Local and global existence theorems for the Einstein equations

Alan D. Rendall
Max-Planck-Institut für Gravitationsphysik
Schlaatzweg 1, 14473 Potsdam, Germany
rendall@aei-potsdam.mpg.de
http://www.aei-potsdam.mpg.de/~rendall/rendall.html

Published 26 January 1998
www.livingreviews.org/Articles/Volume1/1998-4rendall
Living Reviews in Relativity
Published by the Max-Planck-Institute for Gravitational Physics
Albert Einstein Institute, Potsdam, Germany

Abstract

This article is a guide to the literature on existence theorems for the Einstein equations which also draws attention to open problems in the field. The local in time Cauchy problem, which is relatively well understood, is treated first. Next global results for solutions with symmetry are discussed. This is followed by a presentation of global results in the case of small data, and some miscellaneous topics connected with the main theme.

1 Introduction

Many of the mathematical models occurring in physics involve systems of partial differential equations. Only rarely can these equations be solved by explicit formulae. When they cannot, physicists frequently resort to approximations. There is, however, another approach which is complementary. This consists in determining the qualitative behaviour of solutions, without knowing them explicitly. The first and most fundamental step in doing this is to establish the existence of solutions under appropriate circumstances. Unfortunately, this is often hard, and obstructs the way to obtaining more interesting information. It may appear to the outside observer that existence theorems become a goal in themselves to some researchers. It is important to remember that, from a more general point of view, they are only a first step.

The basic partial differential equations of general relativity are Einstein's equations. In general they are coupled to other partial differential equations describing the matter content of spacetime. The Einstein equations are essentially hyperbolic in nature. In other words, the general properties of solutions are similar to those found for the wave equation. It follows that it is reasonable to try to determine a solution by initial data on a spacelike hypersurface. Thus the Cauchy problem is the natural context for existence theorems for the Einstein equations. The Einstein equations are also nonlinear. This means that there is a big difference between the local and global Cauchy problems. A solution evolving from regular data may develop singularities.

A special feature of the Einstein equations is that they are diffeomorphism invariant. If the equations are written down in an arbitrary coordinate system then the solutions of these coordinate equations are not uniquely determined by initial data. Applying a diffeomorphism to one solution gives another solution. If this diffeomorphism is the identity on the chosen Cauchy surface up to first order then the data are left unchanged by this transformation. In order to obtain a system for which uniqueness in the Cauchy problem holds in the straightforward sense it does for the wave equation, some coordinate or gauge fixing must be carried out.

Another special feature of the Einstein equations is that initial data cannot be given freely. They must satisfy constraint equations. To prove the existence of a solution of the Einstein equations, it is first necessary to prove the existence of a solution of the constraints. The usual method of solving the constraints relies on the theory of elliptic equations.

The local existence theory of solutions of the Einstein equations is rather well understood. Section (2) points out some of the things which are not known. On the other hand, the problem of proving general global existence theorems for the Einstein equations is beyond the reach of the mathematics presently available. To make some progress, it is necessary to concentrate on simplified models. The most common simplifications are to look at solutions with various types of symmetry and solutions for small data. These two approaches are reviewed in Sections (3) and (4) respectively. Section (5) collects some miscellaneous results which cannot easily be classified.

The area of research reviewed in the following relies heavily on the theory of differential equations, particularly that of hyperbolic partial differential equations. For the benefit of readers with little background in differential equations, some general references which the author has found to be useful will be listed. A thorough introduction to ordinary differential equations is given in [45]. A lot of intuition for ordinary diffential equations can be obtained from[48]. The article[2] is full of information, in rather compressed form. A classic introductory text on partial differential equations, where hyperbolic equations are well represented, is [52]. Useful texts on hyperbolic equations, some of which explicitly deal with the Einstein equations, are [78, 54, 62, 57, 76, 53].

An important aspect of existence theorems in general relativity which one should be aware of is their relation to the cosmic censorship hypothesis. This point of view was introduced in an influential paper by Moncrief and Eardley[60]. An extended discussion of the idea can be found in[33].

2 Local existence

2.1 The constraints

The unknowns in the constraint equations are the initial data for the Einstein equations. These consist of a three-dimensional manifold S, a Riemannian metric h_{ab} and a symmetric tensor k_{ab} on S, and initial data for any matter fields present. The equations are:

$$R - k_{ab}k^{ab} + (h^{ab}k_{ab})^2 = 16\pi\rho \tag{1}$$

$$\nabla^a k_{ab} - \nabla_b(h^{ac}k_{ac}) = 8\pi j_b \tag{2}$$

Here R is the scalar curvature of the metric h_{ab} and ρ and j_a are projections of the energy-momentum tensor. Assuming matter fields which satisfy the dominant energy condition implies that $\rho \geq (j_a j^a)^{1/2}$. This means that the trivial procedure of making an arbitrary choice of h_{ab} and k_{ab} and defining ρ and j_a by equations (1) and (2) is of no use for producing physically interesting solutions.

The usual method for solving the equations (1) and (2) is the conformal method[15]. In this method parts of the data (the so-called free data) are chosen, and the constraints imply four elliptic equations for the remaining parts. The case which has been studied most is the constant mean curvature (CMC) case, where $\mathrm{tr}k = h^{ab}k_{ab}$ is constant. In that case there is an important simplification. Three of the elliptic equations, which form a linear system, decouple from the remaining one. This last equation, which is nonlinear, but scalar, is called the Lichnerowicz equation. The heart of the existence theory for the constraints in the CMC case is the theory of the Lichnerowicz equation.

Solving an elliptic equation is a non-local problem and so boundary conditions or asymptotic conditions are important. For the constraints the cases most frequently considered in the literature are that where S is compact (so that no boundary conditions are needed) and that where the free data satisfy some asymptotic flatness conditions. In the CMC case the problem is well understood for both kinds of boundary conditions[12, 29, 49]. The other case which has been studied in detail is that of hyperboloidal data [1]. The kind of theorem which is obtained is that sufficiently differentiable free data, in some cases required to satisfy some global restrictions, can be completed in a unique way to a solution of the constraints.

In the non-CMC case our understanding is much more limited although some results have been obtained in recent years (see [51] and references therein.) It is an important open problem to extend these so that an overview is obtained comparable to that available in the CMC case. Progress on this could also lead to a better understanding of the question, when a spacetime which admits a compact, or asymptotically flat, Cauchy surface also admits one of constant mean curvature. Up to now there are only isolated examples which exhibit obstructions to the existence of CMC hypersurfaces[4].

It would be interesting to know whether there is a useful concept of the most general physically reasonable solutions of the constraints representing regular initial configurations. Data of this kind should not themselves contain singularities. Thus it seems reasonable to suppose at least that the metric h_{ab} is complete and that the length of k_{ab}, as measured using h_{ab}, is bounded. Does the existence of solutions of the constraints imply a restriction on the topology of S or on the asymptotic geometry of the data? This question is largely open, and it seems that information is available only in the compact and asymptotically flat cases. In the case of compact S, where there is no asymptotic regime, there is known to be no topological restriction. In the asymptotically flat case there is also no topological restriction implied by the constraints beyond that implied by the condition of asymptotic flatness itself[82]. This shows in particular that any manifold which is obtained by deleting a point from a compact manifold admits a solution of the constraints satisfying the minimal conditions demanded above. A starting point for going beyond this could be the study of data which are asymptotically homogeneous. For instance, the Schwarzschild solution contains interesting CMC hypersurfaces which are asymptotic to the product of a 2-sphere with the real line. More general data of this kind could be useful for the study of the dynamics of black hole interiors[71].

To sum up, the constraints are well understood in the compact, asymptotically flat and hyperboloidal cases under the constant mean curvature assumption, and only in these cases.

2.2 The vacuum evolution equations

The main aspects of the local in time existence theory for the Einstein equations can be illustrated by restricting to smooth (i.e. infinitely differentiable) data for the vacuum Einstein equations. The generalizations to less smooth data and matter fields are discussed in Sections (2.3) and (2.4) respectively. In the vacuum case the data are h_{ab} and k_{ab} on a three-dimensional manifold S, as discussed in Section (2.1). A solution corresponding to these data is given by a four-dimensional manifold M, a Lorentz metric $g_{\alpha\beta}$ on M and an embedding of S in M. Here $g_{\alpha\beta}$ is supposed to be a solution of the vacuum Einstein equations while h_{ab} and k_{ab} are the induced metric and second fundamental form of the embedding, respectively.

The basic local existence theorem says that, given smooth data for the vacuum Einstein equations, there exists a smooth solution of the equations which gives rise to these data[15]. Moreover, it can be assumed that the image of S under the given embedding is a Cauchy surface for the metric $g_{\alpha\beta}$. The latter fact may be expressed loosely, identifying S with its image, by the statement that S is a Cauchy surface. A solution of the Einstein equations with given initial data having S as a Cauchy surface is called a Cauchy development of those data. The existence theorem is local because it says nothing about the size of the solution obtained. A Cauchy development of given data has many open subsets which are also Cauchy developments of that data.

It is intuitively clear what it means for one Cauchy development to be an extension of another. The extension is called proper if it is strictly larger than the other development. A Cauchy development which has no proper extension is called maximal. The standard global uniqueness theorem for the Einstein equations uses the notion of the maximal development. It is due to Choquet-Bruhat and Geroch[14]. It says that the maximal development of any Cauchy data is unique up to a diffeomorphism which fixes the initial hypersurface. It is also possible to make a statement of Cauchy stability which says that, in an appropriate sense, the solution depends continuously on the initial data. Details on this can be found in [15].

A somewhat stronger form of the local existence theorem is to say that the solution exists on a uniform time interval in all of space. The meaning of this is not a priori clear, due to the lack of a preferred time coordinate in general relativity. The following is a formulation which is independent of coordinates. Let p be a point of S. The temporal extent $T(p)$ of a development of data on S is the supremum of the length of all causal curves in the development passing through p. In this way a development defines a function T on S. The development can be regarded as a solution which exists on a uniform time interval if T is bounded below by a strictly positive constant. For compact S this is a straightforward consequence of Cauchy stability. In the case of asymptotically flat data it is less trivial. In the case of the vacuum Einstein equations it is true, and in fact the function T grows at least linearly at infinity [29].

When proving the above local existence and global uniqueness theorems it is necessary to use some coordinate or gauge conditions. At least no explicitly diffeomorphism-invariant proofs have been found up to now. Introducing these extra elements leads to a system of reduced equations, whose solutions are determined uniquely by initial data in the strict sense, and not just uniquely up to diffeomorphisms. When a solution of the reduced equations has been obtained, it must be checked that it is a solution of the original equations. This means checking that the constraints and gauge conditions propagate. There are many methods for reducing the equations. An overview of the possibilities may be found in [43]

2.3 Questions of differentiability

Solving the Cauchy problem for a system of partial differential equations involves specifying a set of initial data to be considered, and determining the differentiability properties of solutions. Thus two regularity properties are involved - the differentiability of the allowed data, and that of the corresponding solutions. Normally it is stated that for all data with a given regularity, solutions with a certain type of regularity are obtained. For instance in the Section (2.2) we chose both types of regularity to be 'infinitely differentiable'. The correspondence between the regularity of data and that of solutions is not a matter of free choice. It is determined by the equations themselves, and in general the possibilities are severely limited. A similar issue arises in the context of the Einstein constraints, where there is a correspondence between the regularity of free data and full data.

The kinds of regularity properties which can be dealt with in the Cauchy problem depends of course on the mathematical techniques available. When solving the Cauchy problem for the Einstein equations it is necessary to deal at least with nonlinear systems of hyperbolic equations. (There may be other types of equations involved, but they will be ignored here.) For general nonlinear systems of hyperbolic equations there is essentially only one technique known, the method of energy estimates. This method is closely connected with Sobolev spaces, which will now be discussed briefly.

Let u be a real-valued function on $\mathbf{R}^n$. Let:

$$\|u\|_s = (\sum_{i=0}^{s} \int |D^i u|^2(x)dx)^{1/2}$$

The space of functions for which this quantity is finite is the Sobolev space $H^s(\mathbf{R}^n)$. Here $|D^i u|^2$ denotes the sum of the squares of all partial derivatives of u of order i. Thus the Sobolev space H^s is the space of functions, all of whose partial derivatives up to order s are square integrable. Similar spaces can be defined for vector valued functions by taking a sum of contributions from the separate components in the integral. It is also possible to define Sobolev spaces on any Riemannian manifold, using covariant derivatives. General information on this can be found in [3]. Consider now a solution u of the wave equation in Minkowski space. Let $u(t)$ be the restriction of this function to a time slice. Then it is easy to compute that, provided u is smooth and $u(t)$ has compact support for each t, the quantity $\|Du(t)\|_s^2 + \|\partial_t u(t)\|_s^2$ is time independent for each s. For $s = 0$ this is just the energy of a solution of the wave equation. For a general nonlinear hyperbolic system, the Sobolev norms are no longer time-independent. The constancy in time is replaced by certain inequalities. Due to the similarity to the energy for the wave equation, these are called energy estimates. They constitute the foundation of the theory of hyperbolic equations. It is because of these estimates that Sobolev spaces are natural spaces of initial data in the Cauchy problem for hyperbolic equations. Due to the locality properties of hyperbolic equations (existence of a finite domain of dependence), it is useful to introduce the spaces $H^s_{\rm loc}$ which are defined by the condition that whenever the domain of integration is restricted to a compact set the integral defining the space H^s is finite.

In the end the solution of the Cauchy problem should be a function which is differentiable enough in order that all derivatives which occur in the equation exist in the usual (pointwise) sense. A square integrable function is in general defined only almost everywhere and the derivatives in the above formula must be interpreted as distributional derivatives. For this reason a connection between Sobolev spaces and functions whose derivatives exist pointwise is required. This is provided by the Sobolev embedding theorem. This says that if a function u on $\mathbf{R}^n$ belongs to the Sobolev space $H^s_{\rm loc}$ and if $k < s - n/2$ then there is a k times continuously differentiable function which agrees with u except on a set of measure zero.

In the existence and uniqueness theorems stated in Section (2.2), the assumptions on the initial data for the vacuum Einstein equations can be weakened to say that h_{ab} should belong to $H^s_{\rm loc}$ and k_{ab} to $H^{s-1}_{\rm loc}$. Then, provided s is large enough, a solution is obtained which belongs to $H^s_{\rm loc}$. In fact its restriction to any spacelike hypersurface also belongs to $H^s_{\rm loc}$, a property which is a priori stronger. The details of how large s must be would be out of place here, since they involve examining the detailed structure of the energy estimates. However there is a simple rule for computing the required value of s. The value of s needed to obtain an existence theorem for the Einstein equations is that for which the Sobolev embedding theorem, applied to spatial slices, just ensures that the metric is continuously differentiable. Thus the requirement is that $s > n/2 + 1 = 5/2$, since $n = 3$. It follows that the smallest possible integer s is three. Strangely enough, uniqueness up to diffeomorphisms is only known to hold for $s \geq 4$. The reason is that in proving the uniqueness theorem a diffeomorphism must be carried out, which need not be smooth. This apparently leads to a loss of one derivative. It would be desirable to show that uniqueness holds for $s = 3$ and to close this gap, which has existed for many years. There exists a definition of Sobolev spaces for an arbitrary real number s, and hyperbolic equations can also be solved in the spaces with s not an integer [77]. Presumably these techniques could be applied to prove local existence for the Einstein equations with s any real number greater than $5/2$. However this has apparently not been done explicitly in the literature.

Consider now C^∞ initial data. Corresponding to these data there is a development of class H^s for each s. It could conceivably be the case that the size of these developments shrinks with increasing s. In that case their intersection might contain no open neighbourhood of the initial hypersurface, and no smooth development would be obtained. Fortunately it is known that the H^s developments cannot shrink with increasing s, and so the existence of a C^∞ solution is obtained for C^∞ data. It appears that the H^s spaces with $s > 5/2$ are the only spaces containing the space of smooth functions for which it has been proved that the Einstein equations are locally solvable.

What is the motivation for considering regularity conditions other than the apparently very natural C^∞ condition? One motivation concerns matter fields and will be discussed in the Section (2.4). Another is the idea that assuming the existence of many derivatives which have no direct physical significance seems like an admission that the problem has not been fully understood. A further reason for considering low regularity solutions is connected to the possibility of extending a local existence result to a global one. If the proof of a local existence theorem is examined closely it is generally possible to give a continuation criterion. This is a statement that if a local solution is such that a certain quantity constructed from the solution is bounded, then the solution can be extended further. If it can be shown that the relevant quantity is bounded on any region where a local solution exists, then global existence follows. It suffices to consider the maximal region on which a solution is defined, and obtain a contradiction if no global solution exists. This description is a little vague, but contains the essence of a type of argument which is often used in global existence proofs. The problem in putting it into practise is that often the quantity whose boundedness has to be checked contains many derivatives, and is therefore difficult to control. If the continuation criterion can be improved by reducing the number of derivatives required, then this can be a significant step towards a global result. Reducing the number of derivatives in the continuation criterion is closely related to reducing the number of derivatives of the data required for a local existence proof.

A striking example is provided by the work of Klainerman and Machedon [55] on the Yang-Mills equations in Minkowski space. Global existence in this case was first proved by Eardley and Moncrief [38], assuming initial data of sufficiently high differentiability. Klainerman and Machedon gave a new proof of this which, though technically complicated, is based on a conceptually simple idea. They prove a local existence theorem for data of finite energy. Since energy is conserved this immediately proves global existence. In this case finite energy corresponds to the Sobolev space H^1 for the gauge potential. Of course a result of this kind cannot be expected for the Einstein equations, since spacetime singularities do sometimes develop from regular initial data. However, some weaker analogue of the result could exist.

2.4 Matter fields

Analogues of the results for the vacuum Einstein equations given above are known for the Einstein equations coupled to many types of matter model. These include perfect fluids, elasticity theory, kinetic theory, scalar fields, Maxwell fields, Yang-Mills fields and combinations of these. An important restriction is that the general results for perfect fluids and elasticity apply only to situations where the energy density is uniformly bounded away from zero on the region of interest. In particular they do not apply to cases representing material bodies surrounded by vacuum. In cases where the energy density, while everywhere positive, tends to zero at infinity, a local solution is known to exist, but it is not clear whether a local existence theorem can be obtained which is uniform in time. In cases where there the fluid has a sharp boundary, ignoring the boundary leads to solutions of the Einstein-Euler equations with low differentiability (cf. Section 2.3), while taking it into account explicitly leads to a free boundary problem. For more discussion of this and references see [74]. In the case of kinetic or field theoretic matter models it makes no difference whether the energy density vanishes somewhere or not. There is apparently little in the literature on the initial value problem for the Einstein equations coupled to fermions, e.g. for the Einstein-Dirac system, although there seems no reason to expect special difficulties in that case. One paper related to this question is [13].

3 Global symmetric solutions

3.1 Stationary solutions

Many of the results on global solutions of the Einstein equations involve considering classes of spacetimes with Killing vectors. A particularly simple case is that of a timelike Killing vector, i.e. the case of stationary spacetimes. In the vacuum case there are very few solutions satisfying physically reasonable boundary conditions. This is related to no hair theorems for black holes and lies outside the scope of this review. More information on the topic can be found in the book of Heusler [47] and in his Living Review[46]. The case of phenomenological matter models has been reviewed in [74] and there has been little further development in that area since then.

The area of stationary solutions of the Einstein equations coupled to field theoretic matter models has been active in recent years as a consequence of the discovery by Bartnik and McKinnon[5] of a discrete family of regular static spherically symmetric solutions of the Einstein-Yang-Mills equations with gauge group $SU(2)$. The equations to be solved are ordinary differential equations and in [5] they were solved numerically by a shooting method. The first existence proof for a solution of this kind is due to Smoller, Wasserman, Yau and McLeod [75] and involves an arduous qualitative analysis of the differential equations. The work on the Bartnik-McKinnon solutions, including the existence theorems, has been extended in many directions. Recently static solutions of the Einstein-Yang-Mills equations which are not spherically symmetric were discovered numerically [56]. It is a challenge to prove the existence of solutions of this kind. Now the ordinary differential equations of the previously known case are replaced by elliptic equations. Moreover, the solutions appear to still be discrete, so that a simple perturbation argument starting from the spherical case does not seem feasible. In another development it was shown that a linearized analysis indicates the existence of stationary non-static solutions [10]. It would be desirable to study the question of linearization stability in this case, which, if the answer were favourable, would give an existence proof for solutions of this kind.

3.2 Spatially homogeneous solutions

A solution of the Einstein equations is called spatially homogeneous if there exists a group of symmetries with three-dimensional spacelike orbits. In this case there are at least three linearly independent spacelike Killing vector fields. For most matter models the field equations reduce to ordinary differential equations. (Kinetic matter leads to an integro-differential equation.) The most important results in this area have been reviewed in a recent book edited by Wainwright and Ellis[81]. See, in particular, part two of the book. There remain a host of interesting and accessible open questions. The spatially homogeneous solutions have the advantage that it is not necessary to stop at just existence theorems; information on the global qualitative behaviour of solutions can also be obtained. An important open question concerns the mixmaster solution, as discussed in [73].

3.3 Spatially inhomogeneous solutions

The most detailed results on global inhomogeneous solutions of the Einstein equations obtained up to now concern spherically symmetric solutions of the Einstein equations coupled to a massless scalar field with asymptotically flat initial data. In a series of papers Christodoulou [18, 17, 20, 19, 21, 22, 23, 16] has proved a variety of deep results on the global structure of these solutions. Particularly notable are his proofs that naked singularities can develop from regular initial data [23] and that this phenomenon is unstable with respect to perturbations of the data [16]. In related work Christodoulou [24, 25, 26] has studied global spherically symmetric solutions of the Einstein equations coupled to a fluid with a special equation of state (the so-called two-phase model).

Solutions of the Einstein equations with cylindrical symmetry which are asymptotically flat in all directions allowed by the symmetry represent an interesting variation on asymptotic flatness. Since black holes are incompatible with this symmetry, one may hope to prove geodesic completeness of solutions under appropriate assumptions. This has been accomplished for the Einstein vacuum equations and for the source-free Einstein-Maxwell equations in[7], building on global existence theorems for wave maps[31, 30]. For a quite different point of view on this question see[83].

In the context of spatially compact spacetimes it is first necessary to ask what kind of global statements are to be expected. In a situation where the model expands indefinitely it is natural to pose the question whether the spacetime is causally geodesically complete towards the future. In a situation where the model develops a singularity either in the past or in the future one can ask what the qualitative nature of the singularity is. It is very difficult to prove results of this kind. As a first step one may prove a global existence theorem in a well-chosen time coordinate. In other words, a time coordinate is chosen which is geometrically defined and which, under ideal circumstances, will take all values in a certain interval (t_-, t_+). The aim is then to show that, in the maximal Cauchy development of data belonging to a certain class, a time coordinate of the given type exists and exhausts the expected interval. The first result of this kind for inhomogeneous spacetimes was proved by Moncrief in [58]. This result concerned Gowdy spacetimes. These are vacuum spacetimes with two commuting Killing vectors acting on compact orbits. The area of the orbits defines a natural time coordinate. Moncrief showed that in the maximal Cauchy development of data given on a hypersurface of constant time, this time coordinate takes on the maximal possible range, namely $(0, \infty)$. This result was extended to more general vacuum spacetimes with two Killing vectors in [6].

Another attractive time coordinate is constant mean curvature (CMC) time. For a general discussion of this see [71]. A global existence theorem in this time for spacetimes with two Killing vectors and certain matter models (collisionless matter, wave maps) was proved in [72]. That the choice of matter model is important for this result was demonstrated by a global non-existence result for dust in [66]. Related results have been obtained for spherical and hyperbolic symmetry [70, 11].

Once global existence has been proved for a preferred time coordinate, the next step is to investigate the asymptotic behaviour of the solution as $t \to t_\pm$. There are few cases in which this has been done successfully. Notable examples are Gowdy spacetimes [32, 50, 35] and solutions of the Einstein-Vlasov system with spherical and plane symmetry[64].

4 Global existence for small data

An alternative to symmetry assumptions is provided by 'small data' results, where solutions are studied which develop from data close to that for known solutions. This leads to some simplification in comparison to the general problem, but with present techniques it is still very hard to obtain results of this kind.

4.1 Stability of de Sitter space

In [40] Friedrich proved a result on the stability of de Sitter space. This concerns the Einstein vacuum equations with positive cosmological constant. His result is as follows. Consider initial data induced by de Sitter space on a regular Cauchy hypersurface. Then all initial data (vacuum with positive cosmological constant) near enough to these data in a suitable (Sobolev) topology have maximal Cauchy developments which are geodesically complete. In fact the result gives much more detail on the asymptotic behaviour than just this and may be thought of as proving a form of the cosmic no hair conjecture in the vacuum case. (This conjecture says roughly that the de Sitter solution is an attractor for expanding cosmological models with positive cosmological constant.) This result is proved using conformal techniques and, in particular, the regular conformal field equations developed by Friedrich.

There are results obtained using the regular conformal field equations for negative or vanishing cosmological constant [42, 39] but a detailed discussion of their nature would be out of place here. (Cf. however Section (5.2).)

4.2 Stability of Minkowski space

The other result on global existence for small data is that of Christodoulou and Klainerman on the stability of Minkowski space[28]. The formulation of the result is close to that given in Section (4.1) but now de Sitter space is replaced by Minkowski space. Suppose then that initial data are given which are asymptotically flat and sufficiently close to those induced by Minkowski space on a hyperplane. Then Christodoulou and Klainerman prove that the maximal Cauchy development of these data is geodesically complete. They also provide a wealth of detail on the asymptotic behaviour of the solutions. The proof is very long and technical. The central tool is the Bel-Robinson tensor which plays an analogous role for the gravitational field to that played by the energy-momentum tensor for matter fields. Apart from the book of Christodoulou and Klainerman itself some introductory material on geometric and analytic aspects of the proof can be found in [8] and [27] respectively.

5 Further results

5.1 Isotropic singularities

The existence and uniqueness results discussed in this section are motivated by Penrose's Weyl curvature hypothesis. Penrose suggests that the initial singularity in a cosmological model should be such that the Weyl tensor tends to zero or at least remains bounded. There is some difficulty in capturing this by a geometric condition and it was suggested by [80] that a clearly formulated geometric condition which, on an intuitive level, is closely related to the original condition, is that the conformal structure should remain regular at the singularity. Singularities of this type are known as conformal or isotropic singularities.

Consider now the Einstein equations coupled to a perfect fluid with the radiation equation of state $p = \rho/3$. Then it has been shown [61, 36] that solutions with an isotropic singularity are determined uniquely by certain free data given at the singularity. The data which can be given is, roughly speaking, half as large as in the case of a regular Cauchy hypersurface. The method of proof is to derive an existence and uniqueness theorem for a suitable class of singular hyperbolic equations. Generalizations of this by Anguige and Tod have been discussed in [79]. Details will be given in Anguige's thesis. Related work was done earlier in a somewhat simpler context by Moncrief[59] who showed the existence of a large class of spacetimes with Cauchy horizons.

5.2 Evolution of hyperboloidal data

In Section (2.1) hyperboloidal initial data were mentioned. They can be thought of as generalizations of the data induced by Minkowski space on a hyperboloid. In the case of Minkowski space the solution admits a conformal compactification where a conformal boundary, null infinity, can be added to the spacetime. It can be shown that in the case of the maximal development of hyperboloidal data a piece of null infinity can be attached to the spacetime. For small data, i.e. data close to that of a hyperboloid in Minkowski space, this conformal boundary also has completeness properties in the future allowing an additional point i_+ to be attached there. (See [41] and references therein for more details.) Making contact between hyperboloidal data and asymptotically flat initial data is much more difficult and there is as yet no complete picture. (An account of the results obtained up to now is given in [39].) If the relation between hyperboloidal and asymptotically flat initial data could be understood it would give a very different approach to the problem treated by Christodoulou and Klainerman (Section (4.2)).

5.3 The Newtonian limit

Most textbooks on general relativity discuss the fact that Newtonian gravitational theory is the limit of general relativity as the speed of light tends to infinity. It is a non-trivial task to give a precise mathematical definition of this statement. Once a definition has been given the question remains whether this definition is compatible with the Einstein equations in the sense that there are general families of solutions of the Einstein equations which have a Newtonian limit in the sense of the chosen definition. A theorem of this kind was proved in [69], where the matter content of spacetime was assumed to be a collisionless gas described by the Vlasov equation. (For another suggestion as to how this problem could be approached see [44].) The essential mathematical problem is that of a family of equations depending continuously on a parameter λ which are hyperbolic for $\lambda \neq 0$ and degenerate for $\lambda = 0$. Because of the singular nature of the limit it is by no means clear a priori that there are families of solutions which depend continuously on λ. That there is an abundant supply of families of this kind is the result of [69]. Asking whether there are families which are k times continuously differentiable in their dependence on λ is related to the issue of giving a mathematical justification of post-Newtonian approximations. The approach of [69] has not even been extended to the case $k = 1$ and it would be desirable to do this. Note however that for k too large serious restrictions arise [68]. The latter fact corresponds to the well-known divergent behaviour of higher order post-Newtonian approximations.

5.4 Newtonian cosmology

Apart from the interest of the Newtonian limit, Newtonian gravitational theory itself may provide interesting lessons for general relativity. This is no less true for existence theorems than for other issues. In this context it is also interesting to consider a slight generalization of Newtonian theory, the Newton-Cartan theory. This allows a nice treatment of cosmological models, which are in conflict with the (sometimes implicit) assumption in Newtonian gravitational theory that only isolated systems are considered. It is also unproblematic to introduce a cosmological constant into the Newton-Cartan theory.

Three global existence theorems have been proved in Newtonian cosmology. The first[9] is an analogue of the cosmic no hair theorem (cf. Section 4.1) and concerns models with a positive cosmological constant. It asserts that homogeneous and isotropic models are nonlinearly stable if the matter is described by dust or a polytropic fluid with pressure. Thus it gives information about global existence and asymptotic behaviour for models arising from small (but finite) perturbations of homogeneous and isotropic data. The second and third results concern collisionless matter and the case of vanishing cosmological constant. The second[65] says that data which constitute a periodic (but not necessarily small) perturbation of a homogeneous and isotropic model which expands indefinitely give rise to solutions which exist globally in the future. The third[63] says that the homogeneous and isotropic models in Newtonian cosmology which correspond to a $k = -1$ Friedmann-Robertson-Walker model in general relativity are non-linearly stable.

5.5 The characteristic initial value problem

In the standard Cauchy problem, which has been the basic set-up for all the previous sections, initial data are given on a spacelike hypersurface. However there is also another possibility, where data are given on one or more null hypersurfaces. This is the characteristic initial value problem. It has the advantage over the Cauchy problem that the constraints reduce to ordinary differential equations. One variant is to give initial data on two smooth spacelike hypersurfaces which intersect transversely in a spacelike surface. A local existence theorem for the Einstein equations with an initial configuration of this type was proved in [67]. Another variant is to give data on a light cone. In that case local existence for the Einstein equations has not been proved, although it has been proved for a class of quasilinear hyperbolic equations which includes the reduced Einstein equations in harmonic coordinates[37].

Another existence theorem which does not use the standard Cauchy problem, and which is closely connected to the use of null hypersurfaces, concerns the

Robinson-Trautman solutions of the vacuum Einstein equations. In that case the Einstein equations reduce to a parabolic equation. Global existence for this equation has been proved by Chruściel[34].

References

[1] Andersson, L., Chruściel, P. T., and Friedrich, H., "On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations", *Commun. Math. Phys.*, **149**, 587–612, (1992).

[2] Arnold, V. I., and Ilyashenko, Yu. S., "Ordinary differential equations", in Anosov, D. V., and Arnold, V. I., eds., *Dynamical Systems I.*, 1–148, (Springer, Berlin, 1988).

[3] Aubin, T., *Nonlinear Analysis on Manifolds. Monge-Ampère equations*, (Springer, Berlin, 1982).

[4] Bartnik, R., "Remarks on cosmological spacetimes and constant mean curvature hypersurfaces", *Commun. Math. Phys.*, **117**, 615–624, (1988).

[5] Bartnik, R., and McKinnon, J., "Particlelike solutions of the Einstein-Yang-Mills equations", *Phys. Rev. Lett.*, **61**, 141–143, (1988).

[6] Berger, B. K., Chruściel, P. T., Isenberg, J., and Moncrief, V., "Global foliations of vacuum spacetimes with T^2 isometry", *Ann. Phys. (NY)*, **260**, 117–148, (1997).

[7] Berger, B. K., Chruściel, P. T., and Moncrief, V., "On asymptotically flat spacetimes with G_2-invariant Cauchy surfaces", *Ann. Phys. (NY)*, **237**, 322–354, (1995).

[8] Bourguignon, J.-P., "Stabilité par déformation non-linéaire de la métrique de Minkowski (d'après D. Christodoulou et S. Klainerman)", *Asterisque*, **201-203**, 321–358, (1992).

[9] Brauer, U., Rendall, A. D., and Reula, O. A., "The cosmic no-hair theorem and the nonlinear stability of homogeneous Newtonian cosmological models", *Class. Quantum Grav.*, **11**, 2283–2296, (1994).

[10] Brodbeck, O., Heusler, M., Straumann, N., and Volkov, M., "Rotating solitons and non-rotating non-static black holes", (July, 1997), [Online Los Alamos Archive Preprint]: cited on 7 January 1998, http://xxx.lanl.gov/abs/gr-qc/9707057.

[11] Burnett, G. A., and Rendall, A. D., "Existence of maximal hypersurfaces in some spherically symmetric spacetimes", *Class. Quantum Grav.*, **13**, 111–123, (1996).

[12] Cantor, M., "A necessary and sufficient condition for York data to specify an asymptotically flat spacetime", *J. Math. Phys.*, **20**, 1741–1744, (1979).

[13] Choquet-Bruhat, Y., "The Cauchy problem in classical supergravity", *Lett. Math. Phys.*, **7**, 459–467, (1983).

[14] Choquet-Bruhat, Y., and Geroch, R., "Global aspects of the Cauchy problem in general relativity", *Commun. Math. Phys.*, **14**, 329–335, (1969).

[15] Choquet-Bruhat, Y., and York, J., "The Cauchy problem", in Held, A., ed., *General relativity and gravitation*, volume 1, 99–172, (Plenum, New York, 1980).

[16] Christodoulou, D., "The instability of naked singularities in the gravitational collapse of a scalar field", Preprint.

[17] Christodoulou, D., "Global existence of generalised solutions of the spherically symmetric Einstein-scalar equations in the large", *Commun. Math. Phys.*, **106**, 587–621, (1986).

[18] Christodoulou, D., "The problem of a self-gravitating scalar field", *Commun. Math. Phys.*, **105**, 337–361, (1986).

[19] Christodoulou, D., "A mathematical theory of gravitational collapse", *Commun. Math. Phys.*, **109**, 613–647, (1987).

[20] Christodoulou, D., "The structure and uniqueness of generalised solutions of the spherically symmetric Einstein-scalar equations", *Commun. Math. Phys.*, **109**, 591–611, (1987).

[21] Christodoulou, D., "The formation of black holes and singularities in spherically symmetric gravitational collapse", *Commun. Pure Appl. Math.*, **44**, 339–373, (1991).

[22] Christodoulou, D., "Bounded variation solutions of the spherically symmetric Einstein-scalar field equations", *Commun. Pure Appl. Math.*, **46**, 1131, (1993).

[23] Christodoulou, D., "Examples of naked singularity formation in the gravitational collapse of a scalar field", *Ann. Math.*, **140**, 607–653, (1994).

[24] Christodoulou, D., "Self-gravitating fluids: a two-phase model", *Arch. Rat. Mech. Anal.*, **130**, 343–400, (1995).

[25] Christodoulou, D., "Self-gravitating fluids: the continuation and termination of a free phase boundary", *Arch. Rat. Mech. Anal.*, **133**, 333–398, (1996).

[26] Christodoulou, D., "Self-gravitating fluids: the formation of a free phase boundary in the phase transition from soft to hard", *Arch. Rat. Mech. Anal.*, **134**, 97–154, (1996).

[27] Christodoulou, D., and Klainerman, S., "Asymptotic properties of linear field equations in Minkowski space", *Commun. Pure Appl. Math.*, **43**, 137–199, (1990).

[28] Christodoulou, D., and Klainerman, S., *The global nonlinear stability of the Minkowski space*, (Princeton University Press, Princeton, 1993).

[29] Christodoulou, D., and Murchadha, Ó, "The boost problem in general relativity", *Commun. Math. Phys.*, **10**, 271–300, (1981).

[30] Christodoulou, D., and Tahvildar-Zadeh, A. S., "On the asymptotic behaviour of spherically symmetric wave maps", *Duke Math. J.*, **71**, 31–69, (1993).

[31] Christodoulou, D., and Tahvildar-Zadeh, A. S., "On the regularity of spherically symmetric wave maps", *Commun. Pure Appl. Math.*, **46**, 1041–1091, (1993).

[32] Chruściel, P. T., "On spacetimes with $U(1) \times U(1)$ symmetric compact Cauchy surfaces", *Ann. Phys. (NY)*, **202**, 100–150, (1990).

[33] Chruściel, P. T., *On the uniqueness in the large of solutions of Einstein's equations (Strong cosmic censorship)*, volume 20 of *Proc. Centre Math. Anal.*, (Australian National University, Canberra, 1991).

[34] Chruściel, P. T., "Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation", *Commun. Math. Phys.*, **137**, 289–313, (1991).

[35] Chruściel, P. T., Isenberg, J., and Moncrief, V., "Strong cosmic censorship in polarised Gowdy spacetimes", *Class. Quantum Grav.*, **7**, 1671–1680, (1990).

[36] Claudel, C., "The Cauchy problem for quasilinear hyperbolic evolution problems with a singularity in the time", Preprint.

[37] Dossa, M., "Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique", *Ann. Inst. Henri Poincare A*, **66**, 37–107, (1997).

[38] Eardley, D., and Moncrief, V., "The global existence of Yang-Mills fields in M^{3+1}", *Commun. Math. Phys.*, **83**, 171–212, (1982).

[39] Friedrich, H., "Gravitational fields near spacelike and null infinity", Preprint AEI 022.

[40] Friedrich, H., "Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant", *J. Geom. Phys.*, **3**, 101–117, (1986).

[41] Friedrich, H., "On the global existence and asymptotic behaviour of solutions to the Einstein-Yang-Mills equations", *J. Diff. Geom.*, **34**, 275–345, (1991).

[42] Friedrich, H., "Einstein equations and conformal structure: Existence of anti-de Sitter-type spacetimes", *J. Geom. Phys.*, **17**, 125–184, (1995).

[43] Friedrich, H., "Hyperbolic reductions of Einstein's field equations", *Class. Quantum Grav.*, **13**, 1451–1469, (1996).

[44] Fritelli, S., and Reula, O., "On the Newtonian limit of general relativity", *Commun. Math. Phys.*, **166**, 221–235, (1994).

[45] Hartman, P., *Ordinary Differential Equations*, (Birkhäuser, Basel, 1982).

[46] Heusler, M., "Black hole spacetimes", (1998), [Article in the Online Journal Living Reviews in Relativity]: cited on 3 January 1998. To appear.

[47] Heusler, M., *Black hole uniqueness theorems*, (Cambridge University Press, Cambridge, 1996).

[48] Hubbard, J. H., and West, B. H., *Differential equations: a dynamical systems approach*, (Springer, Berlin, 1991).

[49] Isenberg, J., "Constant mean curvature solutions of the Einstein constraint equations on closed manifolds", *Class. Quantum Grav.*, **12**, 2249–2274, (1995).

[50] Isenberg, J., and Moncrief, V., "Asymptotic behaviour of the gravitational field and the nature of singularities in Gowdy spacetimes", *Ann. Phys. (NY)*, **199**, 84–122, (1990).

[51] Isenberg, J., and Moncrief, V., "A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds", *Class. Quantum Grav.*, **13**, 1819–1847, (1996).

[52] John, F., *Partial differential equations*, (Springer, Berlin, 1982).

[53] John, F., *Nonlinear wave equations, formation of singularities*, (American Mathematical Society, Providence, 1990).

[54] Kichenassamy, S., *Nonlinear wave equations*, (Marcel Dekker, New York, 1996).

[55] Klainerman, S., and Machedon, M., "Finite energy solutions of the Yang-Mills equations in $\mathbf{R}^{3+1}$", *Ann. Math.*, **142**, 39–119, (1995).

[56] Kleihaus, B., and Kunz, J., "Static axially symmetric Einstein-Yang-Mills-dilaton solutions: I. Regular solutions", (July, 1997), [Online Los Alamos Archive Preprint]: cited on 7 January 1998, http://xxx.lanl.gov/abs/gr-qc/9707045.

[57] Majda, A., *Compressible fluid flow and systems of conservation laws in several space dimensions*, (Springer, Berlin, 1984).

[58] Moncrief, V., "Global properties of Gowdy spacetimes with $T^3 \times R$ topology", *Ann. Phys. (NY)*, **132**, 87–107, (1981).

[59] Moncrief, V., "Neighbourhoods of Cauchy horizons in cosmological spacetimes with one Killing field", *Ann. Phys. (NY)*, **141**, 83–103, (1982).

[60] Moncrief, V., and Eardley, D., "The global existence problem and cosmic censorship in general relativity", *Gen. Relativ. Gravit.*, **13**, 887–892, (1981).

[61] Newman, R. P. A. C., "On the structure of conformal singularities in classical general relativity", *Proc. R. Soc. London*, **443**, A473–492; A493–515, (1993).

[62] Racke, R., *Lectures on nonlinear evolution problems*, (Vieweg, Wiesbaden, 1992).

[63] Rein, G., "Nonlinear stability of homogeneous models in Newtonian cosmology", (March, 1996), [Online Los Alamos Archive Preprint]: cited on 7 January 1998, http://xxx.lanl.gov/abs/gr-qc/9603033.

[64] Rein, G., "Cosmological solutions of the Vlasov-Einstein system with spherical, plane and hyperbolic symmetry", *Math. Proc. Cambridge. Philos. Soc.*, **119**, 739–762, (1996).

[65] Rein, G., and Rendall, A. D., "Global existence of classical solutions to the Vlasov-Poisson system in a three dimensional, cosmological setting", *Arch. Rat. Mech. Anal.*, **126**, 183–201, (1994).

[66] Rendall, A. D., "Existence and non-existence results for global constant mean curvature foliations", (August, 1996), [Online Los Alamos Archive Preprint]: cited on 7 January 1998, http://xxx.lanl.gov/abs/gr-qc/9608045.

[67] Rendall, A. D., "Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations", *Proc. R. Soc. London*, **A427**, 221–239, (1990).

[68] Rendall, A. D., "On the definition of post-Newtonian approximations", *Proc. R. Soc. London*, **A438**, 341–360, (1992).

[69] Rendall, A. D., "The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system", *Commun. Math. Phys.*, **163**, 89–112, (1994).

[70] Rendall, A. D., "Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry", *Class. Quantum Grav.*, **12**, 1517–1533, (1995).

[71] Rendall, A. D., "Constant mean curvature foliations in cosmological spacetimes", *Helv. Phys. Acta*, **69**, 490–500, (1996).

[72] Rendall, A. D., "Existence of constant mean curvature hypersurfaces in spacetimes with two-dimensional local symmetry", *Commun. Math. Phys.*, **189**, 145–164, (1997).

[73] Rendall, A. D., "Global dynamics of the mixmaster model", *Class. Quantum Grav.*, **14**, 2341–2356, (1997).

[74] Rendall, A. D., "Solutions of the Einstein equations with matter", in Francaviglia, M., Longhi, G., Lusanna, L., and Sorace., E., eds., *Proceedings of the 14th International Conference on General Relativity and Gravitation*, 313–335, (World Scientific, Singapore, 1997).

[75] Smoller, J. A., Wasserman, A. G., Yau, S. T., and Mcleod, J. B., "Smooth static solutions of the Einstein-Yang-Mills equations", *Commun. Math. Phys.*, **143**, 115–147, (1991).

[76] Strauss, W., *Nonlinear wave equations*, (American Mathematical Society, Providence, 1989).

[77] Taylor, M. E., *Pseudodifferential operators and nonlinear PDE*, (Birkhäuser, Boston, 1991).

[78] Taylor, M. E., *Partial differential equations*, (Springer, Berlin, 1996).

[79] Tod, K. P., "Isotropic cosmological singularities", to appear in the Proceedings of the XIIth International Congress of Mathematical Physics, Brisbane, 1997.

[80] Tod, K. P., "Isotropic singularities", *Rend. Sem. Mat. Univers. Politecn. Torino*, **50**, 69–93, (1992).

[81] Wainwright, J., and Ellis, G. F. R., *Dynamical Systems in Cosmology*, (Cambridge University Press, Cambridge, 1997).

[82] Witt, D., "Vacuum spacetimes that admit no maximal slice", *Phys. Rev. Lett.*, **57**, 1386–1389, (1986).

[83] Woodhouse, N. M. J., "Integrability and Einstein's equations", *Banach Center Publ.*, **41, 1**, 221–232, (1997).