ON SEQUENCES IN VECTOR LATTICES

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Abstract. In this paper we investigate conditions for a system of sequences of elements of a vector lattice; analogous conditions for systems of sequences of reals were studied by D. E. Peek.

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D. E. Peek [7] investigated a collection $\sigma = \{\mathrm{A1} - \mathrm{A6}\}$ of six conditions for systems of sequences of reals.

In the present paper we deal with a slightly modified collection σ' of conditions that can be applied for systems of sequences in a vector lattice X. If $X = \mathbb{R}$ (the set of reals), then σ and σ' are equivalent.

A question proposed in [7] concerning the squeezing condition was solved independently in [2], [4] and [5].

1. Preliminaries

For vector lattices we use the same notation as in [1]; vector lattices are called Riesz spaces in [6] and K-lineals in [8].

Let V be a vector lattice and let $S(V)$ be the system of all sequences with elements from V. These sequences will be denoted by $(x_n), (y_n), (z_n), \ldots$, or by $X, Y, Z, \ldots$.

Let $\mathbb{R}$ be the vector lattice of all reals (with the usual operations and with the natural linear order).

If $X, Y \in S(V)$ and $k \in \mathbb{R}$, then the symbols $X + Y$ and kX have the obvious meanings.

Let A be a subset of $S(V)$ and let $\equiv$ be a binary relation defined on the set $S(V)$. The system of all such pairs $(A, \equiv)$ will be denoted by $P(V)$.

Consider the following conditions for the pair $(A, \equiv)$:

A1. If $X \in A$ and $k \in \mathbb{R}$, then $kX \in A$.

A2. If X, Y, Z, W, $X + Z$, $Y + W \in A$, $X \equiv Y$ and $Z \equiv W$, then $Y + W \equiv X + Z$.

A3. If $X, Y \in A$, $X \equiv Y$, $Z \in S(V)$ and $x_n \leqslant z_n \leqslant y_n$ for each n, then $Z \in A$ and $Z \equiv X$.

A4. If $X \in A$ and Y, Z are subsequences of X, then $Y, Z \in A$ and $Y \equiv Z$.

A5′. If $x \in V$, $x \neq 0$ and $y_n = (-1)^n x$ for each n, then $(y_n) \notin A$.

A6. If $X \notin A$ and X is bounded, then X has two subsequences Y and Z such that $Y, Z \in A$ and $Y \not\equiv Z$.

A pair $(A, \equiv) \in P(V)$ will be called regular with respect to V if it satisfies the collection of conditions $\sigma' = \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{A4}, \mathrm{A5}', \mathrm{A6}\}$.

In [7], the following condition was considered instead of A5′ for the case $V = \mathbb{R}$:

A5. If $x_n = (-1)^n$ for each n, then $(x_n) \notin A$.

Put $\sigma = \{\mathrm{A1} - \mathrm{A6}\}$. It is easy to verify that in the case $V = \mathbb{R}$ the collections σ and σ' are equivalent (it suffices to apply A1).

If $V = \mathbb{R}$ and $(A, \equiv)$ is regular with respect to V, then in [7] the pair $(A, \equiv)$ is called a convergence system.

We denote by A_0 the set of all $(x_n) \in S(V)$ such that (x_n) is o-convergent. For $(x_n), (y_n) \in S(V)$ we put $(x_n) \equiv_0 (y_n)$ if $(x_n), (y_n)$ are o-convergent and have the same o-limit.

It is well-known that for $V = \mathbb{R}$ the notions of metric convergence and o-convergence coincide. Hence the main result of [7] (expressed originally for the metric convergence) can be formulated as follows:

Theorem 1.1. (Cf. [7].) *Let $V = \mathbb{R}$ and let $(A, \equiv) \in P(V)$. Then*

($*$) *the pair $(A, \equiv)$ is regular with respect to V if and only if $(A, \equiv)$ coincides with $(A_0, \equiv_0)$.*

It seems to be a natural question to ask, for which vector lattices V the condition ($*$) is valid.

We shall deal with this problem in the following section.

The sequence (x_n) in $\mathbb{R}$ such that $x_n = n$ for each $n \in \mathbb{N}$ will be denoted by N. If $(x_n) \in S(V)$, $x \in V$ and $x_n = x$ for each n, then we write $(x_n) = \operatorname{const} x$.

Lemma 2.1. *Let $0 < x \in S(V)$, $y_n = nx$ for each n. Then no subsequence of (y_n) is o-convergent.*

P r o o f. By way of contradiction suppose that there exists a subsequence (z_n) of (y_n) such that (z_n) is o-convergent. There is a subsequence (m_n) of N such that $z_n = m_n x$ for each $n \in N$. In view of the fact that $z_n < z_{n+1}$ for each n we infer that

$$o\text{-}\lim z_n = \bigvee z_n \ .$$

Put $v_n = z_n + x$ for each n. Thus

$$o\text{-}\lim v_n = x + \bigvee z_n > \bigvee z_n \ .$$

Also, since (v_n) is increasing,

$$o\text{-}\lim v_n = \bigvee v_n \ .$$

Next, $z_n < v_n \leqslant z_{n+1}$ for each n, whence $\bigvee v_n = \bigvee z_n$, a contradiction. $\square$

Lemma 2.2. *Let V be a vector lattice satisfying the condition $(*)$. Then V is archimedean.*

P r o o f. By way of contradiction, assume that V fails to be archimedean. Then there are $x, y \in V$ such that $0 < nx < y$ for each n. Hence the sequence (nx) is bounded in V. According to $(*)$, the pair $(A_0, \equiv_0)$ is regular with respect to V. Lemma 2.1 implies that $(nx) \notin A_0$. Hence in view of A6 there exists a subsequence of (nx) which is o-convergent. This contradicts Lemma 2.1. $\square$

Let $1 < a \in \mathbb{R}$. In [3] and [4] the sequence (a^n) was used for constructing examples in connection with the properties of systems of sequences in $\mathbb{R}$.

Lemma 2.3. *Let $(m(1,n))$ and $(m(2,n))$ be subsequences of N, $X = (a^{m(1,n)})$, $Y = (a^{m(2,n)})$, $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{R}$, $Z = k_1 X + k_2 Y$. Then either*

(i) *Z is not bounded,*

or

(ii) *Z o-converges to 0.*

P r o o f. If $k_1 = k_2 = 0$, then (ii) is valid. If $k_1 \neq 0$ and $k_2 = 0$, then (i) is valid; the same holds if $k_1 = 0$ and $k_2 \neq 0$. Suppose that k_1 and k_2 have the same sign; then clearly (i) holds.

Now without loss of generality we can suppose that $k_1 > 0$ and $k_2 < 0$. This case was dealt with in the proof of [4], Lemma 1 and it was proved that under the assumption mentioned either (i) or (ii) is valid. □

The elements x and y in V will be called disjoint if $0 < x$, $0 < y$ and $x \wedge y = 0$. From the well-known properties of disjoint elements we obtain:

Let x and y be disjoint and let $0 < k_1 \in \mathbb{R}$, $0 < k_2 \in \mathbb{R}$, $r_i, s_i \in \mathbb{R}$ $(i = 1, 2)$. Then

(1) $k_1 x$ and $k_2 y$ are disjoint;

(2) $(x - y) \vee 0 = x$ and $(x - y) \wedge 0 = -y$;

(3) $r_1 x + r_2 y \leqslant s_1 x + s_2 y$ if and only if $r_1 \leqslant r_2$ and $s_1 \leqslant s_2$.

Lemma 2.4. *Assume that V is archimedean. Let x and y be disjoint elements in V. Let $k_1, k_2 \in \mathbb{R}$. Let $m(1,n)$ and $m(2,n)$ be subsequences of N and*

$$X_1 = k_1(a^{m(1,n)}x - a^{m(1,n)}y), \quad Y_1 = k_2(a^{m(2,n)}x - a^{m(2,n)}y) ,$$

$$Z = X + Y .$$

Then either Z is not bounded, or Z o-converges to 0.

P r o o f. Denote

$$Z' = ((k_1 a^{m(1,n)} + k_2 a^{m(2,n)})x) ,$$

$$Z'' = (-(k_1 a^{m(1,n)} + k_2 a^{m(2,n)})y) .$$

We have

$$Z = Z' + Z'' .$$

Thus we infer from (1) and (2) that Z is bounded if and only if both Z' and Z'' are bounded.

Next, since V is archimedean, the sequence Z' is bounded if and only if the sequence of reals

(4) $(k_1 a^{m(1,n)} + k_2 a^{m(2,n)})$

is bounded; similarly, Z'' is bounded if and only if the sequence (4) is bounded.

Thus if (4) fails to be bounded, then Z fails to be bounded as well.

If (4) is bounded, then according to Lemma 2.3 the sequence (4) o-converges to 0 and hence both Z' and Z'' o-converge to 0. Therefore Z o-converges to 0, too. □

For $(x_n), (y_n)$ in $S(V)$ we put $(x_n) \leqslant (y_n)$ if $x_n \leqslant y_n$ for each n. Then $\leqslant$ is a partial order on $S(V)$.

Lemma 2.5. *Let x and y be disjoint elements of V. Let Q be the set of all $X_1 \in S(V)$ that can be expressed as*

$$X_1 = k_1(a^{m(1,n)}x - a^{m(1,n)}y) ,$$

where $0 \neq k_1 \in \mathbb{R}$ and $(m(1,n))$ is a subsequence of N. Then any two distinct elements of Q are incomparable.

Proof. Let X_1 be as above and let X_2 be any element of Q,

$$X_2 = k_2(a^{m(2,n)}x - a^{m(2,n)}y) .$$

Assume that $X_1 \leqslant X_2$. Then in view of (3), for each n we have $k_1 a^{m(1,n)} \leqslant k_2 a^{m(2,n)}$ and, at the same time, $-k_1 a^{m(1,n)} \leqslant -k_2 a^{m(2,n)}$. Hence $k_1 a^{m(1,n)} = k_2 a^{m(2,n)}$ for each n and thus $X_1 = X_2$.

Let A_0 be as above. Put $A = A_0 \cup Q$. We define a binary relation $\equiv$ on $S(V)$ as follows: for $X, Y \in S(V)$ we put $X \equiv Y$ if some of the following conditions is valid:

(i) both X and Y belong to A_0;
(ii) both X and Y belong to Q;
(iii) one element of the set $\{X, Y\}$ belongs to Q and the other o-converges to 0.

$\square$

Lemma 2.6. *Assume that V is archimedean and suppose that x and y are disjoint elements in V. Let $(A, \equiv)$ be as above. The pair $(A, \equiv)$ is regular with respect to V.*

Proof. We have to verify that the conditions from σ' are satisfied. The validity of A1, A4 and A6 is obvious. To prove that A2 is valid we apply 2.4 and then proceed analogously as in the proof of Lemma 2, [4]. Next, A3 is a consequence of 2.5, and A5′ is implied by the definition of the set A above. $\square$

Lemma 2.7. *Let V be a vector lattice satisfying the condition $(*)$. Then V is linearly ordered.*

Proof. According to 2.2, V is archimedean. By way of contradiction, assume that V is not linearly ordered. Then there exist disjoint elements x and y in V. Let us construct A and $\equiv$ as above. In view of 2.6, the pair $(A, \equiv)$ is regular with respect to V. Hence according to $(*)$, $(A, \equiv)$ coincides with $(A_0, \equiv_0)$. Since $Q \neq \emptyset$, we obtain $A \neq A_0$, which is a contradiction. $\square$

Theorem 2.8. *Let V be a nonzero vector lattice. Then the following conditions are equivalent:*

(i) *V is isomorphic to $\mathbb{R}$;*
(ii) *V satisfies the condition $(*)$.*

Proof. In view of 1.1, (i) $\Rightarrow$ (ii). Let (ii) be valid. Then according to 2.2 and 2.7, V is archimedean and linearly ordered. Hence V is isomorphic to $\mathbb{R}$. $\square$

3. Concluding remarks

In the proof of 2.2 we have shown that if V is non-archimedean, then the pair $(A_0, \equiv_0)$ fails to be regular with respect to V.

This can be slightly sharpened as follows.

3.1. Proposition. *Let V be a non-archimedean vector lattice. Then no element $(A, \equiv)$ of $P(V)$ is regular with respect to V.*

Proof. By way of contradiction, assume that $(A, \equiv) \in P(V)$ and that $(A, \equiv)$ is regular with respect to V. There exist $x, y \in V$ such that $0 < nx < y$ for each n. Consider $\{\frac{1}{n}y\}$. By reasoning analogous to that in the proof of Theorem 6 of [7] we obtain that $(\frac{1}{n}y) \in A$, $(-\frac{1}{n}y) \in A$ and $(-\frac{1}{n}y) \equiv (\frac{1}{n}y)$ (cf. also Theorem 5, [7]). For each $n \in N$, the relation $-\frac{1}{n}y < (-1)^n x < \frac{1}{n}y$ is valid. Thus according to A3, $((-1)^n x) \in A$. This contradicts A5$'$. $\square$

3.2. Corollary. *Let V be a linearly ordered vector lattice. If $(A, \equiv) \in P(V)$ and $(A, \equiv)$ is regular, then $(A, \equiv)$ coincides with $(A_0, \equiv_0)$.*

Proof. If V is non-archimedean, then the assertion is (trivially) true in view of 3.1. If V is archimedean, then V is isomorphic to $\mathbb{R}$ and so it suffices to apply 1.1. $\square$

3.3. Corollary. *Let V be an archimedean vector lattice. Then the following conditions are equivalent:*

(i) *V is linearly ordered.*
(ii) *If $(A, \equiv) \in P(V)$ and $(A, \equiv)$ is regular, then $(A, \equiv)$ coincides with $(A_0, \equiv_0)$.*

Proof. In view of 3.2, (i) $\Rightarrow$ (ii). Let (ii) be valid. By way of contradiction, suppose that (i) does not hold. Then according to 2.6 and under the notation applied there $(A, \equiv)$ is regular and clearly $(A, \equiv) \neq (A_0, \equiv_0)$. Hence (ii) does not hold, a contradiction. $\square$

Taking into account what was said at the beginning of this section we can ask whether for each archimedean vector lattice V the pair $(A_0, \equiv_0)$ is regular with respect to V. The following example shows that the answer is negative.

3.4. Example. Let V be the vector lattice of all continuous real functions defined on $\mathbb{R}$ (the operations and the partial order are defined componentwise). For each $n \in N$ we define $f_n \in V$ as follows:

$$\begin{aligned} &f_n(t) = 0 \quad \text{for each} \quad t \leqslant 0;\\ &f_n(t) = 1 \quad \text{for each} \quad t \geqslant \frac{1}{n};\\ &f_n \quad \text{is linear on the closed interval} \quad \left[0, \frac{1}{n}\right]. \end{aligned}$$

Then $f_n < f_{n+1}$ for each $n \in N$, hence each subsequence of (f_n) is increasing. (f_n) is a bounded sequence in V. If $(f_{m(n)})$ is a subsequence of (f_n), then $\bigvee_n f_{m(n)}$ does not exist in V, hence $(f_{m(n)})$ fails to be o-convergent. Thus the condition A6 is not satisfied for $(A_0, \equiv_0)$. Therefore $(A_0, \equiv_0)$ fails to be regular with respect to V.

The following question remains open:

Which vector lattices V have the property that $(A_0, \equiv_0)$ is regular with respect to V?

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