MODULARITY AND DISTRIBUTIVITY OF THE LATTICE OF Σ-CLOSED SUBSETS OF AN ALGEBRAIC STRUCTURE

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(Received April 22, 1994)

Summary. Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ and Σ a set of open formulas of the first order language $L(\tau)$. The set $C_\Sigma(\mathscr{A})$ of all subsets of A closed under Σ forms the so called lattice of Σ-closed subsets of $\mathscr{A}$. We prove various sufficient conditions under which the lattice $C_\Sigma(\mathscr{A})$ is modular or distributive.

Keywords: algebraic structure, closure system, Σ-closed subset, modular lattice, distributive lattice, convex subset

MSC 1991: 08A05, 04A05

Modularity and distributivity of subalgebra lattices was investigated by T. Evans and B. Ganter in [4] and by the first author in [1]. However, we can study much more general lattices of closed subsets of an algebra or a relational structure. For convex sublattices of a given lattice this was done by V. I. Marmazajev [6], for convex subsets of monounary algebras or ordered sets see [5] or [3], respectively. A general approach for these considerations was developed by the authors in [2]. By using it, we can state sufficient (and in some cases also necessary) conditions under which a lattice of all Σ-closed subsets of a given algebraic structure is modular or even distributive.

First we recall some concepts. By a *type* we mean a pair of sequences $\tau = \langle\{n_i; i \in I\}, \{m_j; j \in J\}\rangle$ where n_i, m_j are non-negative integers. An *algebraic structure* or briefly a *structure* of type τ is a triplet $\mathscr{A} = (A, F, R)$, where $A \neq \emptyset$ is a set and $F = \{f_i; i \in I\}$, $R = \{\varrho_j; j \in J\}$ such that for each $i \in I$, f_i is an n_i-ary *operation* on A and for each $j \in J$, ϱ_j is an m_j-ary *relation* on A. Denote by $L(\tau)$ the first order language containing operational and relational symbols of type τ. If $R = \emptyset$, the structure $(A, F, \emptyset)$ is denoted briefly by (A, F) and is called *an algebra*. If $F = \emptyset$ then $(A, \emptyset, R)$ is denoted by (A, R) and called *a relational system*; this system

(A,R) is called *binary* if each $\varrho_j \in R$ is binary. A binary relational system (A,R) is said to be *antisymmetrical* if each $\varrho_j \in R$ is an antisymmetrical relation. A binary relational system (A,R) is called an *ordered* (or *quasiordered*) *set* if $R=\{\varrho_1\}$ where ϱ_1 is an *order* on A (or a reflexive and transitive relation, the so called *quasiorder*, respectively).

Let Γ be an index set and for each $\gamma \in \Gamma$ let $G_\gamma(x_1,\dots,x_{k_\gamma},y_1,\dots,y_{s_\gamma},z,f_i)$ be an open formula of a language $L(\tau)$ containing individual variables $x_1,\dots,x_{k_\gamma}$, $y_1,\dots,y_{s_\gamma}$, z and a symbol f_i of n_i-ary term operation. Analogously, let Λ be an index set and for each $\lambda \in \Lambda$ let $G_\lambda(x_1,\dots,x_{k_\lambda},y_1,\dots,y_{s_\lambda},z,\varrho_j)$ be an open formula of the language $L(\tau)$ containing individual variables $x_1,\dots,x_{k_\lambda}$, $y_1,\dots,y_{s_\lambda}$, z and a symbol ϱ_j of m_j-ary relation of type τ. Put $\Sigma=\{G_\gamma;\gamma\in\Gamma\}\cup\{G_\lambda;\lambda\in\Lambda\}$.

Definition 1. A subset B of an algebraic structure $\mathscr{A}=(A,F,R)$ is called Σ-*closed* if for every $\gamma\in\Gamma$, $\lambda\in\Lambda$ and $a_1,\dots,a_{k_\gamma}$, $a'_1,\dots,a'_{k_\lambda}\in B$ and $b_1,\dots,b_{s_\gamma}$, $b'_1,\dots,b'_{s_\lambda}$, c, $c'\in A$, we have $c\in B$ or $c'\in B$ provided $G_\gamma(a_1,\dots,a_{k_\gamma},b_1,\dots,b_{s_\gamma},c,f_i)$ or $G_\lambda(a'_1,\dots,a'_{k_\lambda},b'_1,\dots,b'_{s_\lambda},c',\varrho_j)$ are satisfied in $\mathscr{A}$. Denote by $C_\Sigma(\mathscr{A})$ the set of all Σ-closed subsets of $\mathscr{A}$.

As was proved in [2], the set $C_\Sigma(\mathscr{A})$ of all Σ-closed subsets of a structure $\mathscr{A}=(A,F,R)$ is a complete lattice with respect to set inclusion with the greatest element A. In what follows we will study modularity and distributivity of $C_\Sigma(\mathscr{A})$ depending on the properties of $\mathscr{A}$. For any given structure $\mathscr{A}$ we will suppose that the set of formulas Σ is determined. For a given subset $M\subseteq A$ we denote by $C_{\mathscr{A}}(M)$ the least Σ-closed subset of $\mathscr{A}$ containing M; we say that $C_{\mathscr{A}}(M)$ is *generated* by M. If M is a finite subset, say $M=\{a_1,\dots,a_k\}$, we will write $C_{\mathscr{A}}(a_1,\dots,a_k)$ for $C_{\mathscr{A}}(M)$.

If the set Σ is implicitly known, we will use on the lattice $C_\Sigma(\mathscr{A})$ to specify the closure system. In some more familiar examples of $C_\Sigma(\mathscr{A})$ we will use the common name and notation:

(1) If $\mathscr{A}=(A,F)$ is an algebra, $F=\{f_i\colon i\in I\}$ and $\Sigma=\{G_i\colon i\in I\}$ where $G_i(x_1,\dots,x_{n_i},z,f_i)$ is the formula $\big(f_i(x_i,\dots,x_{n_i})=z\big)$, then Σ-closed subsets of $\mathscr{A}$ are subalgebras of $\mathscr{A}$ and $\emptyset$, and $C_\Sigma(\mathscr{A})=\operatorname{Sub}\mathscr{A}$.

(2) If $\mathscr{L}=(L,\{\vee,\wedge\})$ is a lattice, $\Sigma=\{G_1,G_2\}$ where G_1 is the formula $(x_1\vee x_2=z)$ and G_2 is the formula $(x_1\wedge y_1,z)$, then the Σ-closed subsets of $\mathscr{L}$ are lattice ideals, i.e. $C_\Sigma(\mathscr{L})=\operatorname{Id}\mathscr{L}$.

(3) If $\mathscr{R}=(A,R)$ is a binary relational system with $R=\{\varrho_j;j\in J\}$ and $\Sigma=\{G_j\colon j\in J\}$ where for each $j\in J$ we have

$$G_j \text{ is the formula } (x_1\varrho_j z \text{ and } z\varrho_j x_2),$$

then the Σ-closed subsets of $\mathscr{R}$ are the so called *convex subsets* and $C_\Sigma(\mathscr{R})$ will be denoted by $\operatorname{Conv}\mathscr{R}$.

In particular, if $\mathscr{S} = (S, \leqslant)$ is an ordered set then $\Sigma = \{G\}$ where G is the formula $(x_1 \leqslant z \leqslant x_2)$. Thus Σ-closed subsets of $\mathscr{S}$ are exactly the convex subsets of $\mathscr{S}$ in the usual sense.

(4) If $\mathscr{G} = (G, ., ^{-1}, e)$ is a group and $\Sigma = \{G_1, G_2, G_3, G_4\}$, where $G_1(x_1, x_2, z, .)$ is the formula $(x_1 \cdot x_2 = z)$, $G_2(x_1, z, ^{-1})$ is the formula $(x_1^{-1} = z)$, $G_3(z, e)$ is the formula $(e = z)$ and $G_4(x_1, y_1, z, p)$ is the formula $\big(p(x_1, y_1) = z\big)$ where $p(x_1, y_1)$ is the term operation $y_1 x_1 y_1^{-1}$, then $C_\Sigma(\mathscr{G})$ is the lattice of all normal subgroups of $\mathscr{G}$. It will be denoted simply by $N(\mathscr{G})$.

In what follows we denote join in $C_\Sigma(\mathscr{A})$ by $\vee$, meet evidently coincides with set intersection.

Theorem 1. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure with the system $C_\Sigma(\mathscr{A})$ of Σ-closed subsets satisfying*

(i) *for each X, $Y \in C_\Sigma(\mathscr{A})$, $\emptyset \neq X \neq Y \neq \emptyset$ we have $a \in X \vee Y$ if and only if there exist $x \in X$, $y \in Y$ with $a \in C_\mathscr{A}(x, y)$;*

(ii) *for each x, $y \in A$, if $a \in C_\mathscr{A}(x, y)$ and $C_\mathscr{A}(a) \neq C_\mathscr{A}(x)$ then $y \in C_\mathscr{A}(x, a)$. Then the lattice $\big(C_\Sigma(\mathscr{A}), \subseteq\big)$ is modular.*

Proof. Suppose X, Y, $Z \in C_\Sigma(\mathscr{A})$ and $X \subseteq Z$. If either $X = \emptyset$ or $Y = \emptyset$ the proof is trivial. Also for $X = Y$ we easily obtain the modularity law. Hence, consider $\emptyset \neq X \neq Y \neq \emptyset$. Suppose $a \in (X \vee Y) \cap Z$. Then $a \in Z$ and $a \in X \vee Y$. By (i), there exist $x \in X$, $y \in Y$ such that $a \in C_\mathscr{A}(x, y)$.

If $C_\mathscr{A}(a) = C_\mathscr{A}(x)$ then $a \in C_\mathscr{A}(a) = C_\mathscr{A}(x) \subseteq X \vee (Y \cap Z)$.

If $C_\mathscr{A}(a) \neq C_\mathscr{A}(x)$, then we have $y \in C_\mathscr{A}(x, a)$ by (ii).

However, $x \in X \subseteq Z$, $a \in Z$ thus also $y \in C_\mathscr{A}(x, a) \subseteq Z$. Hence $y \in Y \cap Z$ and $a \in C_\mathscr{A}(x, y) \subseteq X \vee (Y \cap Z)$, which proves modularity of $C_\Sigma(\mathscr{A})$. □

Lemma 1. *Let $\mathscr{A} = (A, \{\varrho\})$ be a binary relational system with only one transitive binary relation and $C_\Sigma(\mathscr{A}) = \operatorname{Conv} \mathscr{A}$. Then $C_\Sigma(\mathscr{A})$ satisfies* (i) *of Theorem* 1.

Proof. The condition (i) of Theorem 1 is equivalent to the following one:

$$C_\mathscr{A}(X) = \bigcup \{C_\mathscr{A}(x_1, x_2); x_1, x_2 \in X\} \quad \text{for each } X \subseteq A.$$

For X, $Y \subseteq A$ put $C^0(X, Y) = X \cup Y$, $C(X, Y) = C^1(X, Y) = \{a \in A; u \varrho a \varrho v$ for some $u, v \in X \cup Y\}$ and $C^{n+1}(X, Y) = C\big(C^n(X, Y)\big)$, where $n \in N_0$ (nonnegative integer). Evidently, $C_\mathscr{A}(X, Y) = \bigcup \big(C^n(X, Y); n \in N_0\big)$. Now, we can prove the following statement by induction on n: "If $a \in C^n(X, Y)$, then there exist u, $v \in X \cup Y$ such that $u \varrho a \varrho v$."

1) For $n = 1$ it is a trivial.

2) Suppose that it is valid for all $k \leqslant n$ and we prove it for $n+1$. Let $a \in C^{n+1}(X,Y)$, i.e. $\alpha\varrho a\varrho\beta$ for some α, $\beta \in C^n(X,Y)$. Clearly, we have the following possibilities:

a) $\alpha \in [x_1, y_1]$, $\beta \in [x_1, y_2]$;

b) $\alpha \in [x_1, y_1]$, $\beta \in [y_2, x_2]$;

c) $\alpha \in [y_1, x_1]$, $\beta \in [x_2, y_2]$;

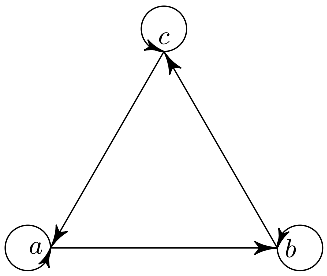
d) $\alpha \in [y_1, x_1]$, $\beta \in [y_2, x_2]$, etc., where x_1, $x_2 \in X$ and y_1, $y_2 \in Y$.

ad a) If $\alpha \in [x_1, y_1]$, $\beta \in [x_1, y_2]$, then $x_1\varrho\alpha a\varrho\beta\varrho y_2$ and $a \in [x_1, y_2]$ by transitivity, i.e. the statement is valid.

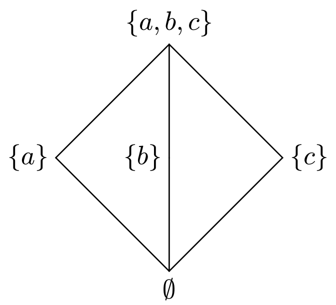
ad b) $\alpha \in [x_1, y_1]$, $\beta \in [y_2, x_2]$ imply $x_1\varrho\alpha a\varrho\beta\varrho x_2$, i.e. $a \in [x_1, x_2]$ and $a \in X$.

Similarly we can easily check the other possibilities. $\square$

E x a m p l e 1. Let $\mathscr{A} = (\{a,b,c\}, \{\varrho\})$ be a binary relational system with the following diagram of ϱ:

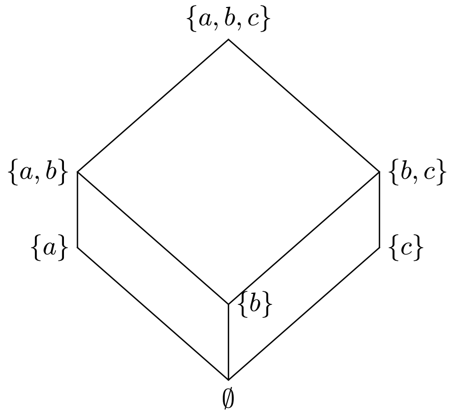


and $C_\Sigma(\mathscr{A}) = \operatorname{Conv}\mathscr{A}$. We can easily check (i) and (ii) of Theorem 1, thus $C_\Sigma(\mathscr{A})$ is modular. We can vizualize the diagram of $C_\Sigma(\mathscr{A})$ in Fig. 2 below:



We can see that it is isomorphic to M_3, hence $C_\Sigma(\mathscr{A})$ is not distributive.

E x a m p l e 2. Let $\mathscr{A} = (\{a,b,c\}, \leqslant)$ be an ordered set which is a chain: $a < b < c$, and let $C_\Sigma(\mathscr{A}) = \operatorname{Conv}\mathscr{A}$. Then it does not satisfy (ii) of Theorem 1 since $b \in C_{\mathscr{A}}(a,c)$, $C_{\mathscr{A}}(b) \neq C_{\mathscr{A}}(a)$ but $c \notin C_{\mathscr{A}}(a,b) = \{a,b\}$. The diagram of $C_\Sigma(\mathscr{A})$ is shown in Fig. 3:



We can see that $C_\Sigma(\mathscr{A})$ is not modular. We are going to show that for some algebraic structures the condition (ii) is really equivalent to modularity of $C_\Sigma(\mathscr{A})$.

Recall from [2] that an algebraic system $\mathscr{A} = (A, F, R)$ is Σ-*separable* if we have $C_{\mathscr{A}}(x) = \{x\}$ for any $x \in A$.

Theorem 2. *Let $\mathscr{A} = (A, F, R)$ be a Σ-separable algebraic structure satisfying* (i) *of Theorem* 1. *The following conditions are equivalent:*

(a) *the lattice $C_\Sigma(\mathscr{A})$ is modular;*

(b) *for each x, $y \in A$, if $a \in C_{\mathscr{A}}(x,y)$ for $a \neq x$ then $y \in C_{\mathscr{A}}(x,a)$.*

P r o o f. Since $\mathscr{A}$ is Σ-separable and $\mathscr{A}$ satisfies (i), we obtain (b) $\Rightarrow$ (a) directly by Theorem 1. Prove (a) $\Rightarrow$ (b). Let $C_\Sigma(\mathscr{A})$ be modular and a, x, $y \in A$, $a \neq x$. Since $\{x,y\} \subseteq C_{\mathscr{A}}(x) \vee C_{\mathscr{A}}(y)$, we have

$$(*) \qquad C_{\mathscr{A}}(x,y) \subseteq C_{\mathscr{A}}(x) \vee C_{\mathscr{A}}(y).$$

Suppose $a \in C_{\mathscr{A}}(x,y)$. Then $a \in C_{\mathscr{A}}(x,y) \cap C_{\mathscr{A}}(a,x)$ and, by $(*)$, also

$$a \in \bigl(C_{\mathscr{A}}(x) \vee C_{\mathscr{A}}(y)\bigr) \cap C_{\mathscr{A}}(a,x).$$

Clearly $C_{\mathscr{A}}(x) \subseteq C_{\mathscr{A}}(a,x)$ and, by modularity of $C_\Sigma(\mathscr{A})$, we conclude

$$a \in C_{\mathscr{A}}(x) \vee \bigl(C_{\mathscr{A}}(y) \cap C_{\mathscr{A}}(a,x)\bigr).$$

However, $\mathscr{A}$ is Σ-separable, thus also $a \in \{x\} \vee \bigl(\{y\} \cap C_{\mathscr{A}}(a,x)\bigr)$. Since $a \neq x$, this yields $\{y\} \cap C_{\mathscr{A}}(a,x) \neq \emptyset$, thus $y \in C_{\mathscr{A}}(a,x)$ which proves (b). $\square$

Definition 2. Let ϱ be a binary relation on A. We say that ϱ is *weakly transitive* if for each pairwise different elements a, b, $c \in A$, $\langle a,b\rangle \in \varrho$ and $\langle b,c\rangle \in \varrho$ imply $\langle c,a\rangle \notin \varrho$.

Corollary 1. *Let $\mathscr{A} = (A, \{\varrho\})$ be an antisymmetrical binary relational system with one weakly transitive relation ϱ and $C_\Sigma(\mathscr{A}) = \operatorname{Conv}\mathscr{A}$. The following conditions are equivalent:*

(1) $\operatorname{Conv}\alpha$ *is modular;*

(2) $\operatorname{Conv}\mathscr{A}$ *is distributive;*

(3) *for any pairwise different elements a, b, $c \in A$ we have $\langle a,b\rangle \notin \varrho$ or $\langle b,c\rangle \notin \varrho$.*

Proof. (3) $\Rightarrow$ (2): If $\mathscr{A}$ satisfies (3) then every subset of A is a convex subset, thus $\operatorname{Conv}\mathscr{A} = \operatorname{Exp} A$, i.e. $\operatorname{Conv}\mathscr{A}$ is distributive. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). Let $\operatorname{Conv}\mathscr{A}$ be modular and let a, x, y be pairwise different elements of A. Suppose $x\varrho a$ and $a\varrho y$. Then $a \in C_{\mathscr{A}}(x,y)$, $a \notin x$ and $y \notin C_{\mathscr{A}}(x,a)$ with respect to antisymmetry and weak transitivity of ϱ. Hence (b) of Theorem 2 is not valid. Moreover, $\mathscr{A}$ is Σ-separable by Theorem 3 in [2] and, by Lemma 1, $C_\Sigma(\mathscr{A})$ satisfies (i) of Theorem 1, thus we have a contradiction. Hence also (3) is satisfied. $\square$

Corollary 2. *Let $\mathscr{S} = (S, \leqslant)$ be an ordered set and $C_\Sigma(\mathscr{S}) = \operatorname{Conv}\mathscr{S}$. The following conditions are equivalent:*

(1) $\operatorname{Conv}\mathscr{S}$ *is modular;*

(2) $\operatorname{Conv}\mathscr{S}$ *is distributive;*

(3) $\mathscr{S}$ *does not contain a chain of length greater than two.*

Proof. Clearly, any order is weakly transitive, and it is almost trivial to show that (3) of Corollary 1 is equivalent to (3) of Corollary 2 for $\varrho = \leqslant$. $\square$

For any group $\mathscr{G}$, the lattice $N(\mathscr{G})$ of all its normal subgroups is modular and it clearly satisfies (i) of Theorem 1 since $\mathscr{G}_1 \vee \mathscr{G}_2 = \mathscr{G}_1 \cdot \mathscr{G}_2$ for each $\mathscr{G}_1$, $\mathscr{G}_2 \in N(\mathscr{G})$. However, it does not satisfy (ii) of Theorem 1: e.g. for the group $(Z,+)$ of all integers we have $4 \in C_{\mathscr{G}}(2,3) = Z$, $C_{\mathscr{G}}(4) \neq C_{\mathscr{G}}(2)$ but $3 \notin C_{\mathscr{G}}(2,4)$. This motivates our effort to give another sufficient condition for modularity of $C_\Sigma(\mathscr{A})$. (Remark that a group $\mathscr{G}$ is not Σ-separable with respect to $C_\Sigma(\mathscr{G}) = N(\mathscr{G})$.)

Definition 3. Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ. By a *binary formula* we mean any formula $G(x_1,x_2,z,f)$ or $G(x_1,x_2,z,\varrho)$ of the language $L(\tau)$ provided f is a binary term operation of $\mathscr{A}$ or ϱ is a binary relation of R.

Theorem 3. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure and let Σ contain a binary formula $G(x_1,x_2,z,f)$ or $G(x_1,x_2,z,\varrho)$ such that the following conditions are* *satisfied:*

(i) *if X, $Y \in C_\Sigma(\mathscr{A})$, $\emptyset \neq X \neq Y \neq \emptyset$, then $a \in X \vee Y$ if and only if there exist $b \in X$, $c \in Y$ such that $G(b,c,a,f)$ or $G(b,c,a,\varrho)$ is satisfied in $\mathscr{A}$;*

(ii) *for each a, b, $c \in A$, $a \neq b$, if the formula $G(b,c,a,f)$ or $G(b,c,a,\varrho)$ is satisfied in $\mathscr{A}$ and $C_\mathscr{A}(a) \neq C_\mathscr{A}(b)$ then $c \in C_\mathscr{A}(a,b)$. Then the lattice $\big(C_\Sigma(\mathscr{A}), \subseteq\big)$ is modular.*

P r o o f. Let X, Y, $Z \in C_\Sigma(\mathscr{A})$ and $X \subseteq Z$. To check modularity of $C_\Sigma(\mathscr{A})$ it is enough to consider $\emptyset \neq X \neq Y \neq \emptyset$. Suppose $a \in (X \vee Y) \cap Z$. By (i) there exist $b \in X$, $c \in Y$ such that some binary formula $G(b,c,a,f)$ or $G(b,c,a,\varrho)$ is satisfied in $\mathscr{A}$. If $C_\Sigma(a) = C_\mathscr{A}(b)$ then

$$a \in C_\mathscr{A}(b) \subseteq X \subseteq X \vee (Y \cap Z).$$

If $C_\mathscr{A}(a) \neq C_\mathscr{A}(b)$ then, by (ii), $c \in C_\mathscr{A}(a,b)$. However, $a \in Z$ and $b \in X \subseteq Z$, thus also $c \in C_\mathscr{A}(a,b) \subseteq Z$. Hence, we conclude by (i)

$$a \in X \vee (Y \cap Z),$$

proving modularity of $C_\Sigma(\mathscr{A})$. □

E x a m p l e 3. If $\mathscr{G} = (A, ., ^{-1}, e)$ is a group and $C_\Sigma(\mathscr{G}) = N(\mathscr{G})$, take a binary formula $(x_1 \cdot x_2 = z)$. Evidently, for $\mathscr{A}_1$, $\mathscr{A}_2 \in N(\mathscr{G})$, $a \in \mathscr{A}_1 \vee \mathscr{A}_2 = \mathscr{A}_1 \cdot \mathscr{A}_2$ if and only if there exist $a_1 \in A_1$, $a_2 \in A_2$ with $a = a_1 \cdot a_2$ and, if $a = b \cdot c$ (i.e. $G(b,c,a,.)$ is satisfied in $\mathscr{G}$) then $c = b^{-1} \cdot a$, thus $c \in C_\mathscr{A}(a,b)$. Hence, both (i), (ii) of Theorem 3 are satisfied.

E x a m p l e 4. It is an easy exercise to verify that the quasiordered set of Example 1 also satisfies the assumptions of Theorem 3 for the binary formula $G(x_1, x_2, z, \varrho) = (x_1 \varrho z$ and $z \varrho x_2)$.

Now, we turn our attention to distributivity of $C_\Sigma(\mathscr{A})$.

Theorem 4. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure with the lattice $C_\Sigma(\mathscr{A})$ of Σ-closed subsets. If there exists a binary term operation $p(x,y)$ of $\mathscr{A}$ such that*

(i) *for B, $C \in C_\Sigma(\mathscr{A})$ we have $a \in B \vee C$ if and only if $a = p(b,c)$ for some $b \in B$, $c \in C$;*

(ii) *if $D \in C_\Sigma(\mathscr{A})$ and $p(b,c) \in D$ for some b, $c \in A$, then b, $c \in D$, then the lattice $\big(C_\Sigma(\mathscr{A}), \subseteq\big)$ is distributive.*

P r o o f. Suppose B, C, $D \in C_\Sigma(\mathscr{A})$ and $a \in D \cap (B \vee C)$. Then $a \in D$ and, by (i), there exist $b \in B$, $c \in C$ with $a = p(b,c)$. Hence also $p(b,c) \in D$ and, by (ii), we have $b \in D$, $c \in D$. Thus $b \in D \cap B$, $c \in D \cap C$ and by (i) again, we conclude $a = p(b,c) \in (D \cap B) \vee (D \cap C)$. □

E x a m p l e 5. If $\mathscr{L}$ is a distributive lattice and $C_\Sigma(\mathscr{L}) = \operatorname{Id}\mathscr{L}$, we can put $p(x,y) = x \vee y$. It is well-known that for I_1, $I_2 \in \operatorname{Id}\mathscr{L}$, $y \in I_1 \vee I_2$ if and only if $y = i_1 \vee i_2$ for some $i_1 \in I_1$, $i_2 \in I_2$. Moreover, if $J \in \operatorname{Id}\mathscr{L}$ and $j_1 \vee j_2 \in J$ for j_1, $j_2 \in \mathscr{L}$ then $j_1 \leqslant j_1 \vee j_2$, $j_2 \leqslant j_1 \vee j_2$ imply also j_1, $j_2 \in J$.

Thus both assumptions of Theorem 4 are satisfied.

Now, let $\mathscr{A} = (A, F, R)$ be an algebraic structure and let $B \in C_\Sigma(\mathscr{A})$ for some given set Σ of open formulas. If there exists an element $b \in A$ such that $B = C_\mathscr{A}(b)$, we say that b is a *generator* of B.

In the remaining part of the paper, denote by Z the set of all integers and suppose $F \neq \emptyset$ for any algebraic structure $\mathscr{A} = (A, F, R)$ under consideration.

Definition 4. An algebraic structure $\mathscr{A} = (A, F, R)$ is called Σ-*cyclic* if there exist an element $d \in A$, a subset $K \subseteq Z$ and binary integral operations φ, ψ: $K \times K \to K$ and unary terms $w_k(x)$ for $k \in K$ of $\mathscr{A}$ such that

(a) for each $B \in C_\Sigma(\mathscr{A})$ there exists $k \in K$ such that $w_k(d)$ is a generator of B;

(b) if $w_m(d)$ or $w_n(d)$ are generators of B or D, respectively, for B, $D \in C_\Sigma(\mathscr{A})$, then $w_{\varphi(m,n)}(d)$ or $w_{\psi(m,n)}(d)$ are generators of $B \vee D$ or $B \cap D$, respectively;

(c) $\psi\bigl(k, \varphi(m,n)\bigr) = \varphi\bigl(\psi(k,m), \psi(k,n)\bigr)$ for every k, m, $n \in K$.

The terms $w_k(x)$ are called *characteristic terms* of $C_\Sigma(\mathscr{A})$.

Theorem 5. *If $\mathscr{A} = (A, F, R)$ is a Σ-cyclic algebraic structure then the lattice $\bigl(C_\Sigma(\mathscr{A}), \subseteq\bigr)$ is distributive.*

P r o o f. Let $\mathscr{A}$ be a Σ-cyclic algebraic structure and let $w_k(x)$ be its characteristic terms for $k \in K \subseteq Z$. Suppose that φ and ψ satisfy (b) and (c) of Definition 4. Let B, C, $D \in C_\Sigma(\mathscr{A})$. Suppose that $d \in A$ and $w_m(d)$ or $w_n(d)$ or $w_k(d)$ are generators of B or C or D, respectively. By (a), (b), (c) of Definition 4, we can easily derive

$$\begin{aligned} D \cap (B \vee C) &= C_\mathscr{A}\bigl(w_k(d)\bigr) \cap \bigl(C_\mathscr{A}\bigl(w_n(d)\bigr) \vee C_\mathscr{A}\bigl(w_m(d)\bigr)\bigr) \\ &= C_\mathscr{A}\bigl(w_{\psi(k,\varphi(m,n))}(d)\bigr) = C_\mathscr{A}\bigl(w_{\varphi(\psi(k,m),\psi(k,n))}(d)\bigr) \\ &= \bigl(C_\mathscr{A}\bigl(w_k(d)\bigr) \cap C_\mathscr{A}\bigl(w_m(d)\bigr)\bigr) \vee \bigl(C_\mathscr{A}\bigl(w_k(d)\bigr) \cap C_\mathscr{A}\bigl(w_n(d)\bigr)\bigr) \\ &= (D \cap B) \vee (D \cap C), \end{aligned}$$

i.e. the lattice $\bigl(C_\Sigma(\mathscr{A}), \subseteq\bigr)$ is distributive. $\square$

E x a m p l e 6. If $\mathscr{G} = (G, .)$ is a cyclic group and $C_\Sigma(\mathscr{G}) = \operatorname{Sub}\mathscr{G}$, put $K = Z$, $w_k = x^k$ and $\varphi(m,n) = GCD(m,n)$, $\psi(m,n) = LCM(m,n)$. As an element $d \in G$ we pick up the generator of $\mathscr{G}$. Evidently, $\mathscr{G}$ is Σ-cyclic.

E x a m p l e 7. If $\mathscr{A} = (A, f)$ is a monounary algebra and $C_\Sigma(\mathscr{A}) = \operatorname{Sub}\mathscr{A}$, we can put $K = N \cup \{0\}$ (non-negative integers), $w_k(x) = f^k(x)$ where $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$ for each $k \in K$. Moreover, put $\varphi(m, n) = \min(m, n)$, $\psi(m, n) = \max(m, n)$. If $\mathscr{A}$ has a unique generator d then $\mathscr{A}$ is Σ-cyclic.

E x a m p l e 8. Suppose $\mathscr{A} = (A, F, R)$ is an algebraic structure with at least two elements such that F contains a nullary operation c and $f(c, \ldots, c) = c$ for each $f \in F$. Further, suppose $C_\Sigma(\mathscr{A}) = \{\{c\}, A\}$ (trivially, $C_\Sigma(\mathscr{A})$ is distributive). Put $K = \{0, 1\}$. If $A \neq \{c\}$, choose $d \neq c$, $d \in A$ and put $w_0(x) = c$, $w_1(x) = d$. Further, let φ and ψ be defined in the same manner as in the foregoing Example 7. Evidently, $\mathscr{A}$ is Σ-cyclic.

References

[1] *Chajda I.*: A note on varieties with distributive subalgebra lattices. Acta Univ. Palack. Olomouc, Fac. Rer. Natur., Matematica *31* (1992), 25–28.

[2] *Chajda I., Emanovský, P.*: Σ-isomorphic algebraic structures. Mathem. Bohemica *120* (1995), 71–81.

[3] *Emanovský, P.*: Convex isomorphic ordered sets. Mathem. Bohemica *118* (1993), 29–35.

[4] *Evans T., Ganter B.*: Varieties with modular subalgebra lattices. Bull. Austral. Math. Soc. *28* (1993), 247–254.

[5] *Jakubíková-Studenovská D.*: Convex subsets of partial monounary algebras. Czech. Math. J. *38* (1988), no. 113, 655–672.

[6] *Marmazajev V.I.*: The lattice of convex sublattices of a lattice. Mezvužovskij naučnyj sbornik 6. Saratov, 1986, pp. 50–58. (In Russian.)

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