referring to the norm topologies, that

(1.2a) $\quad \bar{\Phi}\colon W^{1,p}(\Omega;\mathbb{R}^m) \times H^* \to \mathbb{R}$ is Fréchet differentiable,

(1.2b) $\quad N\colon W^{1,p}(\Omega;\mathbb{R}^m) \to Z$ is continuously Fréchet differentiable,

(1.2c) $\quad K$ is closed convex in H^*,

(1.2d) $\quad L\colon H^* \to Z$ is continuous,

(1.2e) $\quad L(K) = Z$.

The following assertion is a powerful generalization of the Karush-Kuhn-Tucker theorem obtained by Zowe and Kurcyusz [38]; note that it admits a nonlinear Banach-space-valued equality constraint together with K having empty interior, which is just the situation we will meet. Here we only adapt and simplify it for the special structure of our problem; especially, [38] uses a weaker constraint qualification than (1.2e) but in our problems this does not seem to simplify the proof of our stronger but simpler constraint qualification, cf. Lemma 3.3 below. The normal cone to K at $\eta \in K$ is denoted by $N_K(\eta) = \{\eta^* \in H^{**};\, \forall\, \tilde{\eta} \in K\colon\ \langle \eta^*, \tilde{\eta} - \eta\rangle \leqslant 0\}$, where $\langle\cdot,\cdot\rangle$ denotes the canonical duality pairing (here between H^{**} and H^*). Furthermore, $\bar{\Phi}'_u(u,\eta)$ and $\bar{\Phi}'_\eta(u,\eta)$ will denote respectively the derivatives of $\bar{\Phi}(\cdot,\eta)$ and $\bar{\Phi}(u,\cdot)$ at a given point (u,η), and $L^*\colon Z^* \to H^{**}$ will be the adjoint mapping to L, and similarly $[N'(u)]^*\colon Z^* \to W^{1,p}(\Omega;\mathbb{R}^m)^*$, where N' denotes the derivative of N. The variable $\lambda \in Z^*$ will play the role of the Lagrange multiplier with respect to the constraint $N(u) = L\eta$.

Proposition 1.2. (Zowe and Kurcyusz [38], here adapted.) *Let* (1.2) *be valid and let* (u,η) *solve* (AP). *Then there is* $\lambda \in Z^*$ *such that*

$$[N'(u)]^*\lambda + \bar{\Phi}'_u(u,\eta) = 0, \tag{1.3}$$

$$L^*\lambda - \bar{\Phi}'_\eta(u,\eta) \ \in\ N_K(\eta). \tag{1.4}$$

2. Relaxation of (VP)—a general scheme

A concrete relaxation will be determined by a choice of the data H, Z, N, L, K and $\bar{\Phi}$ in the abstract relaxed problem (AP). The general philosophy is that H^* contains some information about oscillations and/or concentrations of the gradient ∇u; cf. (2.2). The choice of H determines essentially the character of the resulted relaxed problem and there is a large freedom in it. Following (and generalizing) the original idea of L. C. Young [35–37], we will always take H as a linear subspace of $\mathrm{Car}^p(\Omega;\mathbb{R}^{nm})$ defined in [27, 30, 31] as the linear space of all Carathéodory functions $h\colon \Omega\times\mathbb{R}^{nm} \to \mathbb{R}$ (that means $h(\cdot,A)$ are measurable and $h(x,\cdot)$ are continuous) with at most p-growth, i.e. $|h(x,A)| \leqslant a_h(x) + b_h|A|^p$ for some $a_h \in L^1(\Omega)$ and $b_h < +\infty$. Such space H can be always normed, e.g. by the (semi) norm

$$|h| \ = \ \inf\Big\{\, \|a\|_{L^1(\Omega)} + b;\, \forall (x,A) \in \Omega\times\mathbb{R}^{nm}\colon |h(x,A)| \leqslant a(x) + b|A|^p \Big\}. \tag{2.1}$$

In particular cases one can use stronger norms with the same effect, however. In any case, the dual space H^* to $(H, |\cdot|)$ is a Banach space if endowed with the standard dual norm $\|\eta\|^* = \sup_{|h|\leqslant 1} \langle \eta, h\rangle$. Furthermore, we define the imbedding i: $W^{1,p}(\Omega;\mathbb{R}^m) \to W^{1,p}(\Omega;\mathbb{R}^m) \times H^*$ by

$$i(u) = (u, i_1(\nabla u)), \tag{2.2}$$

where $i_1\colon L^p(\Omega;\mathbb{R}^{nm}) \to H^*$ is defined by $\langle i_1(y), h\rangle = \int_\Omega h(x, y(x))\,\mathrm{d}x$ for $h \in H$ and $y \in L^p(\Omega;\mathbb{R}^{nm})$. Finally, we put

$$\begin{aligned} Y^p_H(\Omega;\mathbb{R}^{nm}) = \{\eta \in H^*;\ \exists\{y_\alpha\} \text{ a bounded net in } L^p(\Omega;\mathbb{R}^{nm})\colon \\ \text{w*-lim } i_1(y_\alpha) = \eta\}. \end{aligned} \tag{2.3}$$

The following assertion summarizes some selected results from [33].

Proposition 2.1. $Y^p_H(\Omega;\mathbb{R}^{nm})$ *is always a convex, weakly* σ-compact subset of* H^*. *Moreover, if H contains a coercive function, i.e. $\exists h_c \in H$ such that $h_c(x,A) \geqslant |A|^p$, then $Y^p_H(\Omega;\mathbb{R}^{nm})$ is closed, locally weakly* compact. If H is also separable, then $Y^p_H(\Omega;\mathbb{R}^{nm})$ is locally weakly* sequentially compact.*

Occasionally, we will freely address the elements of $Y^p_H(\Omega;\mathbb{R}^{nm})$ as generalized Young functionals, which refers to the fact that, for the choice $H = L^1(\Omega; C_0(\mathbb{R}^{nm}))$ with $C_0(\mathbb{R}^{nm})$ denoting the space of continuous functions $\mathbb{R}^{nm} \to \mathbb{R}$ vanishing at infinity, the set of functionals $Y^p_{L^1(\Omega;C_0(\mathbb{R}^{nm}))}(\Omega;\mathbb{R}^{nm}) \subset L^1(\Omega;C_0(\mathbb{R}^{nm}))^*$, called then Young functionals, is (thanks to the Dunford-Pettis and the Riesz theorems [14]) affinely homeomorphic with a certain set of the classical Young measures which are weakly measurable mappings from Ω to the probability Radon measures on $\mathbb{R}^{nm}$. However, for other choices these generalized Young functionals can have quite different character; e.g. for $H = C(\bar\Omega)\otimes V$ with V a suitable linear subspace of $C_p(\mathbb{R}^{nm}) = \{v\colon \mathbb{R}^{nm} \to \mathbb{R} \text{ continuous};\ \sup_{A\in\mathbb{R}^{nm}} |v(A)|/(1+|A|^p) < +\infty\}$, the set $Y^p_H(\Omega;\mathbb{R}^{nm})$ is homeomorphic to the set of all measures introduced by DiPerna and Majda [13], while for $H = L^{p/(p-1)}(\Omega)\otimes(\mathbb{R}^{nm})^* \equiv \{h \in \mathrm{Car}^p(\Omega;\mathbb{R}^{nm});\ \exists g \in L^{p/(p-1)}(\Omega;\mathbb{R}^{nm})\colon h(x,A) = \sum_{i=1}^n\sum_{j=1}^m g_{ij}(x)[A]_{ij}\}$, the set $Y^p_H(\Omega;\mathbb{R}^{nm})$ is homeomorphic just to $L^p(\Omega;\mathbb{R}^{nm})$ itself, considered in the weak* topology; of course, we have used the standard notation (see, e.g., [34]) $G\otimes V = \{\sum_{\text{finite}} g_l \otimes v_l;\ g_l \in G,\ v_l \in V\}$ with $[g\otimes v](x,A) = g(x)v(A)$ standing for the tensorial product of the functions g and v.

For $k \geqslant 1$ we define the bilinear mapping $(h,\eta) \mapsto h \odot \eta\colon H^k \times H^* \to [C(\bar\Omega)^*]^k \cong \mathrm{rca}(\bar\Omega;\mathbb{R}^k)$, where "rca" stands for "regular countably additive" set functions (= Radon measures, cf. [14]), by $\langle h\odot\eta, g\rangle = \langle \eta, g\cdot h\rangle$ for any $g \in C(\bar\Omega)^k$; note that, if (2.1) is accepted, then

$$\forall h \in H^k\ \exists c_h \in \mathbb{R}^+\ \forall g \in C(\bar\Omega)^k\colon g\cdot h \in H\ \ \&\ \ |g\cdot h| \leqslant c_h\|g\|_{C(\bar\Omega;\mathbb{R}^k)}, \tag{2.4}$$

where $[g\cdot h](x,A)=\sum_{l=1}^{k}g_l(x)h_l(x,A)$, which makes the function $g\mapsto\langle\eta,g\cdot h\rangle$ continuous. The expression $h\odot\eta$ generalizes naturally the substitution of a Young measure ν into an $\mathbb{R}^k$-valued Carathéodory integrand, which is a function $x\mapsto\int_{\mathbb{R}^{nm}}h(x,A)\nu_x(\mathrm{d}A)$. However, as $\eta\in Y_H^p(\Omega;\mathbb{R}^{nm})$ can record also concentration effects, the result $h\odot\eta$ is, in general, a measure on $\bar{\Omega}$. Nevertheless, sometimes $h\odot\eta$ can belong even to $L^q(\Omega;\mathbb{R}^k)$ with some $1<q\leqslant+\infty$. This takes place if $h\in H^k$ satisfies

$$(2.5)\quad \exists c_{h,q}\in\mathbb{R}\ \forall g\in L^{q/(q-1)}(\Omega;\mathbb{R}^k)\colon g\cdot h\in H\ \ \&\ \ |g\cdot h|\ \leqslant c_{h,q}\|g\|_{L^{q/(q-1)}(\Omega;\mathbb{R}^k)}.$$

Then the mapping $\eta\mapsto h\odot\eta\colon H^*\to L^q(\Omega)$ is norm as well as weak* continuous. In particular, it is an easy consequence of the Hölder inequality that (2.5) is valid for $h=1\otimes v$, $q=p/r$, and $|\cdot|$ from (2.1) provided v has the growth not greater than r in the sense $|v(A)|\leqslant C(1+|A|^r)$ with some $0\leqslant r<p$.

The following relaxed variational problem, denoted by (RVP_H), is of general usage as far as the proper relaxation of (VP) is concerned, though for optimality conditions we shall have to modify it a bit. We define the set of "gradient" generalized Young functionals by

$$(2.6)\quad G_H^p(\Omega;\mathbb{R}^{nm})=\{\eta\in Y_H^p(\Omega;\mathbb{R}^{nm});\ \exists\{u_\alpha\}\text{ a bounded net in }W^{1,p}(\Omega;\mathbb{R}^m)\colon\ \text{w*-lim }i_1(\nabla u_\alpha)=\eta\}.$$

Then we define:

$$(\mathrm{RVP}_H)\qquad\begin{cases}\text{minimize} & \bar{\Phi}(u,\eta)\quad\text{for}\ \ (u,\eta)\in W^{1,p}(\Omega;\mathbb{R}^m)\times H^*,\\ \text{subject to} & \nabla u=(1\otimes\mathrm{id})\odot\eta,\\ & \eta\in G_H^p(\Omega;\mathbb{R}^{nm}),\end{cases}$$

where $1\otimes\mathrm{id}\in\mathrm{Car}^p(\Omega;\mathbb{R}^{nm})^{nm}$ with $\mathrm{id}\colon\mathbb{R}^{nm}\to\mathbb{R}^{nm}$ the identity, hence the expression $(1\otimes\mathrm{id})\odot\eta$ uses the operation $\odot\colon[H]^k\times H^*\to[C(\bar{\Omega})^*]^k$ with $k=nm$ now. Obviously, $(\mathrm{RVP}_H)=(\mathrm{AP})$ if one puts $K=G_H^p(\Omega;\mathbb{R}^{nm})$, $Z=L^p(\Omega;\mathbb{R}^{nm})$, $N(u)=\nabla u$, $L\eta=(1\otimes\mathrm{id})\odot\eta$, and defines $\bar{\Phi}$ again by (1.1b). Accepting (2.5), we have (2.5) valid for $h=1\otimes\mathrm{id}$ and $q=p$, which guarantees that L maps H^* actually into $L^p(\Omega;\mathbb{R}^{nm})$.

Proposition 2.2. *Let* (2.1) *be valid, and* φ *be coercive in the sense*

$$(2.7)\qquad \exists a\in L^1(\Omega),\ b,c,r>0\colon\ \varphi(x,u,A)\geqslant a(x)+b|u|^r+c|A|^p.$$

Then (RVP_H) *is a proper relaxation of* (VP).

Proof. We will successively verify the particular items in (1.1).

Let $(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times G^p_H(\Omega;\mathbb{R}^{nm})$ and $\nabla u = (1 \otimes \mathrm{id}) \odot \eta$. Then there is a bounded net $\{u_\alpha\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $i_1(\nabla u_\alpha) \to \eta$ weakly* in H^*. Then $\{u_\alpha\}$ converges weakly (possibly as a subnet only) to some $\tilde{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$. Then $i(u_\alpha - \tilde{u} + u) \to (u,\eta)$, which proves (1.1a).

Obviously, (1.1b) is trivially guaranteed by the definition of $\overline{\Phi}$.

As for (1.1c), let us denote $M_c = \{(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times Y^p_H(\Omega;\mathbb{R}^{nm})\,;\ \overline{\Phi}(u,\eta) \leqslant c\}$. By (2.7), $\Phi(u) \to +\infty$ whenever $\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} \to \infty$, so that $M_c \cap \mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_H)$ is contained in the weak* closure $\overline{B}$ of $i(B)$ for a sufficiently large ball B in $W^{1,p}(\Omega;\mathbb{R}^m)$. This is a weakly* compact set in $W^{1,p}(\Omega;\mathbb{R}^m) \times Y^p_H(\Omega;\mathbb{R}^{nm})$. As $\overline{\Phi}$ is l.s.c., M_c is closed. As $G^p_H(\Omega;\mathbb{R}^{nm}) \cap \mathrm{cl}\, i_1(B)$ is closed and $u \mapsto \nabla u$ and $\eta \mapsto (1 \otimes \mathrm{id}) \odot \eta$ are weakly* continuous, also $\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_H) \cap \overline{B}$ is closed. Hence $M_c \cap \mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_H)$ is a closed (and, if $c > \inf(\mathrm{RVP}_H)$, apparently non-empty) subset of the weakly* compact set $\overline{B}$. Therefore it is compact as well, which proves (1.1c). $\square$

As pointed out already in [27, 31], the general dilemma in the choice of H is the following: a larger H makes evaluation of $\overline{\Phi}$ easier, the solution of the relaxed problems contains more information but is harder to be interpreted and eventually implemented on computers, and the set $G^p_H(\Omega;\mathbb{R}^{nm})$ is more difficult to be described effectively. Taking H smaller leads to just converse effects.

Now we want to compare various choices of H. Let us take two linear subspaces H_1, H_2 such that $H_1 \subset H_2 \subset \mathrm{Car}^p(\Omega;\mathbb{R}^{nm})$. Of course, $\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_l}) = \{(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times G^p_{H_l}(\Omega;\mathbb{R}^{nm})\,;\ (1 \otimes \mathrm{id}) \odot \eta = \nabla u\}$, and

$$\overline{\Phi}_l(u,\eta) = \liminf_{i(u_\alpha)\to(u,\eta)} \int_\Omega \varphi(x, u_\alpha(x), \nabla u_\alpha(x))\,\mathrm{d}x$$

with $i(u_\alpha) \to (u,\eta)$ understood weakly* in $W^{1,p}(\Omega;\mathbb{R}^m) \times H_l^*$, $l = 1,2$. Furthermore, let $Q\colon H_1 \to H_2$ denote the inclusion $H_1 \subset H_2$; without no loss of generality, we can suppose both H_1 and H_2 normed be the (relativized) universal norm (2.1) which makes Q continuous. For $\eta_1 \in G^p_{H_1}(\Omega;\mathbb{R}^{nm})$, we put $J(\eta_1) = \{\eta_2 \in G^p_{H_2}(\Omega;\mathbb{R}^{nm})\,;\ Q^*\eta_2 = \eta_1\}$.

Proposition 2.3. *For any* $(u,\eta_1) \in \mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_1})$,

$$\overline{\Phi}_1(u,\eta_1) = \min_{\eta_2 \in J(\eta_1)} \overline{\Phi}_2(u,\eta_2). \tag{2.8}$$

Moreover, if $G^p_{H_2}(\Omega;\mathbb{R}^{nm})$ *and* $\overline{\Phi}_2|_{\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_2})}$ *are convex, then also* $G^p_{H_1}(\Omega;\mathbb{R}^{nm})$ *and* $\overline{\Phi}_1|_{\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_1})}$ *are convex.*

Proof. Realizing that $Q^*\colon G^p_{H_2}(\Omega;\mathbb{R}^{nm}) \to G^p_{H_1}(\Omega;\mathbb{R}^{nm})$ is weakly* continuous and $\overline{\Phi}_1$ and $\overline{\Phi}_2$ are coercive, the formula (2.8) follows by standard technique, cf. [28; Lemma in Sec. 3].

If $G^p_{H_2}(\Omega;\mathbb{R}^{nm})$ is convex, then so is $G^p_{H_1}(\Omega;\mathbb{R}^{nm})$ because $G^p_{H_1}(\Omega;\mathbb{R}^{nm}) = Q^*(G^p_{H_2}(\Omega;\mathbb{R}^{nm}))$ and Q^* is linear.

As the constraint $(1\otimes\mathrm{id})\odot\eta = \nabla u$ is linear, $\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_1})$ and $\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_2})$ are convex, as well. Suppose $\overline{\Phi}_2$ is convex and take $(u,\eta_1),(\tilde{u},\tilde{\eta}_1)\in\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_1})$. By (2.8) there are $\eta_2,\tilde{\eta}_2\in\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_2})$ such that $Q^*\eta_2=\eta_1$, $Q^*\tilde{\eta}_2=\tilde{\eta}_1$, and $\overline{\Phi}_2(u,\eta_2)=\overline{\Phi}_1(u,\eta_1)$, $\overline{\Phi}_2(\tilde{u},\tilde{\eta}_2)=\overline{\Phi}_1(\tilde{u},\tilde{\eta}_1)$. As $\overline{\Phi}_2$ and $\mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}_{H_2})$ are convex and $Q^*(\frac12\eta_2+\frac12\tilde{\eta}_2)=\frac12\eta_1+\frac12\tilde{\eta}_1$, we can estimate $\overline{\Phi}_1(\frac12 u+\frac12\tilde{u},\frac12\eta_1+\frac12\tilde{\eta}_1)\leqslant\overline{\Phi}_2(\frac12 u+\frac12\tilde{u},\frac12\eta_2+\frac12\tilde{\eta}_2)\leqslant\frac12\overline{\Phi}_2(u,\eta_2)+\frac12\overline{\Phi}_2(\tilde{u},\tilde{\eta}_2)=\frac12\overline{\Phi}_1(u,\eta_1)+\frac12\overline{\Phi}_1(\tilde{u},\tilde{\eta}_1)$. Therefore, $\overline{\Phi}_1$ is convex, as well. $\square$

It is reasonable to consider H containing always integrands linear in terms of the variable A, more precisely $H\supset H_0\equiv L^{p/(p-1)}(\Omega)\otimes(\mathbb{R}^{nm})^*$ where $(\mathbb{R}^{nm})^*$ denotes the space of all linear functionals $\mathbb{R}^{nm}\to\mathbb{R}$. It is natural to define a norm on H_0 by $\|h\|=\|g\|_{L^{p/(p-1)}(\Omega;\mathbb{R}^{nm})}$ for $h=\sum_{i=1}^n\sum_{j=1}^m g_{ij}\otimes v_{ij}$ with $v_{ij}(A)=[A]_{ij}$. The reader can easily verify that H_0^* is isometrically isomorphic to $L^p(\Omega;\mathbb{R}^{nm})$ via the adjoint mapping to $L^{p/(p-1)}(\Omega;\mathbb{R}^{nm})\to H_0\colon g=(g_{ij})\mapsto h=\sum_{i=1}^n\sum_{j=1}^m g_{ij}\otimes v_{ij}$, and $Y^p_{H_0}(\Omega;\mathbb{R}^{nm})$ is $L^p(\Omega;\mathbb{R}^{nm})$ itself. Also note that $G^p_{H_0}(\Omega;\mathbb{R}^{nm})$ is convex. Then it is well known (see [12]) that the corresponding l.s.c. relaxation $\overline{\Phi}$ of Φ is given by $\overline{\Phi}(u,y)=\int_\Omega\varphi^\#(x,u(x),y(x))\,\mathrm{d}x$ with $\varphi^\#(x,u,A)=[\varphi(x,u,\cdot)]^\#(A)$ where, for a function $v\colon\mathbb{R}^{nm}\to\mathbb{R}$, $v^\#\colon\mathbb{R}^{nm}\to\mathbb{R}$ denotes its quasiconvexification defined by $v^\#(A)=\inf_{u\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)}\int_\Omega v(A+\nabla u(x))\,\mathrm{d}x$.

Corollary 2.1. *If $H\supset H_0$ and the quasiconvexification $\Phi^\#$ defined by $\Phi^\#(u)=\int_\Omega\varphi^\#(x,u(x),\nabla u(x))\,\mathrm{d}x$ is nonconvex, then the relaxation* (RVP_H) *of* (VP) *cannot be convex, which means that inevitably $\overline{\Phi}$ is non-convex or $G^p_H(\Omega;\mathbb{R}^{nm})$ is non-convex.*

Proof. Take $H_1=H_0\equiv L^{p/(p-1)}(\Omega)\otimes(\mathbb{R}^{nm})^*$ and $H_2=H$. We have just $\overline{\Phi}_1(u,\eta)=\Phi^\#(u)$ provided $(1\otimes\mathrm{id})\odot\eta=\nabla u$. As $H_1\subset H_2$, we can use Propostition 2.3. If the relaxed problem (RVP_H) had both $\overline{\Phi}$ and $G^p_H(\Omega;\mathbb{R}^{nm})$ convex, then, by Proposition 2.3, $\Phi^\#$ would have to be convex, as well. This is a contradiction. $\square$

3. The optimality conditions

From the viewpoint of the optimality conditions of the type (1.3)–(1.4), the problem (RVP_H) does not suit satisfactorily even if H is so small that $G^p_H(\Omega;\mathbb{R}^{nm})$ is convex. This is due to the fact that $G^p_H(\Omega;\mathbb{R}^{nm})$ is always too small so that the normal cone in (1.4) is too large and, as a result, (1.4) is not enough informative; cf. also [29]. For H larger, $G^p_H(\Omega;\mathbb{R}^{nm})$ may even fail to be convex and Proposition 1.2 cannot be used at all.

Therefore we face a necessity to choose a larger set for K than $G^p_H(\Omega;\mathbb{R}^{nm})$ from (2.6). The point is not to require the elements of K to be attainable by gradients. Even if $m=1$ (except the coarser cases like $H=H_0$) it requires to adopt a certain "non-concentration" concept: we say that $\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})$ is p-nonconcentrating if there is a net $\{y_\alpha\} \subset L^p(\Omega;\mathbb{R}^{nm})$ such that $i_1(y_\alpha) \to \eta$ weakly* in H^* and simultaneously the set $\{u_\alpha\}$ is relatively weakly compact in $L^1(\Omega)$. This notion is very natural because, as shown below, coercive problems have typically p-nonconcentrating solutions; for special problems it was observed recently by Kinderlehrer and Pedregal [19], using essentially deep results by Acerbi and Fusco [1]. Moreover, we must inevitably (again except coarser cases like $H=H_0$) enrich the system of constraints $L\eta = N(u)$. This is generally a very delicate matter. Nevertheless, if one accepts some restrictions on H (see (3.1)–(3.2) below), it is possible to construct the following relaxed problem:

$$(\mathrm{RVP}'_H) \qquad \begin{cases} \text{minimize} & \bar{\Phi}(u,\eta) \qquad \text{for} \quad (u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times H^*, \\ \text{subject to} & \mathrm{adj}_s \nabla u = (1 \otimes \mathrm{adj}_s) \odot \eta \qquad \text{with} \quad s = 1,\dots,\min(n,m), \\ & \eta \in Y^p_H(\Omega;\mathbb{R}^{nm}); \end{cases}$$

note that $1 \otimes \mathrm{adj}_s \in \mathrm{Car}^p(\Omega;\mathbb{R}^{nm})^{\sigma(s)}$ if $p \geqslant s$. Now $(\mathrm{RVP}'_H) = (\mathrm{AP})$ if one puts $K = Y^p_H(\Omega;\mathbb{R}^{nm})$, $Z = \prod_{s=1}^{\min(n,m)} L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$, $N(u) = (\mathrm{adj}_s \nabla u)_{s=1}^{\min(n,m)}$, $L\eta = ((1\otimes \mathrm{adj}_s)\odot\eta)_{s=1}^{\min(n,m)}$, and defines $\bar{\Phi}$ again by (1.1b). The property (2.5) is now valid for $h = 1 \otimes \mathrm{adj}_s$ and $q = p/s$ provided the universal choice (2.1) is accepted.

To guarantee that (RVP'_H) will be a proper relaxation of (VP), we must require H to satisfy

$$(3.1) \qquad H \ \text{ contains densely } \ C(\bar{\Omega}) \otimes V,$$

where

$$(3.2\mathrm{a}) \qquad V \ \text{ is a separable linear subspace of } \ C_p(\mathbb{R}^{nm}),$$

$$(3.2\mathrm{b}) \qquad \forall\, 1 \leqslant s \leqslant \min(n,m): \ \mathrm{adj}_s \in V^{\sigma(s)},$$

$$(3.2\mathrm{c}) \qquad \forall v \in V: v^\# \neq -\infty \ \Rightarrow \ v^\# \ \text{ is polyconvex },$$

where $C_p(\mathbb{R}^{nm}) = \{v\colon \mathbb{R}^{nm} \to \mathbb{R} \text{ continuous};\ \sup_{A\in\mathbb{R}^{nm}} |v(A)|/(1+|A|^p) < +\infty\}$ is endowed with the norm $\|v\| = \sup_{A\in\mathbb{R}^{nm}} |v(A)|/(1+|A|^p)$. Several examples of subspaces satisfying (3.2) have been shown in [31], where it was also proved that every such subspace is contained in a maximal subspace with these properties. Realizing that each H satisfying (3.1)–(3.2) is always separable, the following assertion can be found in [31]:

Proposition 3.1. *Let $p \geqslant \min(n,m)$, let H satisfy* (3.1) *with V satisfying* (3.2), *and let $\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})$ be p-nonconcentrating. If there is $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\forall 1 \leqslant s \leqslant \min(n,m):\ (1 \otimes \mathrm{adj}_s) \odot \eta = \mathrm{adj}_s(\nabla u) \ \text{ in } L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)}), \tag{3.3}$$

then $\eta \in G^p_H(\Omega;\mathbb{R}^{nm})$.

More precisely, H was considered in [31] with a (generally) coarser topology, but this makes only H^* smaller without changing $Y^p_H(\Omega;\mathbb{R}^{nm})$, cf. also [33; Proposition 3.2]. Also, [31] used $L^\infty(\Omega)$ in place of $C(\bar{\Omega})$, but this change requires only a few technical modifications in [31; proofs of Lemmas 3.2 and 3.3].

We will now have to specify $\bar{\Phi}$, confining ourselves to a continuous extension; this means $\bar{\Phi}\colon W^{1,p}(\Omega;\mathbb{R}^m) \times H^* \to \mathbb{R}$ is continuous (in weak×weak* topology) and $\bar{\Phi}\circ i = \Phi$. Let us only remark that, for coarser relaxation, this requirement might be too restrictive and should be better replaced only by the norm continuity, possibly without insisting on $\bar{\Phi}\circ i = \Phi$, but then the necessity of evaluation of a lower weakly* semi-continuous envelope of Φ in (1.1b) may appear if Φ fails to be lower weakly* semi-continuous. The weakly* continuous extension we want to treat here does exist provided

$$\forall u \in L^r(\Omega;\mathbb{R}^m)\colon \varphi\circ u \in H \ \ \&\ \ u \mapsto \varphi\circ u\colon L^r(\Omega;\mathbb{R}^m) \to H \quad \text{norm continuous}, \tag{3.4}$$

where $[\varphi\circ u](x,A) = \varphi(x,u(x),A)$ and $r \geqslant 1$ is arbitrary if $p > n$ or $r < np/(n-p)$ if $p \leqslant n$, so that we always have the compact imbedding $W^{1,p}(\Omega;\mathbb{R}^m) \subset L^r(\Omega;\mathbb{R}^m)$. Note that (3.4) together with (3.2c) basically requires $\varphi(x,u,\cdot)$ to have a polyconvex quasiconvexification. If it is actually so, then a suitable subspace V does exist. Yet, as already mentioned, this requirement is not much realistic, which certainly urges further theoretical research to be able to handle real problems from, e.g., nonlinear elasticity.

Lemma 3.1. *If φ satisfies* (3.4)*, then Φ admits a continuous extension, given by the formula*

$$\bar{\Phi}(u,\eta) = \langle \eta, \varphi\circ u\rangle = \int_{\bar{\Omega}} [(\varphi\circ u)\odot\eta]\,\mathrm{d}x. \tag{3.5}$$

P r o o f. We obviously have $\bar{\Phi}\circ i(u) = \langle i_1(u), \varphi\circ u\rangle = \int_\Omega \varphi(x,u,\nabla u(x))\,\mathrm{d}x = \Phi(u)$ and the weak* continuity follows from the compact imbedding of $W^{1,p}(\Omega;\mathbb{R}^m)$ into $L^r(\Omega;\mathbb{R}^m)$, from (3.4), and from the joint continuity of the canonical bilinear pairing $\langle\cdot,\cdot\rangle\colon (H^*,\text{weak}*) \times (H,\text{norm}) \to \mathbb{R}$. $\square$

The following notion will be useful: $\mathring{\eta} \in Y_H^p(\Omega;\mathbb{R}^{nm})$ is called a p-nonconcentrating modification of $\eta \in Y_H^p(\Omega;\mathbb{R}^{nm})$ if $\mathring{\eta}$ is p-nonconcentrating and if $\langle\mathring{\eta},h\rangle = \langle\eta,h\rangle$ whenever $h \in H$ has slower growth than p in the sense $|h(x,A)| \leqslant a(x) + o(|A|^p)$ with some $a \in L^1(\Omega)$ and $o(r)/r \to 0$ for $r \to \infty$. It was shown in [32] that, if H is separable, every $\eta \in Y_H^p(\Omega;\mathbb{R}^{nm})$ admits precisely one p-nonconcentrating modification $\mathring{\eta}$. Moreover, $\langle\eta-\mathring{\eta},h\rangle \geqslant 0$ provided $h \in H$ such that $h(x,s) \geqslant a_0(x)$ for some $a_0 \in L^1(\Omega)$, and also $\langle\eta-\mathring{\eta},h\rangle > 0$ provided $\eta \neq \mathring{\eta}$ and $h \in H$ is coercive in the sense $h(x,s) \geqslant a_0(x) + b|s|^p$ with some $a_0 \in L^1(\Omega)$ and $b > 0$.

Proposition 3.2. *Let $p > \min(n,m)$, (2.7), (3.1), (3.2), and (3.4) be valid. Then* (RVP'_H) *is a proper relaxation of* (VP).

Proof. Let us take a solution (u,η) to (RVP'_H). Suppose, for a moment, that η is not p-nonconcentrating, i.e. $\mathring{\eta} \neq \eta$ with $\mathring{\eta} \in Y_H^p(\Omega;\mathbb{R}^{nm})$ being the p-nonconcentrating modification of η; here we have used the fact that, due to (3.1)–(3.2), H is separable to ensure the existence of $\mathring{\eta}$. Since the integrands $1\otimes\mathrm{adj}_s$ have the growth $s \leqslant \min(n,m)$ strictly less than p, we have $(1\otimes\mathrm{adj}_s)\odot\mathring{\eta} = (1\otimes\mathrm{adj}_s)\odot\eta$. Therefore, the couple $(u,\mathring{\eta})$ is admissible for (RVP'_H). Besides, (2.7) makes the integrand $\varphi\circ u$ coercive so that $(\varphi\circ u)\odot\mathring{\eta} < (\varphi\circ u)\odot\eta$. This contradicts the assumption that (u,η) is a minimizer of (RVP'_H). Thus we have shown that η is inevitably p-nonconcentrating.

Now let us observe that both (RVP_H) and (RVP'_H) have the same cost functionals as well as the admissible domains if one confines oneself to p-nonconcentrating functionals. Indeed, $(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times Y_H^p(\Omega;\mathbb{R}^{nm})$ with η p-nonconcentrating and $(1\otimes\mathrm{adj}_s)\odot\eta = \mathrm{adj}_s(\nabla u)$ implies $(u,\eta) \in W^{1,p}(\Omega;\mathbb{R}^m) \times G_H^p(\Omega;\mathbb{R}^{nm})$ by Proposition 3.1, so that (u,η) is admissible for (RVP_H). Conversely, let (u,η) be admissible for (RVP_H), i.e. $\nabla u = (1\otimes\mathrm{id})\odot\eta$ and $\eta \in G_H^p(\Omega;\mathbb{R}^{nm})$. As in the proof of Proposition 2.2, we can get a sequence $\{u_\alpha\} \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $i(u_\alpha) \to (u,\eta)$ weakly*. In particular, $\mathrm{adj}_s(\nabla u_\alpha) = (1\otimes\mathrm{adj}_s)\odot i_1(\nabla u_\alpha) \to (1\otimes\mathrm{adj}_s)\odot\eta$. Simultaneously, by the continuity of minors of gradients [25] (see also [2, 3]) we have also $\mathrm{adj}_s(\nabla u_\alpha) \to \mathrm{adj}_s(\nabla u)$ weakly in $L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$. This shows that $\mathrm{adj}_s(\nabla u) = (1\otimes\mathrm{adj}_s)\odot\eta$ so that the pair (u,η) is admissible for (RVP'_H), too. $\square$

Now we want to treat the optimality conditions for (VP). To guarantee (1.2a), we must still impose some data qualifications,

$$\text{(3.6a)}\qquad \forall u \in L^r(\Omega;\mathbb{R}^m)\colon \left\{\left[\frac{\partial\varphi}{\partial u}\circ u\right]\cdot\tilde{u};\ \|\tilde{u}\|_{L^r(\Omega;\mathbb{R}^m)} \leqslant 1\right\} \subset H \ \text{ bounded,}$$

(3.6b)

$$u \mapsto \left[\frac{\partial\varphi}{\partial u}\circ u\right]\cdot\tilde{u} : L^r(\Omega;\mathbb{R}^m) \to H \quad \text{norm equi-continuous for } \|\tilde{u}\|_{L^r(\Omega;\mathbb{R}^m)} \leqslant 1.$$

Lemma 3.2. *If φ satisfies* (3.6), *then the continuous extension $\bar{\Phi}$ of Φ, defined by* (3.5), *is continuously differentiable and*

(3.7)
$$\bar{\Phi}'(u,\eta) \equiv \left(\bar{\Phi}'_u(u,\eta), \bar{\Phi}'_\eta(u,\eta)\right) = \left(\Big(\frac{\partial\varphi}{\partial u}\circ u\Big)\odot\eta, \varphi\circ u\right) \in L^{r/(r-1)}(\Omega;\mathbb{R}^m)\times H.$$

P r o o f. For the evaluation of the expression (3.7) as a Gâteaux derivative of $\bar{\Phi}$ using the geometry from $\mathrm{Car}^p(\Omega;\mathbb{R}^{nm})$ we refer to [30]. Its norm continuity here clearly follows from (3.4) and (3.6) when one estimates

$$\begin{aligned}
&\left\|\Big(\frac{\partial\varphi}{\partial u}\circ u_1\Big)\odot\eta_1 - \Big(\frac{\partial\varphi}{\partial u}\circ u\Big)\odot\eta\right\|_{L^{r/(r-1)}(\Omega;\mathbb{R}^m)}\\
&= \sup_{\|\tilde{u}\|_{L^r(\Omega;\mathbb{R}^m)}\leqslant 1}\left|\left\langle\Big(\frac{\partial\varphi}{\partial u}\circ u_1\Big)\odot\eta_1 - \Big(\frac{\partial\varphi}{\partial u}\circ u\Big)\odot\eta, \tilde{u}\right\rangle\right|\\
&= \sup_{\|\tilde{u}\|_{L^r(\Omega;\mathbb{R}^m)}\leqslant 1}\left|\left\langle\eta_1, \left(\Big(\frac{\partial\varphi}{\partial u}\circ u_1\Big) - \Big(\frac{\partial\varphi}{\partial u}\circ u\Big)\right)\cdot\tilde{u}\right\rangle + \left\langle\eta_1-\eta, \Big(\frac{\partial\varphi}{\partial u}\circ u\Big)\cdot\tilde{u}\right\rangle\right|
\end{aligned}$$

by using (3.6b) and (3.6a) respectively for the first and the second terms. Thus, for $\eta_1\to\eta$ and $u_1\to u$, we get $(\frac{\partial\varphi}{\partial u}\circ u_1)\odot\eta_1 \to (\frac{\partial\varphi}{\partial u}\circ u)\odot\eta$ in $L^{r/(r-1)}(\Omega;\mathbb{R}^m)$. $\square$

The following surjectivity assertion basically says, in other words, that a sequence of functions $y_\alpha\in L^p(\Omega;\mathbb{R}^{nm})$ can always oscillate in such a way that all its minors $(\mathrm{adj}_s y_\alpha)_{s=1}^{\min(n,m)}$ weakly approach arbitrary values in $Z = \prod_{s=1}^{\min(n,m)} L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$. However, it is expressed in terms of limits, and also the proof works only with the limits, which reduces the employed technique to algebraic manipulations only.

Lemma 3.3. (The constraint qualification.) *If $p\geqslant\min(n,m)$, then*

$$L(Y^p_H(\Omega;\mathbb{R}^{nm})) = Z.$$

In other words, for any $z = (z_1,\ldots,z_{\min(n,m)}) \in \prod_{s=1}^{\min(n,m)} L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$, there is $\eta\in Y^p_H(\Omega;\mathbb{R}^{nm})$ such that $(1\otimes\mathrm{adj}_s)\odot\eta = z_s$ for any $s=1,\ldots,\min(n,m)$.

P r o o f. Given $z\in\prod_{s=1}^{\min(n,m)} L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})$, we will construct explicitly some $\eta\in Y^p_H(\Omega;\mathbb{R}^{nm})$, having a Young-measure representation in the form of a convex combination of the Dirac measures a.e., such that $L\eta = z$. As the general procedure is not easy to observe, we will prove only the special cases $n=m=2$ or 3, and outline the general case.

We will use the general notation A_s^ξ for an $s\times s$-matrix with its entries defined by

$$[A_s^\xi]_{ij} = \begin{cases} \operatorname{sign}(\xi)|\xi|^{1/s} & \text{if } i=j=1, \\ |\xi|^{1/s} & \text{if } i=j>1, \\ 0 & \text{otherwise,} \end{cases}$$

where ξ is a real parameter. Besides, we denote by D the diagonal matrix with the first element equal to 1 and all the others equal to -1. Then we define an $s\times s$-matrix $\overline{A}_s^\xi$ by

$$\overline{A}_s^\xi = \begin{cases} -A_s^\xi & \text{if } s \text{ is even,} \\ DA_s^\xi & \text{if } s \text{ is odd.} \end{cases}$$

Note that these matrices are designed so that always $\det A_s^\xi = \xi = \det\overline{A}_s^\xi$, and $A_s^\xi, \overline{A}_s^\xi \in L^p(\Omega;\mathbb{R}^{nm})$ provided $\xi \in L^{p/s}(\Omega)$. Besides, we define a "compensation" matrix $C_s^\xi = -A_s^\xi - \overline{A}_s^\xi$, which has obviously the only nonvanishing entry $-2\operatorname{sign}(\xi)|\xi|^{1/s}$ at the position $i=j=1$ provided s is odd.

Let $n=m=2$ and $z_1 \in L^p(\Omega;\mathbb{R}^{2\times 2})$, $z_2 \in L^{p/2}(\Omega)$. Define $\eta \in Y_H^p(\Omega;\mathbb{R}^{nm})$ by the following convex combination:

$$\eta = \frac{1}{3}\left[i_1(3z_1) + i_1(A_2^\xi) + i_1(\overline{A}_2^\xi)\right].$$

Then we can evaluate $(1\otimes\mathrm{id})\odot\eta = \frac{1}{3}(3z_1 + A_2^\xi - A_2^\xi) = z_1$ and $(1\otimes\det)\odot\eta = (1\otimes\mathrm{adj}_2)\odot\eta = 3\det z_1 + \frac{2}{3}\xi$. Taking ξ appropriately, namely $\xi = \frac{3}{2}z_2 - \frac{9}{2}\det z_1$, we get $(1\otimes\det)\odot\eta = z_2$. Note that $\xi\in L^{p/2}(\Omega)$ so that $A_2^\xi \in L^p(\Omega;\mathbb{R}^{nm})$. As z_1 and z_2 were arbitrary, $L(Y_H^p(\Omega;\mathbb{R}^{nm})) = Z$ has been proved. Note that it holds even for $p=2$ thanks to our non-concentration concept.

Let us go on to the case $n=m=3$. Take arbitrary $z_1 \in L^p(\Omega;\mathbb{R}^{3\times 3})$, $z_2 \in L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$, and $z_3 \in L^{p/3}(\Omega)$. Define $\eta \in Y_H^p(\Omega;\mathbb{R}^{nm})$ by

$$\eta = \frac{1}{22}\Big[i_1(22z_1) + i_1(A_3^\xi) + i_1(\overline{A}_3^\xi) + i_1(C_3^\xi) + \sum_{i,j=1}^{3}\big(i_1(B_{ij}^{\xi_{ij}}) + i_1(\overline{B}_{ij}^{\xi_{ij}})\big)\Big]$$

with ξ_{ij} real parameters and B_{ij}^ξ and $\overline{B}_{ij}^\xi$ 3×3-matrices with, respectively, A_2^ξ and $\overline{A}_2^\xi$ appearing at the place of the 2×2-minor indexed by ij while the other row and column vanishing. Then we obviously have $(1\otimes\mathrm{id})\odot\eta = \frac{1}{22}(22z_1 + A_3^\xi + \overline{A}_3^\xi + C_3^\xi) = z_1$. Also we can easily evaluate $(1\otimes\det)\odot\eta = 22^2\det z_1 + \frac{1}{11}\xi$. From this we can determine $\xi \in L^{p/3}(\Omega)$ so that $(1\otimes\det)\odot\eta = z_3$, namely $\xi = 11(z_3 - 22^2\det z_1)$. Denoting the particular components of adj_s by adj_{ij}, we can calculate $(1\otimes\mathrm{adj}_{ij})\odot\eta = 22\mathrm{adj}_{ij}z_1 + \frac{1}{11}\xi_{ij}$ if $i\neq 1\neq j$ and $(1\otimes\mathrm{adj}_{ij})\odot\eta = 22\mathrm{adj}_{ij}z_1 + \frac{1}{11}\xi_{ij} + \frac{1}{11}|\xi|^{2/3}$ if $i=j=1$. Having ξ already chosen, we can easily determine each $\xi_{ij} \in L^{p/2}(\Omega)$ so that $(1 \otimes \mathrm{adj}_{ij}) \odot \eta = [z_2]_{ij}$. As z_1, z_2 and z_3 were arbitrary, $L(Y_H^p(\Omega; \mathbb{R}^{nm})) = Z$ has been proved in this case.

The general case can be treated analogously: First, determine η as an appropriate convex combination; note that at each odd level except the first the compensation matrices C_s^ξ do not vanish and must be included into this combination. Then begin at the level $s = \min(n,m)$ and determine all parameters ξ; i.e. $1+|n-m|$ parameters. Afterwards, successively continue for lower levels, always determining all parameters ξ at a current level, and stop at the level $s = 2$. The point is that the convex combination can always be designed so that the current parameter (let us denote it by $\xi_{s,l}$, $1 \leqslant l \leqslant \sigma(s)$) can influence only $(1 \otimes \mathrm{adj}_{s'}) \odot \eta$ with $2 \leqslant s' < s$ and the component $[(1 \otimes \mathrm{adj}_s) \odot \eta]_l$ itself. □

Now we can almost readily use Proposition 1.1.

Theorem 3.1. (The integral maximum principle.) *Let $p \geqslant \min(n,m)$, (2.7), (3.1), (3.2), (3.4), and (3.6) be fulfilled. If (u, η) solves (RVP'_H), then there are $\lambda_s \in L^{p/(p-s)}(\Omega; \mathbb{R}^{\sigma(s)})$ such that*

$$\text{(3.8a)} \quad \sum_{s=1}^{\min(n,m)} \operatorname{div}\Big(\lambda_s \frac{\partial \mathrm{adj}_s}{\partial A}(\nabla u)\Big) = \Big(\frac{\partial \varphi}{\partial u} \circ u\Big) \odot \eta \qquad \text{in} \quad W^{-1,p/(p-1)}(\Omega; \mathbb{R}^m)$$

$$\text{(3.8b)} \quad \sum_{s=1}^{\min(n,m)} n_1 \cdot \Big(\lambda_s \frac{\partial \mathrm{adj}_s}{\partial A}(\nabla u)\Big)\Big|_{\partial\Omega} = 0 \qquad \text{in} \quad W^{-1+1/p,p/(p-1)}(\partial\Omega; \mathbb{R}^m),$$

where n_1 denotes the unit outward normal to the boundary $\partial\Omega$, and, for the Hamiltonian $\mathcal{H}_{u,\lambda}$ defined in (0.3) with $\lambda = (\lambda_1, \ldots, \lambda_{\min(n,m)})$, the following maximum principle is satisfied:

$$\text{(3.9)} \qquad \langle \eta, \mathcal{H}_{u,\lambda} \rangle = \sup_{y \in L^p(\Omega; \mathbb{R}^{nm})} \int_\Omega \mathcal{H}_{u,\lambda}(x, y(x))\, \mathrm{d}x.$$

P r o o f. The assumption (1.2a) was already verified in Lemma 3.2 (which asserted even more) and the assumption (1.2c) is ensured via Proposition 2.1 since, by (2.7) and (3.4), H must contain a coercive integrand, namely $\varphi \circ u$. It remains to verify (1.2b), i.e. the continuous dependence on u of the differential $N'(u)\colon W^{1,p}(\Omega; \mathbb{R}^m) \to Z\colon \tilde{u} \mapsto ((\partial \mathrm{adj}_s / \partial A)(\nabla u) \cdot \nabla \tilde{u})_{s=1}^{\min(n,m)}$. However, this follows easily from the estimate

$$
\begin{aligned}
&\big|[N'(u_1)]_s - [N'(u_2)]_s\big|_{\mathcal{L}(W^{1,p}(\Omega;\mathbb{R}^m),L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)}))}\\
&\qquad = \sup_{\|\tilde u\|_{W^{1,p}(\Omega;\ m)}\leqslant 1} \left\| \frac{\partial \mathrm{adj}_s}{\partial A}(\nabla u_1)\cdot\nabla\tilde u - \frac{\partial \mathrm{adj}_s}{\partial A}(\nabla u_2)\cdot\nabla\tilde u \right\|_{L^{p/s}(\Omega;\mathbb{R}^{\sigma(s)})}\\
&\qquad \leqslant \sup_{\|\tilde u\|_{W^{1,p}(\Omega;\ m)}\leqslant 1} \big\| a_s |\nabla u_1 - \nabla u_2|^{s-1} |\nabla \tilde u| \big\|_{L^{p/s}(\Omega)}\\
&\qquad \leqslant \sup_{\|\tilde u\|_{W^{1,p}(\Omega;\ m)}\leqslant 1} \big\| a_s |\nabla u_1 - \nabla u_2|^{s-1} \big\|_{L^{p/(s-1)}(\Omega)} \|\nabla\tilde u\|_{L^p(\Omega;\mathbb{R}^{nm})}\\
&\qquad \leqslant a_s \|\nabla u_1 - \nabla u_2\|^{s-1}_{L^p(\Omega;\mathbb{R}^{nm})},
\end{aligned}
$$

where a_s denotes the constant from the estimate $|(\partial\mathrm{adj}_s/\partial A)(A_1) - (\partial\mathrm{adj}_s/\partial A)(A_2)| \leqslant a_s|A_1 - A_2|^{s-1}$.

Now it only remains to evaluate the particular terms in (1.3)–(1.4).

To evaluate $[N'(u)]^*$, we use Green's formula and identify, as usual, $W^{1,p}(\Omega;\mathbb{R}^m)^*$ with $W^{-1,p/(p-1)}(\Omega;\mathbb{R}^m)\times W^{-1+1/p,p/(p-1)}(\partial\Omega;\mathbb{R}^m)$. Then

$$
[N'(u)]^*\colon Z^* \to W^{1,p}(\Omega;\mathbb{R}^m)^*
$$

is given by

$$
\lambda = (\lambda_1,\dots,\lambda_{\min(n,m)}) \mapsto \left[-\sum_{s=1}^{\min(n,m)} \nabla\cdot\Big(\lambda_s \frac{\partial\mathrm{adj}_s}{\partial A}(\nabla u)\Big),\ n_1\cdot\Big(\lambda_s\frac{\partial\mathrm{adj}_s}{\partial A}(\nabla u)\Big)\right].
$$

Let us evaluate L^*. To this aim, we define a linear mapping

$$
S\colon \prod_{s=1}^{\min(n,m)} L^{p/(p-s)}(\Omega;\mathbb{R}^{\sigma(s)}) \to H\colon \lambda \mapsto \sum_{s=1}^{\min(n,m)} \lambda_s\otimes\mathrm{adj}_s.
$$

Accepting (2.1), S is continuous. As $p<+\infty$, $S^{**}=S$. For any $\eta\in H^*$, we can calculate: $\langle S^*\eta,\lambda\rangle = \langle\eta,S\lambda\rangle = \langle\eta,\sum_{s=1}^{\min(n,m)}\lambda_s\otimes\mathrm{adj}_s\rangle = \langle[(1\otimes\mathrm{adj}_s)\odot\eta]_{s=1}^{\min(n,m)},\lambda\rangle$. As λ is arbitrary, we get $L=S^*$. Hence $L^*=S$.

Both $\overline{\Phi}'_\eta$ (see Lemma 3.2) and $L^*=S$ have values in H and not in $H^{**}\setminus H$, hence we can investigate only the trace on H of the normal cone $N_K(\eta)\subset H^{**}$ with $K=Y^p_H(\Omega;\mathbb{R}^{nm})$. Obviously,

$$
\begin{aligned}
N_K(\eta)\cap H &= \{h\in H\,;\ \forall\tilde\eta\in K\colon \langle\tilde\eta-\eta,h\rangle\leqslant 0\}\\
&= \Big\{h\in H\,;\ \langle\eta,h\rangle = \sup_{y\in L^p(\Omega;\mathbb{R}^{nm})}\int_\Omega h(x,y(x))\,\mathrm{d}x\Big\}.
\end{aligned}
$$

Using, moreover, the expression for $\overline{\Phi}'$ from Lemma 3.2, we can substitute everything into (1.3) and (1.4), which then results respectively in (3.8) and (3.9). $\square$

Theorem 3.2. (The pointwise maximum principle.) *If* $p > \min(n,m)$, (2.7) *and* (3.4) *are valid, then* (3.9) *is equivalent to*

$$\big[\mathcal{H}_{u,\lambda} \odot \eta\big](x) \;=\; \max_{A\in\mathbb{R}^{nm}} \mathcal{H}_{u,\lambda}(x,A) \qquad \text{for a.a. } x \in \Omega. \tag{3.10}$$

P r o o f. As it only modifies [30; proof of Theorem 3.2], we will abridge it.

First, note that the left-hand side of (3.10) is in $L^1(\Omega)$ thanks to our non-concentration hypothesis, and that the maximum on the right-hand side is actually attained due to the coercivity of $-\mathcal{H}_{u,\lambda}(x,\cdot)$, which follows from the assumed coercivity of $\varphi \circ u$ (see (2.7)) and $p > \min(n,m)$. Then the implication (3.10) $\Rightarrow$ (3.9) is easy.

The only essential point for the converse implication is whether every measurable $y\colon \Omega \to \mathbb{R}^{nm}$, satisfying $\mathcal{H}_{u,\lambda}(x,y(x)) = \max_{A\in\mathbb{R}^{nm}} \mathcal{H}_{u,\lambda}(x,A)$, belongs to $L^p(\Omega;\mathbb{R}^{nm})$. For such y, we can estimate:

$$\begin{aligned} a_u(x) + b_u|y(x)|^p &\leqslant \varphi(x,u(x),y(x)) \\ &= \sum_{s=1}^{\min(n,m)} \lambda_s(x)\mathrm{adj}_s y(x) \\ &\qquad + \min_{A\in\mathbb{R}^{nm}} \Big(\varphi(x,u(x),A) - \sum_{s=1}^{\min(n,m)} \lambda_s(x)\mathrm{adj}_s A\Big) \\ &\leqslant \alpha|y(x)|^p + (C_\alpha + C) \sum_{s=1}^{\min(n,m)} |\lambda_s(x)|^{p/(p-s)} + a(x) \end{aligned} \tag{3.11}$$

with $\alpha > 0$ arbitrarily small and $C_\alpha \in \mathbb{R}$ depending on α, and with $a \in L^1(\Omega)$ and $C \in \mathbb{R}$ from the estimate

$$\begin{aligned} \min_{A\in\mathbb{R}^{nm}} &\Big(\varphi(x,u(x),A) - \sum_{s=1}^{\min(n,m)} \lambda_s(x)\mathrm{adj}_s A\Big) \\ &\leqslant \min_{A\in\mathbb{R}^{nm}} \Big(a(x) + b|A|^p + \sum_{s=1}^{\min(n,m)} c_{s,n,m}|\lambda_s(x)|\;|A|^s\Big) \\ &\leqslant a(x) + C \sum_{s=1}^{\min(n,m)} |\lambda_s(x)|^{p/(p-s)}. \end{aligned} \tag{3.12}$$

Choosing $\alpha < b_u$, the first right-hand-side term can be absorbed in the left-hand side of (3.11), which gives $y \in L^p(\Omega;\mathbb{R}^{nm})$. $\square$

Note that both (3.11) and (3.12) would collapse in the limit case $p = \min(n,m)$ unless one can guarantee $\lambda_{\min(n,m)}$ sufficiently small in the L^∞-norm.

The following assertion points out the selectivity of our optimality conditions.

Theorem 3.3. (A partial sufficiency.) *Let $p > \min(n,m)$, (2.7), (3.1), (3.2) and (3.4) be valid, let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ solve the quasiconvexified problem, i.e. u minimizes $\Phi^\#(u)$, and let $\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})$ satisfy (3.3) and (3.9) with some λ. Then (u,η) solves the relaxed problem (RVP'_H).*

Proof. For u fixed, let us consider an auxiliary problem: minimize $\bar{\Phi}(u,\cdot)$ over the set $\mathcal{D}_u = \{\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})\,;\,(3.3)\text{ holds}\}$. Let us denote by I_u its infimum. It is obvious that $I_u \geqslant \min(\mathrm{RVP}'_H)$. In view of Proposition 2.3, we also know that for any u with $\Phi^\#(u) = \inf(\mathrm{VP})$, there is some $\eta \in G^p_H(\Omega;\mathbb{R}^{nm})$ such that $(1\otimes\mathrm{id})\odot\eta = \nabla u$ and $\bar{\Phi}(u,\eta) = \inf(\mathrm{VP}) = \min(\mathrm{RVP}'_H)$ by Proposition 3.2. Thanks to the coercivity of φ and $p > \min(n,m)$, $-\mathcal{H}_{u,\lambda}$ must be also coercive, and therefore each η satisfying (3.9) must be p-nonconcentrating. Also (3.3) is valid thanks to the weak continuity of minors of gradients. Therefore $\eta \in \mathcal{D}_u$. Thus we have shown that $I_u = \min(\mathrm{RVP}'_H)$.

If η solves the above auxiliary problem, then there is $\lambda \in Z^*$ such that (1.4) is satisfied. As this auxiliary problem is convex, (1.4) is also sufficient for $\eta \in \mathcal{D}_u$ to solve this problem. It was proved in Theorem 3.1 that (1.4) takes the form (3.9). Thus our assumptions guarantee η to solve this problem. As $I_u = \min(\mathrm{RVP}'_H)$, we have $\bar{\Phi}(u,\eta) = \min(\mathrm{RVP}'_H)$, which proves (u,η) to be a solution of (RVP'_H) when one realizes that $\eta \in \mathcal{D}_u$ means just $(u,\eta) \in \mathcal{D}_{\mathrm{ad}}(\mathrm{RVP}'_H)$. $\square$

Remark 3.1. If H is separable, then every $\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})$ p-nonconcentrating admits a Young-measure representation $\nu = \{\nu_x\}_{x\in\Omega} \in L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{nm}))$ (the subscript "w" denotes "weakly measurable") in the sense

$$\forall h \in H\colon\ h\odot\eta = \langle \nu, h\rangle \quad \text{in} \quad L^1(\Omega), \tag{3.13}$$

where $\nu_x \in \mathrm{rca}(\mathbb{R}^{nm})$ is a probability measure for a.a. $x \in \Omega$ and $\langle\nu,h\rangle$ denotes, as usual, the function $x \mapsto \langle \nu_x, h(x,\cdot)\rangle = \int_{\mathbb{R}^{nm}} h(x,A)\nu_x(\,\mathrm{d}A)$. This justifies the notation from Sec. 0.

Indeed, by separability of H (hence metrizability of the weak* topology on bounded sets in H^*), there is always a sequence $\{y_k\}$ bounded in $L^p(\Omega;\mathbb{R}^{nm})$ such that $i_1(y_k) \to \eta$ weakly* in H^*. Then there is its subsequence $\{y_{k_1}\}$ determining a Young measure in the classical sense. As η is assumed p-nonconcentrating, $h(y_k) \to h\odot\eta$ weakly in $L^1(\Omega)$ for any $h \in H$. In particular, $\{h(y_k)\}$ is sequentially weakly relatively compact in $L^1(\Omega)$, and therefore by Ball [4] $h(y_{k_1}) \to \langle\nu,h\rangle$ weakly in $L^1(\Omega)$, which proves (3.13). Of course, ν need not be determined uniquely by a given $\eta \in Y^p_H(\Omega;\mathbb{R}^{nm})$.

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