HAMILTONIAN CONNECTEDNESS AND A MATCHING IN POWERS OF CONNECTED GRAPHS

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(Received March 24, 1994)

Abstract. In this paper the following results are proved:
1. Let P_n be a path with n vertices, where $n \geqslant 5$ and $n \neq 7, 8$. Let M be a matching in P_n. Then $(P_n)^4 - M$ is hamiltonian-connected.
2. Let G be a connected graph of order $p \geqslant 5$, and let M be a matching in G. Then $G^5 - M$ is hamiltonian-connected.

Keywords: power of a graph, matching, hamiltonian connectedness

MSC 1991: 05C70, 05C45

1. Introduction

By a graph we mean a finite undirected graph with no loops or multiple edges (a graph in the sense of [1] and [2]). If G is a graph, then we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the diameter of G, respectively. The number $|V(G)|$ is called the order of G. If $u, v, w \in V(G)$, then the degree of u in G and the distance between v and w in G will be denoted by $\deg_G u$ and $d_G(v, w)$, respectively. If $W \subseteq V(G)$, then we denote by $\langle W\rangle_G$ the subgraph of G induced by W.

A path connecting vertices u and v in G is referred to as $u - v$ path in G. We say that a graph G is hamiltonian-connected if for every pair of distinct vertices u and v of G, there exists a hamiltonian $u - v$ path in G.

If a spanning subgraph F of G is a regular graph of degree one, then we say that F is a 1-factor of G. A set $M \subseteq E(G)$ is called a matching in G if no two edges in M are incident with the same vertex. We denote by $\mathcal{M}(G)$ and $\mathcal{H}(G)$ the set of matchings in G and the set of hamiltonian paths of G, respectively.

For every integer $n \geqslant 1$, by the n-th power G^n of G we mean the graph with $V(G^n) = V(G)$ and

$$E(G^n) = \{uv\,;\, u, v \in V(G) \quad \text{and} \quad 1 \leqslant d_G(u, v) \leqslant n\}.$$

We now mention some results concerning hamiltonian properties of powers of connected graphs.

Theorem A. [5] *If G is a nontrivial connected graph, then G^3 is hamiltonian-connected.*

Theorem B. [6] *Let G be a connected graph of order $p \geqslant 4$ and let M be a matching in G. Then there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.*

Theorem C. [3] *Let G be a connected graph of order $p \geqslant 4$. Then for every matching M in G^4 there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.*

2. Results

In the present paper we prove the following two theorems:

Theorem 1. *Let P_n be a path with n vertices, where $n \geqslant 5$ and $n \neq 7, 8$. Let M be a matching in P_n. Then $(P_n)^4 - M$ is hamiltonian-connected.*

Theorem 2. *Let G be a connected graph of order $p \geqslant 5$ and let M be a matching in G. Then $G^5 - M$ is hamiltonian-connected.*

To prove Theorem 1 we will use two lemmas and five remarks. The following lemma immediately follows from Theorem B.

Lemma 1. *Let M be a matching in a complete graph K_n, where $n \geqslant 5$. Then $K_n - M$ is hamiltonian-connected.*

The following notation will be useful for us.

Let $n \geqslant 1$ be an integer, and let $w_1, \ldots, w_n$ be mutually distinct vertices. We denote by A_n the path with

$$V(A_n) = \{w_1, \ldots, w_n\} \quad \text{and} \quad E(A_n) = \{w_i w_{i+1}\,;\, 1 \leqslant i \leqslant n-1\}.$$

A permutation $(k_1, k_2, \ldots, k_n)$ of the set $\{1, 2, \ldots, n\}$ with the property that $|k_i - k_{i+1}| \leqslant k$ for every $i \in \{1, 2, \ldots, n-1\}$ determines the hamiltonian path $P \in \mathcal{H}((A_n)^k)$ with $E(P) = \{w_{k_1}w_{k_2}, w_{k_2}w_{k_3}, \ldots, w_{k_{n-1}}w_{k_n}\}$. The path P is a $w_{k_1} - w_{k_n}$ path of $(A_n)^k$ and also a $w_{k_n} - w_{k_1}$ path of $(A_n)^k$.

Finally, we define

$$A_{n*} = A_n - w_{n-1}w_n + w_{n-2}w_n \quad \text{for } n \geqslant 3.$$

Remark 1. Let M be a matching in A_4. Then there exist hamiltonian $w_1 - w_3$, $w_2 - w_3$ and $w_2 - w_4$ paths of $(A_4)^3 - M$.

Let T be a tree of order $p = 4$ which is not isomorphic to A_4. Then T is isomorphic to A_{4*}. For the sake of simplicity we will assume that $T = A_{4*}$. Let M be a matching in T. For every j, $j \in \{1, 3, 4\}$, there exists a hamiltonian $w_2 - w_j$ path of $T^2 - M$.

Remark 2. Let M be a matching in A_5. Clearly, $(A_5)^4$ is the complete graph. It follows from Lemma 1 that $(A_5)^4 - M$ is hamiltonian-connected.

We define the following matchings in A_5 :

$$M_1 = \{w_1w_2, w_3w_4\}, \quad M_2 = \{w_1w_2, w_4w_5\}, \quad M_3 = \{w_2w_3, w_4w_5\}.$$

For every matching $M' \in \mathcal{M}(A_5)$ there exists $k \in \{1, 2, 3\}$ such that $M' \subseteq M_k$.
The permutations

$$\begin{aligned}
&(1,3,5,4,2),\ (1,4,5,2,3),\ (1,3,2,5,4),\ (1,4,2,3,5),\ (2,4,1,3,5),\\
&(3,1,4,2,5),\ (4,1,3,2,5) \quad \text{for } k = 1,\\[1ex]
&(1,4,3,5,2),\ (1,4,2,5,3),\ (1,3,5,2,4),\ (1,4,3,2,5),\ (2,4,1,3,5),\\
&(3,1,4,2,5),\ (4,1,3,2,5) \quad \text{for } k = 2,\\[1ex]
&(1,4,3,5,2),\ (1,4,2,5,3),\ (1,3,5,2,4),\ (1,3,4,2,5),\ (2,1,4,3,5),\\
&(3,4,1,2,5),\ (4,2,1,3,5) \quad \text{for } k = 3
\end{aligned}$$

of the set $\{1, 2, 3, 4, 5\}$ determine in $(A_5)^3 - M_k$ the hamiltonian $w_1 - w_j$ and $w_i - w_5$ paths, where $1 \leqslant i < j \leqslant 5$.

Hence for every i, j, $i \in \{1, 2, 3, 4\}$ and $j \in \{2, 3, 4, 5\}$ there exist hamiltonian $w_i - w_5$ and $w_1 - w_j$ paths of $(A_5)^3 - M$.

Remark 3. Let M be a matching in A_6. The permutations

$(1,4,6,3,5,2)$, $(1,4,6,2,5,3)$, $(1,3,5,2,6,4)$, $(1,3,6,4,2,5)$, $(1,3,5,2,4,6)$, $(2,5,1,4,6,3)$, $(2,6,3,5,1,4)$, $(2,6,4,1,3,5)$, $(2,5,3,1,4,6)$, $(3,6,2,5,1,4)$, $(3,6,2,4,1,5)$, $(3,5,1,4,2,6)$, $(4,1,3,6,2,5)$, $(4,1,3,5,2,6)$, $(5,2,4,1,3,6)$
of the set $\{1,\ldots,6\}$ determine the hamiltonian $w_i - w_j$ paths of $(A_6)^4 - M$, where $1 \leqslant i < j \leqslant 6$.

This means that $(A_6)^4 - M$ is hamiltonian-connected.

Remark 4. Let M be a matching in A_7. The permutations

$(1,4,6,3,7,5,2)$, $(1,4,6,2,5,7,3)$, $(1,3,7,5,2,6,4)$, $(1,3,7,4,6,2,5)$, $(1,3,7,5,2,4,6)$, $(1,3,6,4,2,5,7)$, $(2,6,4,1,5,7,3)$, $(2,6,3,7,5,1,4)$, $(2,6,4,1,3,7,5)$, $(2,5,7,3,1,4,6)$, $(2,6,4,1,3,5,7)$, $(6,2,4,1,5,7,3)$, $(6,2,5,7,3,1,4)$, $(6,2,4,1,3,7,5)$, $(6,2,4,1,5,3,7)$, $(7,5,1,4,2,6,3)$, $(7,5,2,6,3,1,4)$, $(7,3,6,2,4,1,5)$
of the set $\{1,\ldots,7\}$ determine the hamiltonian $w_i - w_j$ paths of $(A_7)^4 - M$, where $i \in \{1,2,6,7\}$, $j \in \{1,2,\ldots,7\}$ and $i \neq j$.

The permutations
(3,6,2,7,5,1,4), (3,6,2,7,4,1,5), (4,1,3,6,2,7,5) of the set $\{1,2,\ldots,7\}$ determine the hamiltonian $w_3 - w_4$, $w_3 - w_5$, $w_4 - w_5$ paths of $(A_7)^5 - M$.

If $M = \{w_1w_2, w_6w_7\}$, then there exist no hamiltonian $w_3 - w_4$, $w_3 - w_5$, $w_4 - w_5$ paths of $(A_7)^4 - M$.

This means that $(A_7)^5 - M$ is hamiltonian-connected and for $i \in \{1,2,6,7\}$, $j \in \{1,2,\ldots,7\}$, $i \neq j$ there exist hamiltonian $w_i - w_j$ paths of $(A_7)^4 - M$.

Remark 5. Let M be a matching in A_8.

1. We denote

$$M_1 = E(A_8 - w_1) \cap M.$$

Then $M_1 \in \mathcal{M}(A_8 - w_1)$. It follows from Remark 4 that for every j, $j \in \{2,4,5,6,7,8\}$, there exists a hamiltonian $w_3 - w_j$ path $P_1 \in \mathcal{H}((A_8 - w_1)^4 - M_1)$. Then

$P = P_1 + w_1w_3$ is a hamiltonian $w_1 - w_j$ path of $(A_8)^4 - M$,

$P = P_1 + w_1w_j$ is a hamiltonian $w_1 - w_3$ path of $(A_8)^4 - M$ if $j = 4$.

Analogously we can show that for every j, $j \in \{1,2,\ldots,7\}$, there exists a hamiltonian $w_8 - w_j$ path of $(A_8)^4 - M$.

2. We denote

$$M_1 = E(A_8 - w_1 - w_2 - w_3) \cap M.$$

Then $M_1 \in \mathcal{M}(A_8 - w_1 - w_2 - w_3)$. It follows from Remark 2 that for every $j, j \in \{4,6,7,8\}$, there exists a hamiltonian $w_5 - w_j$ path $P_1 \in \mathcal{H}((A_8 - w_1 - w_2 - w_3)^4 - M_1)$.

We put

$$P = P_1 + w_5w_3 + w_3w_1 + w_1w_2 \quad \text{if} \quad w_1w_2 \notin M,$$
$$P = P_1 + w_5w_1 + w_1w_3 + w_3w_2 \quad \text{if} \quad w_1w_2 \in M.$$

Then P is a hamiltonian $w_2 - w_j$ path of $(A_8)^4 - M$.
Further, we put

$$P = P_1 + w_jw_1 + w_1w_3 + w_3w_2 \quad \text{if} \quad j = 4 \quad \text{and} \quad w_2w_3 \notin M,$$
$$P = P_1 + w_jw_3 + w_3w_1 + w_1w_2 \quad \text{if} \quad j = 4 \quad \text{and} \quad w_2w_3 \in M.$$

Then P is a hamiltonian $w_2 - w_5$ path of $(A_8)^4 - M$.
The path

$$P = P_1 + w_5w_1 + w_1w_3 + w_2w_j \quad \text{if} \quad j = 4$$

is a hamiltonian $w_2 - w_3$ path of $(A_8)^4 - M$.

Analogously we can show that for every j, $j \in \{1, 2, \ldots, 6, 8\}$, there exists a hamiltonian $w_7 - w_j$ path of $(A_8)^4 - M$.

3. The permutations

$$(3,8,6,2,7,5,1,4),\ (3,8,6,2,7,4,1,5),\ (3,8,5,2,7,4,1,6),$$
$$(4,1,3,8,6,2,7,5),\ (4,1,3,8,5,7,2,6),\ (5,1,3,8,4,7,2,6)$$

of the set $\{1, \ldots, 8\}$ determine the hamiltonian $w_i - w_j$ paths of $(A_8)^5 - M$, where $3 \leqslant i < j \leqslant 6$.

4. If $M = \{w_1w_2, w_3w_4, w_5w_6, w_7w_8\}$, then for i, j, $3 \leqslant i < j \leqslant 6$ there exists no hamiltonian $w_i - w_j$ path of $(A_8)^4 - M$.

This means that $(A_8)^5 - M$ is hamiltonian-connected and for $i \in \{1, 2, 7, 8\}$, $j \in \{1, 2, \ldots, 8\}$, $i \neq j$ there exists a hamiltonian $w_i - w_j$ path of $(A_8)^4 - M$.

Lemma 2. *Let $n \geqslant 9$, and let M be a matching in A_n. Then $(A_n)^4 - M$ is hamiltonian-connected.*

P r o o f. We distinguish the following cases and subcases:

1. Let $n = 9$. In $(A_9)^4 - M$ we shall construct hamiltonian $w_i - w_j$ paths, where $1 \leqslant i < j \leqslant 9$. Denote

$$W_1 = \{w_1, \ldots, w_5\},\ W_2 = \{w_5, \ldots, w_9\},$$
$$G_1 = \langle W_1 \rangle_{A_9} \ \text{and} \ G_2 = \langle W_2 \rangle_{A_9}.$$

Moreover, denote by M_1 and M_2 the matchings with the properties

$$M_1 \in \mathcal{M}(G_1),\ M_2 \in \mathcal{M}(G_2) \text{ and } M_1 \cup M_2 = M.$$

1.1. $1 \leqslant i < j \leqslant 5$ or $5 \leqslant i < j \leqslant 9$.
We prove the proposition of Lemma 2 for the case $1 \leqslant i < j \leqslant 5$.
If $5 \leqslant i < j \leqslant 9$, then the proof is analogous.

It follows from Remark 2 that there exists a hamiltonian $w_i - w_j$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_5 - w_6$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. If $w_j = w_5$, then according to Remark 2 there exists a hamiltonian $w_i - w_5$ path $P_1 \in \mathcal{H}((G_1)^3 - M_1)$. This implies that there exists $x \in V(G_1)$ such that $xw_5 \in E(P_1)$ and $x \neq w_1$.
Then $d_{A_9}(x, w_6) \leqslant 4$. We put

$$P = (P_1 \cup P_2) - xw_5 + xw_6.$$

Then P is a hamiltonian $w_i - w_j$ path of $(A_9)^4 - M$.

1.2. $1 \leqslant i < 5$ and $5 < j \leqslant 9$.
According to Lemma 1 there exists a hamiltonian $w_i - w_5$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_5 - w_j$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. We put

$$P = P_1 \cup P_2.$$

Then P is a hamiltonian $w_i - w_j$ path of $(A_9)^4 - M$.

From these two subcases it follows that $(A_9)^4 - M$ is hamiltonian-connected.

2. Let $n \geqslant 10$. Assume that for every tree A_m, where $9 \leqslant m < n$, it is proved that $(A_m)^4 - M^*$ is hamiltonian-connected for any matching $M^* \in \mathcal{M}(A_m)$.
In $(A_n)^4 - M$ we shall construct hamiltonian $w_i - w_j$ paths, where $1 \leqslant i < j \leqslant n$.

2.1. $1 \leqslant i < j \leqslant 5$ or $(n-4) \leqslant i < j \leqslant n$.
We prove the proposition of Lemma 2 for the case $1 \leqslant i < j \leqslant 5$. If $(n-4) \leqslant i < j \leqslant n$, then the proof is analogous. Denote

$$W_1 = \{w_1, \ldots, w_5\},\ W_2 = \{w_5, \ldots, w_n\},$$
$$G_1 = \langle W_1 \rangle_{A_n} \text{ and } G_2 = \langle W_2 \rangle_{A_n}.$$

Moreover, denote by M_1 and M_2 the matchings with the properties

$$M_1 \in \mathcal{M}(G_1),\ M_2 \in \mathcal{M}(G_2) \text{ and } M_1 \cup M_2 = M.$$

It follows from the induction hypothesis and Remarks 3, 4, 5 that there exists a hamiltonian $w_5 - w_6$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. It follows from Remark 2 that there exists a hamiltonian $w_i - w_j$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and if $w_j = w_5$, then $P_1 \in \mathcal{H}((G_1)^3 - M_1)$. This implies that there exists $x \in V(G_1)$ such that $xw_5 \in E(P_1)$ and $x \neq w_1$. Then $d_{A_n}(x, w_6) \leqslant 4$ and

$$P = (P_1 \cup P_2) - xw_5 + xw_6$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^4 - M$.

2.2. $1 \leqslant i \leqslant 4$ and $6 \leqslant j \leqslant n$ or $5 \leqslant i < j \leqslant n-4$ or $5 \leqslant i \leqslant n-5$ and $n-3 \leqslant j \leqslant n$.

2.2.1. There exists $w_k \in V(A_n)$ with the property

$$i < k < j \quad \text{and} \quad 5 \leqslant k \leqslant n-4. \tag{1}$$

Denote

$$W_1 = \{w_1, \ldots, w_k\},\ W_2 = \{w_k, w_{k+1}, \ldots, w_n\},$$
$$G_1 = \langle W_1 \rangle_{A_n} \quad \text{and} \quad G_2 = \langle W_2 \rangle_{A_n}.$$

Further, denote by M_1 and M_2 the matchings with the properties

$$M_1 \in \mathcal{M}(G_1),\ M_2 \in \mathcal{M}(G_2) \quad \text{and} \quad M_1 \cup M_2 = M.$$

According to the induction hypothesis and Remarks 2, 3, 4, 5 there exists a hamiltonian $w_i - w_k$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_k - w_j$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. Then

$$P = P_1 \cup P_2$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^4 - M$.

2.2.2. There exists no $w_k \in V(A_n)$ with the property (1). Then $w_iw_j \in E(A_n)$ and $5 \leqslant i < j \leqslant n-4$. Hence $w_j = w_{i+1}$.

We denote by G_1 or G_2 the component of $A_n - w_iw_{i+1}$ which contains w_i or w_{i+1}, respectively. Further, we denote by M_1 and M_2 the matchings with the properties

$$M_1 \in \mathcal{M}(G_1),\ M_2 \in \mathcal{M}(G_2),\ M_1 = M \cap E(G_1) \quad \text{and} \quad M_2 = M \cap E(G_2).$$

It follows from the induction hypothesis and Remarks 2, 3, 4, 5 that there exists a hamiltonian $w_{i-1} - w_i$ path $P_1 \in \mathcal{H}((G_1)^4 - M_1)$ and a hamiltonian $w_{i+1} - w_{i+2}$ path $P_2 \in \mathcal{H}((G_2)^4 - M_2)$. Then

$$P = P_1 \cup P_2 + w_{i+1}w_{i+2}$$

is a hamiltonian $w_i - w_j$ path of $(A_n)^4 - M$.

From this subcases it follows that $(A_n)^4 - M$ is hamiltonian-connected. Thus the proof of Lemma 2 is complete. $\square$

Theorem 1 immediately follows from Lemma 2 and Remarks 2 and 3.

To prove Theorem 2 we will use the previous lemmas and remarks as well as the two following lemmas.

Lemma 3. *Let T be a tree of order $p \geqslant 5$ and let M be a matching in T. Then $T^5 - M$ is hamiltonian-connected.*

Proof. The cases when $p \in \{5, 6, 7\}$ follows immediately from Lemma 1 and Remark 4.

Let $p = 8$. If T is isomorphic to A_8, or $\delta(T) \leqslant 5$, then the proposition of Lemma 3 follows from Remark 5 and Lemma 1.
Denote

$$\begin{aligned} T_1 &= A_{8*}, \\ T_2 &= A_8 - w_7w_8 + w_5w_8, \\ T_3 &= A_8 - w_7w_8 + w_4w_8, \\ \mathcal{T} &= \{T_1, T_2, T_3\}. \end{aligned}$$

If T is not isomorphic to A_8 and $\delta(T) > 5$, then T is isomorphic to one of the elements of $\mathcal{T}$. For the sake of simplicity we shall assume that $T \in \mathcal{T}$. Further, we denote

$$M_0 = E(T - w_8) \cap M.$$

Then $T - w_8 = A_7$ and $M_0 \in \mathcal{M}(A_7)$. It follows from Remark 4 that there exists a hamiltonian $w_i - w_j$ path $P_0 \in \mathcal{H}((A_7)^5 - M_0)$, where $i, j \in \{1, \ldots, 7\}, i \neq j$. Since $|E(P_0)| = 6$, there exist integers k, l, $k, l \in \{1, \ldots, 7\}, k \neq l$, such that $w_kw_l \in E(P_0)$ and

$$\begin{aligned} &k,\, l \notin \{1, 6\} \quad \text{if} \quad T = T_1, \\ &k,\, l \neq 5 \quad \text{if} \quad T = T_2, \\ &k,\, l \neq 4 \quad \text{if} \quad T = T_3. \end{aligned}$$

Then
$P = P_0 - w_kw_l + w_kw_8 + w_1w_8$ is a hamiltonian $w_i - w_j$ path of $T^5 - M$, where $i, j \in \{1, \ldots, 7\}$,
$P = P_0 + w_jw_8$ is a hamiltonian $w_i - w_8$ path of $T^5 - M$ if $j = 3$ and $i \in \{1, 2, 4, 5, 6, 7\}$,
$P = P_0 + w_iw_8$ is a hamiltonian $w_3 - w_8$ path of $T^5 - M$ if $i = 2$ and $j = 3$.

This means that for $p = 8$ the statement of Lemma 3 is correct.

Let $p \geqslant 9$. Assume that for every tree T^* of order p^*, where $5 \leqslant p^* < p$, it is proved that $(T^*)^5 - M^*$ is hamiltonian-connected for any matching $M^* \in \mathcal{M}(T^*)$.

If T is isomorphic to A_p, or if $\delta(T) \leqslant 5$, then the result follows from Lemma 2 or Lemma 1. We shall assume that T is not isomorphic to A_p and $\delta(T) > 5$.

Let x and y be arbitrary distinct vertices of T. We shall construct a hamiltonian $x - y$ path P of $T^5 - M$.

We denote by t_x, t_y the vertices of T with the following properties:

(1) $t_x t_y \in E(T)$,

(2) t_x, t_y belong to the $x - y$ path in T,

(3) $0 \leqslant d_T(t_x, x) < d_T(t_y, x)$.

Then $T - t_x t_y$ has two components. We denote by T_x or T_y the component of $T - t_x t_y$ which contains x, t_x or y, t_y, respectively. Further, we denote by M_x and M_y the matching with the properties

$$M_x \in \mathcal{M}(T_x),\ M_y \in \mathcal{M}(T_y),\ M_x = M \cap E(T_x) \ \text{ and } \ M_y = M \cap E(T_y).$$

We define graphs T_1 and T_2:

$$T_1 = T_x \quad \text{and} \quad V(T_2) = V(T_y) \cup \{t_x\},\ E(T_2) = E(T_y) \cup \{t_x t_y\}.$$

Finally, we denote by M_1 and M_2 the matchings with the properties

$$M_1 \in \mathcal{M}(T_1),\ M_2 \in \mathcal{M}(T_2),\, M_1 = M_x \ \text{ and } \ M_2 = M \cap E(T_2).$$

We distinguish the following cases and subcases:

1. There exist $t_x, t_y \in V(T)$ with the properties (1)–(3) such that $|V(T_x)| \geqslant 5$ and $|V(T_y)| \geqslant 5$. Then $|V(T_1)| \geqslant 5$ and $|V(T_2)| \geqslant 5$.

1.1. Let $t_x \neq x$. According to the induction hypothesis there exists a hamiltonian $x - t_x$ path $P_1 \in \mathcal{H}((T_1)^5 - M_1)$ and a hamiltonian $t_x - y$ path $P_2 \in \mathcal{H}((T_2)^5 - M_2)$. We put

$$P = P_1 \cup P_2.$$

1.2. Let $t_x = x$. We denote by x_1 the vertex of T_x with the property that $xx_1 \in E(T_x)$. If $t_y = y$, then we denote by y_1 the vertex of T_y with the property that $yy_1 \in E(T_y)$. Then $d_T(x_1, t_y) = 2$ and $d_T(x_1, y_1) = 3$. It follows from the induction hypothesis that there exists a hamiltonian $x - x_1$ path $P_1 \in \mathcal{H}((T_x)^5 - M_x)$ and a hamiltonian path $P_2 \in \mathcal{H}((T_y)^5 - M_y)$. Let us suppose that

$$\begin{aligned} &P_2 \text{ is a hamiltonian } t_y - y \text{ path} &&\text{if} \quad t_y \neq y \\ &P_2 \text{ is a hamiltonian } y_1 - y \text{ path} &&\text{if} \quad t_y = y. \end{aligned}$$

We put

$$P = P_1 \cup P_2 + x_1 t_y \quad \text{if} \quad t_y \neq y$$
$$P = P_1 \cup P_2 + x_1 y_1 \quad \text{if} \quad t_y = y.$$

2. For every two vertices t_x, t_y with the properties (1)–(3) we have $|V(T_x)| < 5$ or $|V(T_y)| < 5$. We put $t_y = y$. Without loss of generality we assume that $|V(T_y)| < 5$.

2.1. Let $|V(T_y)| = 1$. Then $V(T_y) = \{y\}$ and $|V(T_x)| \geqslant 8$. There exists $u \in V(T_x)$ such that $u \neq x, u \neq t_x$ and $1 \leqslant d_T(u, t_x) \leqslant 2$. Then $2 \leqslant d_T(u, y) \leqslant 3$. It follows from the induction hypothesis that there exists a hamiltonian $x - u$ path $P_1 \in \mathcal{H}((T_x)^5 - M_x)$. We put

$$P = P_1 + uy.$$

2.2. Let $|V(T_y)| = 4$. According to Remark 1 there exists a hamiltonian $y - v$ path $P_2 \in \mathcal{H}((T_y)^5 - M_y)$, where $v \in V(T_y)$ and

$$d_T(v, y) = 1 \quad \text{if} \quad T_y \text{ is not isomorphic to } A_4,$$
$$d_T(v, y) = 2 \quad \text{if} \quad T_y \text{ is isomorphic to } A_4.$$

Since $|V(T_y)| = 4$ and $p \geqslant 9$, we have $|V(T_x)| \geqslant 5$. We denote by u the vertex with the properties

$$u \in V(T_x), \quad u \neq x \quad \text{and} \quad d_T(u, y) \leqslant 2.$$

Then $d_T(u, v) \leqslant 4$. It follows from the induction hypothesis that there exists a hamiltonian $x - u$ path $P_1 \in \mathcal{H}((T_x)^5 - M_x)$. We put

$$P = P_1 \cup P_2 + vu.$$

2.3. Let $1 < |V(T_y)| < 4$. Let $S_1, \ldots, S_m$ be all components of $T - t_x$ which are different from T_y. We denote by $L_1, \ldots, L_m$ the matchings in $S_1, \ldots, S_m$ such that $L_j = M \cap E(S_j)$ for $j = 1, \ldots, m$.

2.3.1. There exists i, $i \in \{1, \ldots, m\}$ such that $|V(S_i)| \geqslant 5$.
Then there exist $u_1, u_2 \in V(S_i)$ such that $u_1 \neq u_2 \neq x$, $d_T(u_1, t_x) \leqslant 2$, $1 < d_T(u_2, t_x) \leqslant 3$, and if $x \notin V(S_i)$, then $d_T(u_1, t_x) = 1$. According to the induction hypothesis there exists a hamiltonian path $P_1 \in \mathcal{H}((S_i)^5 - L_i)$. Let us suppose that

$$P_1 \text{ is a hamiltonian } u_1 - u_2 \text{ path} \quad \text{if} \quad x \notin V(S_i),$$
$$P_1 \text{ is a hamiltonian } u_2 - x \text{ path} \quad \text{if} \quad x \in V(S_i).$$

Denote

$$T_0 = T - V(S_i).$$

Then T_0 is a tree, $|V(T_0)| \geqslant 3$ and $y \in V(T_0)$. Further we denote by M_0 the matching in T_0 such that $M_0 = M \cap E(T_0)$.

2.3.1.1. Let $|V(T_0)| = 3$. Then $m = i = 1$ and there exists $v \in V(T_0)$ such that $V(T_0) = \{t_x, y, v\}$ and $E(T_0) = \{t_x y, yv\}$. If $x \notin V(S_1)$, then $x = t_x$. We put

$$\begin{aligned} P &= P_1 + u_1 v + vx + u_2 y \quad \text{if} \quad x \notin V(S_1),\\ P &= P_1 + u_2 v + v t_x + t_x y \quad \text{if} \quad x \in V(S_1) \text{ and } t_x y \notin M,\\ P &= P_1 + u_2 t_x + t_x v + vy \quad \text{if} \quad x \in V(S_1) \text{ and } t_x y \in M. \end{aligned}$$

2.3.1.2. Let $|V(T_0)| = 4$. Assume that $x \in V(S_i)$. Then according to Remark 1 there exists a hamiltonian $y - v$ path $P_2 \in \mathcal{H}((T_0)^3 - M_0)$, where $v \in V(T_0)$, $v \neq y$ and

$$\begin{aligned} d_T(t_x, v) &= 2 \quad \text{if} \quad \deg_{T_0} t_x = 1,\\ d_T(t_x, v) &= 1 \quad \text{if} \quad \deg_{T_0} t_x = 2. \end{aligned}$$

Then $d_T(v, u_2) \leqslant 5$. We put

$$P = P_1 \cup P_2 + u_2 v.$$

Let $x \notin V(S_i)$. There exist $v_1, v_2 \in V(T_0)$ such that $v_1 \neq v_2 \neq t_x \neq y$. Then $V(T_0) = \{t_x, y, v_1, v_2\}$. We put

$$\begin{aligned} P &= P_1 + u_1 v_2 + v_2 y + u_2 v_1 + v_1 x \quad \text{if} \quad x = t_x \text{ and } E(T_0) = \{xy, yv_1, v_1 v_2\},\\ P &= P_1 + u_1 v_2 + v_2 v_1 + v_1 x + u_2 y \quad \text{if} \quad x = t_x \text{ and } E(T_0) = \{xy, yv_1, yv_2\}\\ &\qquad \text{or} \quad \text{if} \quad x = t_x \text{ and } E(T_0) = \{xy, yv_1, xv_2\},\\ P &= P_1 + u_1 y + u_2 t_x + t_x v_1 + v_1 x \quad \text{if} \quad x = v_2 \text{ and } E(T_0) = \{x t_x, t_x y, y v_1\}. \end{aligned}$$

2.3.1.3. Let $|V(T_0)| \geqslant 5$. Since $|V(T_x)| < 5$ or $|V(T_y)| < 5$ for every two vertices t_x, t_y of T with the properties (1)–(3), we have $x \notin V(S_i)$. It follows from the induction hypothesis that there exists a hamiltonian $x - y$ path $P_2 \in \mathcal{H}((T_0)^5 - M_0)$. Since $|V(T_y)| < 4$, there exists $v \in V(T_0)$ such that $vy \in E(P_2)$ and $d_T(v, t_x) \leqslant 4$. We put

$$\begin{aligned} P &= P_1 \cup P_2 - yv + u_1 v + u_2 y \quad \text{if} \quad v \neq t_x,\\ P &= P_1 \cup P_2 - yv + u_2 v + u_1 y \quad \text{if} \quad v = t_x. \end{aligned}$$

2.3.2. For every i, $i \in \{1, \ldots, m\}$ we have $|V(S_i)| < 5$. Denote

$$T_0 = T - V(T_y), \quad M_0 = M \cap E(T_0).$$

Then $|V(T_0)| > 5$, $M_0 \in \mathcal{M}(T_0)$, $x \in V(T_0)$ and for every i, $i \in \{1, \ldots, m\}$, we have $V(S_i) \subset V(T_0)$. There exists $v \in V(T_0)$ such that $v \neq x$ and $1 \leqslant d_T(v, t_x) \leqslant 2$. It follows from the induction hypothesis that there exists a hamiltonian $x-v$ path $P_0 \in \mathcal{H}((T_0)^5 - M_0)$. Since $|V(T_y)| \in \{2, 3\}$ and $\delta(T) > 5$, there exists k, $k \in \{1, \ldots, m\}$, such that S_k is isomorphic to one of the elements of $\mathcal{A}$, where

$$\mathcal{A} = \{A_3, A_4, A_{4*}\}.$$

For the sake of simplicity we shall assume that $S_k \in \mathcal{A}$. Then

$$V(S_k) = \{w_1, \ldots, w_n\}, \quad \text{where } n \in \{3, 4\},$$
$$d_T(w_j, t_x) = j, \quad \text{for every } j,\ j \in \{1, 2, 3\},$$
$$d_T(w_4, t_x) = 4 \quad \text{if} \quad S_k = A_4 \quad \text{and} \quad d_T(w_4, t_x) = 3 \quad \text{if} \quad S_k = A_{4*}.$$

Let a_2 and a_3 be distinct vertices of T_0 such that a_2w_2, $a_3w_3 \in E(P_0)$. If $S_k = A_4$, then there exists h, $h \in \{2, 3\}$, such that $a_h \neq w_4$. Then $d_T(a_h, t_x) \leqslant 3$. The component T_y is isomorphic to one of the elements of $\mathcal{B}$, where

$$\mathcal{B} = \{A_2, A_3, A_{3*}\}.$$

We denote the vertices of T_y by $t_1, \ldots, t_n$ ($n \in \{2, 3\}$) so that

$$d_T(t_jt_x) = j \quad \text{if} \quad j \in \{1, 2\},$$
$$d_T(t_3t_x) = 3 \quad \text{if} \quad T_y \text{ is isomorphic to } A_3,$$
$$d_T(t_3t_x) = 2 \quad \text{if} \quad T_y \text{ is isomorphic to } A_{3*}.$$

Then $t_1 = y$, $d_T(a_h, t_2) \leqslant 5$, $d_T(w_2, t_2) = 4$, $d_T(w_3, t_2) = 5$ and $d_T(v, t_3) \leqslant 5$. We put

$$P = P_0 - a_hw_h + vy + a_ht_2 + w_ht_2 \quad \text{if} \quad T_y \text{ is isomorphic to } A_2,$$
$$P = P_0 - a_hw_h + vt_3 + t_3y + a_ht_2 + w_ht_2 \quad \text{if} \quad T_y \text{ is isomorphic to } A_3,$$
$$P = P_0 - a_hw_h + vy + a_ht_2 + t_2t_3 + t_3w_h \quad \text{if} \quad T_y \text{ is isomorphic to } A_{3*}.$$

We can see that in each subcase P is the hamiltonian $x - y$ path of $T^5 - M$. Thus the proof of Lemma 3 is complete. □

Lemma 4. ([4] p.63) *Let G be a connected graph and let L be a subgraph of G which contains no cycle. Then there exists a spanning tree T of G such that L is a subgraph of T.*

P r o o f of Theorem 2. Let G be a graph satisfying the conditions of Theorem 2 and let M be an arbitrary matching in G. As follows from Lemma 4, there exists a spanning tree T of G such that M is a matching in T. According to Lemma 3, $T^5 - M$ is hamiltonian-connected. Thus $G^5 - M$ is also hamiltonian-connected. □

R e m a r k 6 . Let $n \geqslant 1$ be an integer, and let G be the tree of order $p = 4n + 4$ which is given in Fig. 1. Let

$$M = \{u_{i1}u_{i2}, u_{i3}u_{i4}\,;\ 1 \leqslant i \leqslant n\} \cup \{xy, w_3w_4\}$$

be a matching in G. Then there exists no hamiltonian $x - y$ path of $G^4 - M$.

This means that the value 5 of the power in Theorem 2 is the best possible.

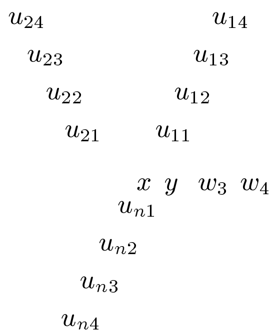

Fig. 1

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