PROJECTIONS OF RELATIONS

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Abstract. A projection of a relation is defined as a relation of reduced arity. The paper deals with projections of relations in coherence with their reflexivity, symmetry, completeness, regularity, cyclicity and other properties. Relationships between projections of hulls and hulls of projections are also studied.

Keywords: relation, n-decomposition, diagonal, $(\mathcal{K}, \varphi)$-modification, (p)-hull, (q, X)-projection

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0. Introduction

The paper is a continuation of [2] where a modification of relational axioms was presented. Roughly speaking, a projection of a relation is a certain relation of reduced arity. The concept of projection has been used, among others, in a special case by V. Novák in [3]. Projections have been studied by J. Šlapal in [10] as well.

Let G, H be nonempty sets. By a *relation* (with the *carrier* G and the *index set* H) we understand a set $R \subseteq G^H$ where G^H denotes the set of all mappings of the set H into the set G. $\mathbb{N}$ will denote the set of all positive integers. For any $n \in \mathbb{N}$ we denote $(n] = \{m \in \mathbb{N};\, m \leqslant n\}$. In the case of a finite set H of cardinality k we shall not distinguish between mappings of the set H into the set G and k-tuples of elements of the set G. For any $n \in \mathbb{N}$, $q \in (n]$, we denote by S_n the set of all permutations of the set $(n]$, by $S_n(q)$ the set of all permutations of the set $(n]$ mapping q onto itself (preserving q). For any $\varphi \in S_n$ and $m \in \mathbb{N}$, φ^m denotes the m-th iteration of the permutation φ, for any φ, $\psi \in S_n$, $\varphi\psi$ denotes the composition of the permutations φ and ψ. The symbol id denotes the identical permutation of the set $(n]$, π denotes the permutation of the set $(n]$ defined by

$$\pi(i) = \begin{cases} i+1 & \text{for all } i \in (n-1] \\ 1 & \text{for } i = n. \end{cases}$$

For any $n \in \mathbb{N}$, an *n-decomposition* of the set H means a sequence of $n+1$ sets $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ such that

(i) $\bigcup\limits_{i=1}^{n+1} K_i = H$,

(ii) $K_i \cap K_j = \emptyset$ for all i, $j \in (n+1]$, $i \neq j$.

If $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ is an n-decomposition of the set H, then the relation

$$E_{\mathcal{K}} = \{f \in G^H\,;\ f(K_i) = f(K_j) \text{ for all } i, j \in (n]\}$$

is called the *diagonal with regard to* $\mathcal{K}$. If, moreover, $R \subseteq G^H$ is a relation, $\varphi \in S_n$, then the relation

$$R_{\mathcal{K},\varphi} = \big\{f \in G^H\,;\ \exists g \in R\colon f(K_i) = g\big(K_{\varphi(i)}\big) \text{ for all } i \in (n], f(K_{n+1}) = g(K_{n+1})\big\}$$

is called the *$(\mathcal{K}, \varphi)$-modification* of the relation R. We put

$$R^1_{\mathcal{K},\varphi} = R_{\mathcal{K},\varphi},$$
$$R^m_{\mathcal{K},\varphi} = (R^{m-1}_{\mathcal{K},\varphi})_{\mathcal{K},\varphi}$$

for any $m \in \mathbb{N}$, $m \geqslant 2$.

If $R \subseteq G^H$ is a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $\varphi \in S_n$, then R is called

(iii) *reflexive (irreflexive) with regard to* $\mathcal{K}$ if $E_{\mathcal{K}} \subseteq R$ ($R \cap E_{\mathcal{K}} = \emptyset$),

(iv) *symmetric (asymmetric, antisymmetric) with regard to* $\mathcal{K}$ *and* φ if $R_{\mathcal{K},\varphi} \subseteq R$ ($R \cap R_{\mathcal{K},\varphi} = \emptyset$, $R \cap R_{\mathcal{K},\varphi} \subseteq E_{\mathcal{K}}$),

(v) *complete with regard to* $\mathcal{K}$ if $f \in G^H$, $f(K_i) \neq f(K_j)$ for all i, $j \in (n]$, $i \neq j$ imply the existence of $\psi \in S_n$ such that $f \in R_{\mathcal{K},\psi}$,

(vi) *regular with regard to* $\mathcal{K}$ if $f \in R$, $g \in G^H$, $f(K_i) = g(K_i)$ for all $i \in (n+1]$ imply $g \in R$,

(vii) *cyclic with regard to* $\mathcal{K}$ if $R_{\mathcal{K},\pi} \subseteq R$.

If $R \subseteq G^H$ is a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$, (p) any of the properties (iii)–(vii), then a relation $Q \subseteq G^H$ is called the *(p)-hull of R with regard to* $\mathcal{K}$ (*and* φ) if

(viii) $R \subseteq Q$,

(ix) Q has the property (p),

(x) if $T \subseteq G^H$ is any relation having the property (p) and such that $R \subseteq T$, then $Q \subseteq T$.

We recall some results from [2] that will be essential in the sequel:

Lemma 1. ([2], 1.14(1), 1.15(4), 2.3) *Let R, $T \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of the set H, φ, $\psi \in S_n$. Then:*

(1) $(R_{\mathcal{K},\varphi})_{\mathcal{K},\psi} = R_{\mathcal{K},\varphi\psi}$.

(2) $R \subseteq T$ *implies* $R_{\mathcal{K},\varphi} \subseteq T_{\mathcal{K},\varphi}$.

(3) $R_{\mathcal{K},\varphi}$ *is regular with regard to* $\mathcal{K}$.

(4) *If R is symmetric with regard to $\mathcal{K}$ and φ, then R is regular with regard to $\mathcal{K}$.*

Lemma 2. ([2], 3.2) *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Let* (p) *be any of the properties* (iii)–(vii)*. Then R has the property (p) if and only if the (p)-hull Q of R with regard to $\mathcal{K}$ (and φ) exists and $R = Q$.*

Lemma 3. ([2], 3.6, 3.7) *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Let $\varphi \in S_n$, $r \in \mathbb{N}$ be such that $\varphi^r = \mathrm{id}$. Then the following relation exist:*

(5) *the reflexive hull $R^{(r)}_{\mathcal{K}}$ of R with regard to $\mathcal{K}$ and we have*

$$R^{(r)}_{\mathcal{K}} = R \cup E_{\mathcal{K}},$$

(6) *the symmetric hull $R^{(s)}_{\mathcal{K},\varphi}$ of R with regard to $\mathcal{K}$ and φ and we have*

$$R^{(s)}_{\mathcal{K},\varphi} = \bigcup_{i=1}^{r} R^{i}_{\mathcal{K},\varphi},$$

(7) *the regular hull $R^{(g)}_{\mathcal{K}}$ of R with regard to $\mathcal{K}$ and we have*

$$R^{(g)}_{\mathcal{K}} = R_{\mathcal{K},\mathrm{id}}.$$

1. Definition and properties of projections

Definition 4. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, $X \subseteq G$. Then we define a relation $R_{q,X,\mathcal{K}} \subseteq G^{H-K_q}$ as follows:

$$R_{q,X,\mathcal{K}} = \{g \in G^{H-K_q};\ \exists f \in R\colon f(K_q) = X, f(K_i) = g(K_i) \\ \text{for each } i \in (n+1], i \neq q\}.$$

$R_{q,X,\mathcal{K}}$ is called the (q,X)-*projection* of R with regard to $\mathcal{K}$.

R e m a r k 5. Let $T \subseteq G^H$ be a relation, $H = \{1,2,3\}$ (i.e. T is ternary), $\mathcal{K} = \{\{1\},\{2\},\{3\},\emptyset\}$, $q = 1$, $X = \{x_0\}$. Then $T_{1,X,\mathcal{K}}$ coincides with the binary relation ϱ_T defined in [3], 1.9.

Definition 6. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $X \subseteq G$. Then X is said *to fulfill the condition* (α) with regard to $\mathcal{K}$ and q if $0 < \operatorname{card} X \leqslant \operatorname{card} K_q$ or $X = K_q = \emptyset$.

R e m a r k 7. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$, $q \in (n]$, $K_q \subset H$, $X \subseteq G$, let X fail to fulfill the condition (α) with regard to $\mathcal{K}$ and q (i.e $X = \emptyset \neq K_q$ or $\operatorname{card} K_q < \operatorname{card} X$). Obviously, then $R_{q,X,\mathcal{K}} = \emptyset$.

N o t a t i o n 8. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ be an n-decomposition of the set H, $n \geqslant 2$, $q \in (n]$, $K_q \subset H$. Then $\mathcal{K}_{(q)} = \{\tilde{K}_i\}_{i=1}^{n}$ denotes the $(n-1)$-decomposition of the set $H - K_q$ defined as follows:

$$\tilde{K}_i = \begin{cases} K_i & \text{for all } i \in (q-1] \\ K_{i+1} & \text{for all } i \in (n] - (q-1]. \end{cases}$$

Let, moreover, $\varphi \in S_n(q)$. Denote by $\varphi_{(q)}$ the permutation of the set $(n-1]$ defined as follows:

$$\varphi_{(q)}(i) = \begin{cases} \varphi(i) & \text{if } i < q,\ \varphi(i) < q \\ \varphi(i) - 1 & \text{if } i < q,\ \varphi(i) > q \\ \varphi(i+1) & \text{if } i \geqslant q,\ \varphi(i+1) < q \\ \varphi(i+1) - 1 & \text{if } i \geqslant q,\ \varphi(i+1) > q. \end{cases}$$

Lemma 9. *Let J be a nonempty set, $j_0 \in J$, let R, T, T_j for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ be an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, $X \subseteq G$, $\varphi \in S_n(q)$, $m \in \mathbb{N}$. Denote $\mathcal{K}_{(q)} = \{\tilde{K}_i\}_{i=1}^{n}$. Then:*

(8) $(E_{\mathcal{K}})_{q,X,\mathcal{K}} \subseteq E_{\mathcal{K}_{(q)}}$.

(9) *If* $\operatorname{card} K_i < \operatorname{card} X$ *or* $X = \emptyset \neq K_i$ *for some* $i \in (n]$, *then* $(E_{\mathcal{K}})_{q,X,\mathcal{K}} = \emptyset$.

(10) *If* $f \in E_{\mathcal{K}_{(q)}}$ *is such that* $f(\tilde{K}_1) = X$ *and* X *fulfills the condition* (α) *with regard to* $\mathcal{K}$ *and* q, *then* $f \in (E_{\mathcal{K}})_{q,X,\mathcal{K}}$.

(11) $(R^m_{\mathcal{K},\varphi})_{q,X,\mathcal{K}} = (R_{q,X,\mathcal{K}})^m_{\mathcal{K}_{(q)},\varphi_{(q)}}$.

(12) *If* $R \subseteq T$, *then* $R_{q,X,\mathcal{K}} \subseteq T_{q,X,\mathcal{K}}$.

(13) $\Big(\bigcup_{j\in J} T_j\Big)_{q,X,\mathcal{K}} = \bigcup_{j\in J} (T_j)_{q,X,\mathcal{K}}$.

(14) $\Big(\bigcap_{j\in J} T_j\Big)_{q,X,\mathcal{K}} \subseteq \bigcap_{j\in J} (T_j)_{q,X,\mathcal{K}}$.

(15) *If* T_j *is regular for each* $j \in J - \{j_0\}$, *then* (14) *becomes the equality.*

Proof. The statements follow directly from the definitions. For example, let us prove (14) and (15).

(14) As $\bigcap_{j\in J} T_j \subseteq T_j$ for any $j \in J$, we obtain, by (12), $\Big(\bigcap_{j\in J} T_j\Big)_{q,X,\mathcal{K}} \subseteq (T_j)_{q,X,\mathcal{K}}$ for any $j \in J$. Hence $\Big(\bigcap_{j\in J} T_j\Big)_{q,X,\mathcal{K}} \subseteq \bigcap_{j\in J}(T_j)_{q,X,\mathcal{K}}$.

(15) Let $f \in \bigcap_{j\in J}(T_j)_{q,X,\mathcal{K}}$. Then there exist $g_j \in T_j$ such that $g_j(K_q) = X$ for all $j \in J$, $f(K_i) = g_j(K_i)$ for all $j \in J$ and all $i \in (n+1] - \{q\}$. Thus, we have $g_j(K_i) = g_{j_0}(K_i)$ for all $j \in J$ and all $i \in (n+1]$. As T_j is regular for each $j \in J - \{j_0\}$, $g_j \in T_j$, $g_j(K_i) = g_{j_0}(K_i)$ for each $j \in J$ and each $i \in (n+1]$, we have $g_{j_0} \in T_j$ for each $j \in J$, so that $g_{j_0} \in \bigcap_{j\in J} T_j$. This yields $f \in \Big(\bigcap_{j\in J} T_j\Big)_{q,X,\mathcal{K}}$, and we get $\bigcap_{j\in J}(T_j)_{q,X,\mathcal{K}} \subseteq \Big(\bigcap_{j\in J} T_j\Big)_{q,X,\mathcal{K}}$. $\square$

Theorem 10. *Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, $X \subseteq G$, $\varphi \in S_n(q)$. Then:*

(16) $R_{q,X,\mathcal{K}}$ is regular with regard to $\mathcal{K}_{(q)}$.

(17) If R is symmetric (asymmetric, antisymmetric, cyclic) with regard to $\mathcal{K}$ (and φ), then $R_{q,X,\mathcal{K}}$ has the same property with regard to $\mathcal{K}_{(q)}$ (and $\varphi_{(q)}$).

Proof. The assertion (16) is obvious.

(17) The case of symmetry follows from (12) and (11). Let R be asymmetric with regard to $\mathcal{K}$ and φ. By (3), $R_{\mathcal{K},\varphi}$ is regular with regard to $\mathcal{K}$. By (11) and (15), we have $R_{q,X,\mathcal{K}} \cap (R_{q,X,\mathcal{K}})_{\mathcal{K}_{(q)},\varphi_{(q)}} = R_{q,X,\mathcal{K}} \cap (R_{\mathcal{K},\varphi})_{q,X,\mathcal{K}} = (R \cap R_{\mathcal{K},\varphi})_{q,X,\mathcal{K}} = \emptyset_{q,X,\mathcal{K}} = \emptyset$, so that $R_{q,X,\mathcal{K}}$ is asymmetric with regard to $\mathcal{K}_{(q)}$ and $\varphi_{(q)}$. The case of antisymmetry follows from (3), (11), (15), (12), and (8). The cyclicity is a special case of symmetry. $\square$

Remark 11. In contrast to [10], 4.5, a statement analogous to (17) cannot be formulated for the reflexivity or irreflexivity. In order to show it, we present two counterexamples. Put $G = \{a, b\}$, $H = \{1, 2, 3\}$, $\mathcal{K} = \big\{\{1\}, \{2\}, \{3\}, \emptyset\big\}$, $q = 3$, $X = \{a\}$. Then the diagonal $E_{\mathcal{K}} = \{(a,a,a), (b,b,b)\}$ is reflexive with regard to $\mathcal{K}$, while $(E_{\mathcal{K}})_{3,\{a\},\mathcal{K}} = \{(a,a)\}$ is not reflexive with regard to $\mathcal{K}_{(3)}$. Further, the relation $R = \{(b,b,a)\}$ is irreflexive with regard to $\mathcal{K}$, while $R_{3,\{a\},\mathcal{K}} = \{(b,b)\}$ is not irreflexive with regard to $\mathcal{K}_{(3)}$.

Lemma 12. *Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $q \in (n]$. Let R be cyclic and complete with regard to $\mathcal{K}$. Let $f \in G^H$ be a mapping such that $f(K_i) \neq f(K_j)$ for all $i, j \in (n]$, $i \neq j$. Then $f \in \bigcup_{\varphi\in S_n(q)} R_{\mathcal{K},\varphi}$.*

Proof. As R is complete with regard to $\mathcal{K}$, there exists a permutation $\psi \in S_n$ such that $f \in R_{\mathcal{K},\psi}$. Denote $\psi(q) = k$, $\chi = \pi^{n-k+q}\psi$. Then $\chi \in S_n(q)$ and

$\psi = \pi^{n+k-q}\chi$. As R is cyclic with regard to $\mathcal{K}$, we have $R_{\mathcal{K},\pi} \subseteq R$. It follows that, by (1) and (2), $R \supseteq R_{\mathcal{K},\pi} \supseteq (R_{\mathcal{K},\pi})_{\mathcal{K},\pi} = R_{\mathcal{K},\pi^2} \supseteq (R_{\mathcal{K},\pi^2})_{\mathcal{K},\pi} = R_{\mathcal{K},\pi^3} \supseteq \ldots \supseteq R_{\mathcal{K},\pi^{n+k-q}}$, so that $R_{\mathcal{K},\psi} = R_{\mathcal{K},\pi^{n+k-q}\chi} = (R_{\mathcal{K},\pi^{n+k-q}})_{\mathcal{K},\chi} \subseteq R_{\mathcal{K},\chi}$, hence $f \in R_{\mathcal{K},\chi} \subseteq \bigcup\limits_{\varphi \in S_n(q)} R_{\mathcal{K},\varphi}$. $\square$

N o t a t i o n 13. Let $R \subseteq G^H$ be a relation and $F \subseteq G$ a non-empty set. The restriction of R to F is denoted by $R\big|F$, i.e. $R\big|F \subseteq F^H$ is the relation defined by $R\big|F = R \cap F^H$.

Theorem 14. *Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$, $\operatorname{card} G \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, let $X \subseteq G$ fulfil the condition (α) with regard to $\mathcal{K}$ and q. Let R be cyclic and complete with regard to $\mathcal{K}$. Then $R_{q,X,\mathcal{K}}\big|(G - \{x\})$ is complete with regard to $\mathcal{K}_{(q)}$ for each element $x \in X$.*

P r o o f. For $X = \emptyset$ the statement is trivial. Let $X \neq \emptyset$. Let $x \in X$ be an element, let $\mathcal{K}_{(q)} = \{\tilde{K}_i\}_{i=1}^{n}$, and $f \in (G - \{x\})^{H-K_q}$ be a mapping such that $f(\tilde{K}_i) \neq f(\tilde{K}_j)$ for each i, $j \in (n-1]$, $i \neq j$. We are going to find a permutation $\psi \in S_{n-1}$ such that $f \in \big(R_{q,X,\mathcal{K}}\big|(G - \{x\})\big)_{\mathcal{K}_{(q)},\psi}$, i.e. a mapping $g \in R_{q,X,\mathcal{K}}\big|(G - \{x\})$ with $f(\tilde{K}_i) = g(\tilde{K}_{\psi(i)})$ for each $i \in (n-1]$, $f(\tilde{K}_n) = g(\tilde{K}_n)$. Let $h \in G^H$ be a mapping such that $h(K_q) = X$ and $h(K_i) = f(K_i)$ for each $i \in (n+1] - \{q\}$. Then, by the preceding, $h(K_i) = f(K_i) \neq f(K_j) = h(K_j)$ for each i, $j \in (n] - \{q\}$, $i \neq j$. Further, $h(K_i) = f(K_i) \subseteq G - \{x\}$ for any $i \in (n] - \{q\}$, while $h(K_q) = X$, so that $h(K_i) \neq h(K_q)$ for any $i \in (n] - \{q\}$. This implies that $h(K_i) \neq h(K_j)$ for each i, $j \in (n]$, $i \neq j$. By 12, we obtain $h \in \bigcup\limits_{\varphi \in S_n(q)} R_{\mathcal{K},\varphi}$. Therefore there exists $\varphi \in S_n(q)$ such that $h \in R_{\mathcal{K},\varphi}$, i.e. there exists $\ell \in R$ with $h(K_i) = \ell(K_{\varphi(i)})$ for each $i \in (n]$, $h(K_{n+1}) = \ell(K_{n+1})$. In particular, we have $\ell(K_q) = h(K_q) = X$. Now, let $g \in G^{H-K_q}$ be a mapping such that $g(K_i) = \ell(K_i)$ for each $i \in (n+1] - \{q\}$. Then $g \in R_{q,X,\mathcal{K}}$ and we have $f(K_i) = h(K_i) = \ell(K_{\varphi(i)}) = g(K_{\varphi(i)})$ for each $i \in (n] - \{q\}$. It suffices to put $\psi = \varphi_{(q)}$. Hence $R_{q,X,\mathcal{K}}\big|(G - \{x\})$ is complete with regard to $\mathcal{K}_{(q)}$ for each $x \in X$. $\square$

Lemma 15. *Let R, $T \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, and let T be regular with regard to $\mathcal{K}$. If $R_{q,X,\mathcal{K}} \subseteq T_{q,X,\mathcal{K}}$ for any set $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then $R \subseteq T$.*

P r o o f. Let $f \in R$ and let $g \in G^{H-K_q}$ be such that $f(K_i) = g(K_i)$ for each $i \in (n+1] - \{q\}$. Putting $X = f(K_q)$ we have $g \in R_{q,X,\mathcal{K}}$. Thus $g \in T_{q,X,\mathcal{K}}$ and consequently there exists a mapping $h \in T$ such that $h(K_q) = X$ and $h(K_i) = g(K_i)$ for each $i \in (n+1] - \{q\}$. But now $f(K_i) = h(K_i)$ for each $i \in (n+1]$, hence $f \in T$ because T is regular with regard to $\mathcal{K}$. The inclusion $R \subseteq T$ is proved. $\square$

Theorem 16. *Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, let $\varphi \in S_n(q)$. Then:*

(18) *If $R_{q,X,\mathcal{K}}$ is irreflexive (asymmetric, complete) with regard to $\mathcal{K}_{(q)}$ (and $\varphi_{(q)}$) for any subset $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then R has the same property with regard to $\mathcal{K}$ (and φ).*

(19) *If R is regular with regard to $\mathcal{K}$ and if $R_{q,X,\mathcal{K}}$ is reflexive (symmetric) with regard to $\mathcal{K}_{(q)}$ (and $\varphi_{(q)}$) for any subset $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then R has the same property with regard to $\mathcal{K}$ (and φ).*

Proof. (18) The cases of irreflexivity and asymmetry follow from (8), (14), and (11). Let $R_{q,X,\mathcal{K}}$ be complete with regard to $\mathcal{K}_{(q)}$ for any subset $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q. Let $f \in G^H$ be a mapping such that $f(K_i) \neq f(K_j)$ for each i, $j \in (n]$, $i \neq j$. Put $X = f(K_q)$. Then X fulfils the condition (α) with regard to $\mathcal{K}$ and q. Let $g \in G^{H-K_q}$ be a mapping fulfilling $g(K_i) = f(K_i)$ for each $i \in (n+1] - \{q\}$. As $g(K_i) = f(K_i) \neq f(K_j) = g(K_j)$ for each $i, j \in (n]$, $i \neq j$, there exists $\chi \in S_{n-1}$ such that $g \in (R_{q,X,\mathcal{K}})_{\mathcal{K}_{(q)},\chi}$. Let $\psi \in S_n$ be such that $\chi = \psi_{(q)}$. By (11) (for $m = 1$), $(R_{q,X,\mathcal{K}})_{\mathcal{K}_{(q)},\chi} = (R_{q,X,\mathcal{K}})_{\mathcal{K}_{(q)},\psi_{(q)}} = (R_{\mathcal{K},\psi})_{q,X,\mathcal{K}}$. Thus $g \in (R_{\mathcal{K},\psi})_{q,X,\mathcal{K}}$. Consequently, there exists $h \in R_{\mathcal{K},\psi}$ such that $h(K_i) = g(K_i)$ for each $i \in (n+1] - \{q\}$, $h(K_q) = X$. Hence $f(K_i) = g(K_i) = h(K_i)$ for each $i \in (n+1] - \{q\}$, and $f(K_q) = X = h(K_q)$. As $R_{\mathcal{K},\psi}$ is regular with regard to $\mathcal{K}$ by (3), we obtain $f \in R_{\mathcal{K},\psi}$, so that R is complete with regard to $\mathcal{K}$.

(19) follows from (8), (11), and 15. □

Remark 17. Here we give an example of a relation $R \subseteq G^H$ that is not antisymmetric with regard to $\mathcal{K}$ and φ, while its (q, X)-projection $R_{q,X,\mathcal{K}}$ is antisymmetric with regard to $\mathcal{K}_{(q)}$ and $\varphi_{(q)}$ for any subset $X \subseteq G$. Put $G = \{a, b\}$, $H = \{1, 2, 3\}$, $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$, $q = 3$, $\varphi \in S_3(3)$ such that $\varphi(1) = 2$, $\varphi(2) = 1$. Then the relation $R = \{(a, a, b)\}$ is not antisymmetric with regard to $\mathcal{K}$ and φ, because $R_{\mathcal{K},\varphi} = \{(a, a, b)\} = R$, so that $R \cap R_{\mathcal{K},\varphi} \nsubseteq E_{\mathcal{K}} = \{(a, a, a), (b, b, b)\}$, but the relations $R_{3,\emptyset,\mathcal{K}} = R_{3,\{a\},\mathcal{K}} = R_{3,G,\mathcal{K}} = \emptyset$, $R_{3,\{b\},\mathcal{K}} = \{(a, a)\}$ are all antisymmetric with regard to $\mathcal{K}_{(3)}$ and $\varphi_{(3)}$.

Theorem 18. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, let $X \subseteq G$, $\varphi \in S_n(q)$. Then:*

(20) $(R^{(r)})_{q,X,\mathcal{K}} \subseteq (R_{q,X,\mathcal{K}})^{(r)}_{\mathcal{K}_{(q)}}$.

(21) $(R^{(s)}_{\mathcal{K},\varphi})_{q,X,\mathcal{K}} = (R_{q,X,\mathcal{K}})^{(s)}_{\mathcal{K}_{(q)},\varphi_{(q)}}$.

(22) $(R^{(g)}_{\mathcal{K}})_{q,X,\mathcal{K}} = R_{q,X,\mathcal{K}}$.

Proof. (20) follows from (5), (13), and (8).

(21) follows from (6), (11), and (13).

(22) follows from (7), (16), (17), and (11). □

R e m a r k 19. The inclusion in (20) can be strict, which we show by the following example. Put $G = \{a, b\}$, $H = \{1, 2, 3\}$, $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$, $q = 3$, $X = \{a\}$, $R = E_{\mathcal{K}}$. Then $R = \{(a,a,a), (b,b,b)\} = R_{\mathcal{K}}^{(r)}$, $(R_{3,\{a\},\mathcal{K}})_{\mathcal{K}_{(3)}}^{(r)} = \{(a,a)\}_{\mathcal{K}_{(3)}}^{(r)} = \{(a,a), (b,b)\}$, and

$$(R_{\mathcal{K}}^{(r)})_{3,\{a\},\mathcal{K}} = R_{3,\{a\},\mathcal{K}} = \{(a,a)\} \subset \{(a,a),(b,b)\} = (R_{3,\{a\},\mathcal{K}})_{\mathcal{K}_{(3)}}^{(r)}.$$

Corollary 20. *Let Q, $R \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, let $X \subseteq G$, $\varphi \in S_n(q)$. If Q is the symmetric hull of R with regard to $\mathcal{K}$ and φ, then $Q_{q,X,\mathcal{K}}$ is the symmetric hull of $R_{q,X,\mathcal{K}}$ with regard to $\mathcal{K}_{(q)}$ and $\varphi_{(q)}$.*

P r o o f is obvious.

Theorem 21. *Let Q, $R \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 2$. Let $q \in (n]$, $K_q \subset H$, $\varphi \in S_n(q)$. Let Q be regular with regard to $\mathcal{K}$. Then:*

(23) *If $Q_{q,X,\mathcal{K}}$ is the reflexive hull of $R_{q,X,\mathcal{K}}$ with regard to $\mathcal{K}_{(q)}$ for any $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then $R_{\mathcal{K}}^{(r)} \subseteq Q$.*

(24) *If $Q_{q,X,\mathcal{K}}$ is the symmetric hull of $R_{q,X,\mathcal{K}}$ with regard to $\mathcal{K}_{(q)}$ and $\varphi_{(q)}$ for any $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then Q is the symmetric hull of R with regard to $\mathcal{K}$ and φ.*

P r o o f. The first assertion is a consequence of (5), (8), (13), and 15, the second assertion follows from (6), (11), (13), (4), and 15. □

R e m a r k 22. Under the assumptions of 21, item (23), Q need not be the reflexive hull of R with regard to $\mathcal{K}$. In order to show it, put $G = \{a, b\}$, $H = \{1, 2, 3\}$, $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$, $q = 3$, $R = \emptyset$, $Q = \{(a,a,a), (a,a,b), (b,b,a), (b,b,b)\}$; then $\{a\}$ and $\{b\}$ are exactly all the subsets of G fulfilling the condition (α) with regard to $\mathcal{K}$ and q, and $Q_{3,\{a\},\mathcal{K}} = \{(a,a),(b,b)\} = (R_{3,\{a\},\mathcal{K}})_{\mathcal{K}_{(3)}}^{(r)}$, $Q_{3,\{b\},\mathcal{K}} = \{(a,a),(b,b)\} = (R_{3,\{b\},\mathcal{K}})_{\mathcal{K}_{(3)}}^{(r)}$, but $R_{\mathcal{K}}^{(r)} = \{(a,a,a),(b,b,b)\} \subset Q$.

Theorem 23. *Let Q, $R \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $n \geqslant 3$. Let $K_i \subset H$ for each $i \in (n]$. Let Q and R be regular with regard to $\mathcal{K}$. If $Q_{q,X,\mathcal{K}}$ is the reflexive hull of $R_{q,X,\mathcal{K}}$ with regard to $\mathcal{K}_{(q)}$ for any $q \in (n]$ and any $X \subseteq G$ fulfilling the condition (α) with regard to $\mathcal{K}$ and q, then Q is the reflexive hull of R with regard to $\mathcal{K}$.*

P r o o f. By (23), we have $R_{\mathcal{K}}^{(r)} \subseteq Q$. Admit that $R_{\mathcal{K}}^{(r)} \subset Q$, i.e. there exists $f \in Q - R_{\mathcal{K}}^{(r)}$. By (5), we obtain $f \in Q$, $f \notin R$, $f \notin E_{\mathcal{K}}$. As $f \notin E_{\mathcal{K}}$, there exist i, $j \in (n]$ such that $f(K_i) \neq j(K_j)$. Let $q \in (n]$, $i \neq q \neq j$. Put $X = f(K_q)$. Then $X \subseteq G$

fulfils the condition (α) with regard to $\mathcal{K}$ and q. Let $g \in G^{H-K_q}$ be a mapping such that $g(K_i) = f(K_i)$ for each $i \in (n+1]$, $i \neq q$. Then $g \in Q_{q,X,\mathcal{K}} = R_{q,X,\mathcal{K}} \cup E_{\mathcal{K}_{(q)}}$, again by (5). Thus $g \in R_{q,X,\mathcal{K}}$ or $g \in E_{\mathcal{K}_{(q)}}$. In the former case, there exists $h \in R$ such that $g(K_i) = h(K_i)$ for each $i \in (n+1]$, $i \neq q$, $h(K_q) = X$. This implies that $f(K_i) = h(K_i)$ for each $i \in (n+1]$, so that $f \in R$, for R is regular with regard to $\mathcal{K}$, which is a contradiction. In the latter case, $f(K_i) = g(K_i) = g(K_j) = g(K_i)$ for each $i, j \in (n]$, $i \neq q \neq j$, which is again a contradiction. Hence $Q = R^{(r)}_{\mathcal{K}}$. □

Remark 24. For $n = 2$, the preceding theorem need not be valid. In order to show it, put $G = \{a, b\}$, $H = \{1, 2\}$, $\mathcal{K} = \big\{\{1\}, \{2\}, \emptyset\big\}$, $R = \emptyset$, $Q = \{(a,a), (a,b), (b,a), (b,b)\}$. Then $Q_{q,X,\mathcal{K}} = (R_{q,X,\mathcal{K}})^{(r)}_{\mathcal{K}_{(q)}} = \{a, b\}$ for $q = 1$, 2 and $X = \{a\}$, $\{b\}$, but $R^{(r)}_{\mathcal{K}} = \{(a,a), (b,b)\} \subset Q$.

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