

Md. Azizul Baten

**ON THE SMOOTHNESS OF SOLUTIONS OF
LINEAR-QUADRATIC REGULATOR FOR DEGENERATE
DIFFUSIONS**

(submitted by A. Lapin)

ABSTRACT. The paper studies the smoothness of solutions of the degenerate Hamilton-Jacobi-Bellman (HJB) equation associated with a linear-quadratic regulator control problem. We establish the existence of a classical solution of the degenerate HJB equation associated with this problem by the technique of viscosity solutions, and hence derive an optimal control from the optimality conditions in the HJB equation.

1. INTRODUCTION

We are concerned with the quadratic control problem to minimize the expected cost with discount factor $\beta > 0$:

$$J(c) = E\left[\int_0^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt\right] \quad (1)$$

over $c \in \mathcal{A}$ subject to the degenerate stochastic differential equation

$$dx_t = [Ax_t + c_t]dt + \sigma x_t dw_t, \quad x_0 = x \in \mathbf{R}, \quad t \geq 0, \quad (2)$$

Key words and phrases. Stochastic differential equation, Hamilton-Jacobi-Bellman equation, Linear-Quadratic problem, Viscosity solutions, Applications to control theory.

2000 Mathematical Subject Classification. 60H10, 49N10, 49J15, 49L25, 58E25.

for non-zero constants $A, \sigma \neq 0$, and a continuous function h on \mathbf{R} , where w_t is a one-dimensional standard Brownian motion on a complete probability space (Ω, \mathcal{F}, P) endowed with the natural filtration \mathcal{F}_t generated by $\sigma(w_s, s \leq t)$, and \mathcal{A} denotes the class of all \mathcal{F}_t -progressively measurable processes $c = (c_t)$ with $J(c) < \infty$.

This kind of stochastic control problem has been studied by many authors [3, 6] for non-degenerate diffusions to (1) and (2). We also assume that h satisfies the following properties:

$$h(x) \geq 0 : \text{convex}; \quad (3)$$

$$\text{There exists } C > 0 \text{ such that } h(x) \leq C(1 + |x|^n), x \in \mathbf{R}, \quad (4)$$

for some constant $C > 0, n \geq 2$. We refer to [5] for the quadratic case of degenerate diffusions related to Riccati equations in case of $h(x) = Cx^2$ and $n = 2$ with infinite horizon.

The purpose of this paper is to show the existence of a smooth solution u of the associated Hamilton-Jacobi-Bellman (in short, HJB) equation of the form:

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' + \min_{r \in \mathbf{R}}(r^2 + ru') + h(x) = 0 \quad \text{in } \mathbf{R}, \quad (5)$$

and to give a synthesis of optimal control. Our method consists in finding the viscosity solution u of (5) [4, 6], by the limit of the solution $v = v_L, L > 0$, to the HJB equation

$$-\beta v_L + \frac{1}{2}\sigma^2 x^2 v_L'' + Axv_L' + \min_{|r| \leq L}(r^2 + rv_L') + h(x) = 0 \quad \text{in } \mathbf{R}, \quad (6)$$

as $L \rightarrow \infty$, and then in considering the smoothness of u by its convexity. To show the existence of the viscosity solution v_L , we assume that h has the following property: there exists $C_\rho > 0$, for any $\rho > 0$, such that

$$|h(x) - h(y)| \leq C_\rho |x - y|^n + \rho(1 + |x|^n + |y|^n), \quad \forall x, y \in \mathbf{R}, \quad (7)$$

for a fixed integer $n \geq 2$.

This condition acts as the uniform continuity of h with order n , and plays an important role for the existence of viscosity solutions [7, 8]. We notice that (7) holds for $h(x) = |x|^{\bar{n}}, \bar{n} \in [2, n]$,

In §2 we show that $u(x) := \lim_{L \rightarrow \infty} v_L(x)$ is a viscosity solution of (5), as $L \rightarrow \infty$. §3 is devoted to the study of smoothness of u . Finally in §4 we present an optimal control to the optimization problem (1) and (2).

2. CONVERGENCES OF THE VALUE FUNCTION

In this subsection we show that $v_L(x)$ is a viscosity solution of the Bellman equation (5) for any fixed $L > 0$, and then converges to a viscosity solution $u(x)$ of the Bellman equation (5). In order to introduce solutions in the viscosity sense, given a continuous and degenerate elliptic map $H : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, we recall by [4] the definition of viscosity solutions of

$$H(x, w, w', w'') = 0 \quad \text{in } \mathbf{R}. \quad (8)$$

Definition 2.1. $w \in C(\mathbf{R})$ is called a viscosity subsolution (resp., supersolution) of (8) if, whenever for $\varphi \in C^2(\mathbf{R})$, $w - \varphi$ attains its local maximum (resp., minimum) at $x \in \mathbf{R}$, then

$$H(x, w(x), \varphi'(x), \varphi''(x)) \leq 0 \quad (9)$$

$$\text{resp., } H(x, w(x), \varphi'(x), \varphi''(x)) \geq 0. \quad (10)$$

We also call $w \in C(\mathbf{R})$ a viscosity solution of (8) if it is both viscosity sub- and supersolution of (8).

According to Crandall, Ishii and Lions [4] and Fleming and Soner [6] this definition is equivalent to the following: for any $x \in \mathbf{R}$,

$$H(x, w(x), p, q) \leq 0 \quad \text{for } (p, q) \in J^{2,+}w(x)$$

$$H(x, w(x), p, q) \geq 0 \quad \text{for } (p, q) \in J^{2,-}w(x),$$

where $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets defined by

$$\begin{aligned} J^{2,+}w(x) &= \{(p, q) \in \mathbf{R}^2 : \limsup_{y \rightarrow x} \frac{w(y) - w(x) - p(y-x) - \frac{1}{2}q|y-x|^2}{|y-x|^2} \leq 0\}, \\ J^{2,-}w(x) &= \{(p, q) \in \mathbf{R}^2 : \liminf_{y \rightarrow x} \frac{w(y) - w(x) - p(y-x) - \frac{1}{2}q|y-x|^2}{|y-x|^2} \geq 0\}. \end{aligned}$$

Let us define the value function $v_L(x) = \inf_{c \in \mathcal{A}_L} J(c)$, where $\mathcal{A}_L = \{c = (c_t) \in \mathcal{A} : |c_t| \leq L \text{ for all } t \geq 0\}$. We assume that there exists $\beta_0 \in (0, \beta)$ satisfying

$$-\beta_0 + \sigma^2 n(2n-1) + 2n|A| < 0, \quad (11)$$

and we set $f_k(x) = \gamma + |x|^k$ for any $2 \leq k \leq 2n$ and a constant $\gamma \geq 1$ chosen later.

Lemma 2.2. Assume (11). Then there exist $\gamma \geq 1$ and $\eta > 0$, depending on L, k , such that

$$-\beta_0 f_k + \frac{1}{2} \sigma^2 x^2 f_k'' + A x f_k' + \max_{|r| \leq L} (r^2 + r f_k') + \eta f_k \leq 0 \quad (12)$$

Further

$$E\left[\int_0^\tau e^{-\beta_0 s} \eta f_k(x_s) ds + e^{-\beta_0 \tau} f_k(x_\tau)\right] \leq f_k(x) \quad \text{for } 2 \leq k \leq 2n, \quad (13)$$

$$E[\sup_t e^{-\beta_0 t} f_k(x_t)] < \infty \quad \text{for } 2 \leq k \leq n, \quad (14)$$

where τ is any stopping time and x_t is the response to $(c_t) \in \mathcal{A}_L$.

Proof. By (11), we choose $\eta \in (0, \beta_0)$ such that

$$-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta < 0, \quad (15)$$

and then $\gamma \geq 1$ such that

$$(-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta)|x|^k + Lk|x|^{k-1} + (L^2 + \eta\gamma - \beta_0\gamma) \leq 0.$$

Then (12) is immediate. By (12) and Ito's formula, we deduce (13). Moreover, by moment inequalities for martingales we get

$$\begin{aligned} E[\sup_t e^{-\beta_0 t} f_k(x_t)] &\leq f_k(x) + E[\sup_t |\int_0^t e^{-\beta_0 s} f'_k(x_s) \sigma x_s dw_s|] \\ &\leq f_k(x) + KE[(\int_0^\infty e^{-2\beta_0 s} \sigma^2 |x_s|^{2k} ds)^{1/2}], \end{aligned}$$

for some constant $K > 0$. Therefore (14) follows from this relation together with (13).

Theorem 2.3. *We assume (3), (4), (7) and (11). Then*

$$v_L \text{ satisfies (3), (4), (7),} \quad (16)$$

and the dynamic programming principle holds, i.e.,

$$v_L(x) = \inf_{c \in \mathcal{A}_L} E\left[\int_0^\tau e^{-\beta t} \{h(x_t) + |c_t|^2\} dt + e^{-\beta \tau} v_L(x_\tau)\right] \quad (17)$$

for any stopping time τ .

Proof. We suppress L of v_L for simplicity. The convexity of v follows from the same line as [5, Chap. 4, Lemma 10.6]. Let x_t^0 be the unique solution of

$$dx_t^0 = Ax_t^0 dt + \sigma x_t^0 dw_t, \quad x_0^0 = x. \quad (18)$$

Then, by (13) and (4)

$$v(x) \leq E\left[\int_0^\infty e^{-\beta t} h(x_t^0) dt\right] \leq CE\left[\int_0^\infty e^{-\beta_0 t} f_n(x_t^0) dt\right] \leq Cf_n(x)/\eta. \quad (19)$$

For the solution y_t of (2) with $y_0 = y$, it is clear that $x_t - y_t$ satisfies (18) with initial condition $x - y$. We note by (15) with $k = n$ and Ito's formula that

$$E[e^{-\beta_0 t} |x_t^0|^n] \leq |x|^n.$$

Thus by (7) and (13)

$$\begin{aligned} |v(x) - v(y)| &\leq \sup_{c \in \mathcal{A}_L} E\left[\int_0^\infty e^{-\beta t} |h(x_t) - h(y_t)| dt\right] \\ &\leq \sup_{c \in \mathcal{A}_L} E\left[\int_0^\infty e^{-\beta t} \left\{C_\rho |x_t - y_t|^n + \rho(1 + |x_t|^n + |y_t|^n)\right\} dt\right] \\ &\leq \sup_{c \in \mathcal{A}_L} \int_0^\infty e^{-\beta t} \left\{C_\rho |x - y|^n e^{\beta_0 t} + \rho(h_n(x) + h_n(y)) e^{\beta_0 t}\right\} dt \\ &\leq \frac{1}{\beta - \beta_0} [C_\rho |x - y|^n + 2\rho\gamma(1 + |x|^n + |y|^n)] \quad (20) \end{aligned}$$

Therefore we get (16).

To prove (17), we denote by $v^r(x)$ the right hand side of (17). By the formal Markov property

$$\begin{aligned} E\left[\int_\tau^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt \middle| \mathcal{F}_\tau\right] &= E\left[\int_0^\infty e^{-\beta(t+\tau)} \{h(x_{\tau+t}) + |c_{\tau+t}|^2\} dt \middle| \mathcal{F}_\tau\right] \\ &= e^{-\beta\tau} J_{\tilde{c}}(x_\tau), \end{aligned}$$

with \tilde{c} equal to c shifted by τ . Thus

$$\begin{aligned} J_c(x) &= E\left[\int_0^\tau + \int_\tau^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt\right] \\ &= E\left[\int_0^\tau e^{-\beta t} \{h(x_t) + |c_t|^2\} dt\right] + E\left[\int_\tau^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt \middle| \mathcal{F}_\tau\right] \\ &\geq E\left[\int_0^\tau e^{-\beta t} \{h(x_t) + |c_t|^2\} dt + e^{-\beta\tau} v_L(x_\tau)\right]. \end{aligned}$$

It is known in [6, 9] that this formal argument can be verified, and we deduce $v_L(x) \geq v^r(x)$.

To prove the reverse inequality, let $\rho > 0$ be arbitrary. We set

$$V_c(x) = E\left[\int_0^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt\right] \quad (21)$$

By the same calculation as (20), there exists $C_\rho > 0$ such that

$$|V_c(x) - V_c(y)| \leq C_\rho |x - y|^n + \rho(1 + |x|^n + |y|^n).$$

Take $0 < \delta < 1$ with $C_\rho \delta^n < \rho$. Then, we have for $|x - y| < \delta$,

$$\begin{aligned}
|v(x) - v(y)| &\leq \sup_{c \in \mathcal{A}_L} |V_c(x) - V_c(y)| \\
&\leq \rho(2 + |x|^n + |y|^n) \\
&\leq \rho[2 + |x|^n + 2^n(1 + |x|^n)] \\
&= \rho[(2 + 2^n) + (1 + 2^n)|x|^n] \\
&\leq \Xi_\rho(x) := \rho(2^n + 2)(1 + |x|^n).
\end{aligned}$$

Let $\{S_i\}$ be a sequence of disjoint subsets of \mathbf{R} such that

$$\text{diam}(S_i) < \delta \quad \text{and} \quad \cup_i S_i = \mathbf{R}.$$

For any i , we take $x^{(i)} \in S_i$ and $c^{(i)} \in \mathcal{A}_L$ such that

$$V_{c^{(i)}}(x^{(i)}) \leq \inf_{c \in \mathcal{A}_L} V_c(x^{(i)}) + \rho.$$

Define $c^\tau \in \mathcal{A}_L$ by

$$c_t^\tau = c_t 1_{\{t < \tau\}} + c_{t-\tau}^{(i)} 1_{\{x_\tau \in S_i\}} 1_{\{t \geq \tau\}}, \quad \text{for } x_\tau \in S_i.$$

Hence,

$$\begin{aligned}
V_{c^{(i)}}(x_\tau) &= V_{c^{(i)}}(x_\tau) - V_{c^{(i)}}(x^{(i)}) + V_{c^{(i)}}(x^{(i)}) \\
&\leq \Xi_\rho(x_\tau) + V_{c^{(i)}}(x^{(i)}) \\
&\leq \Xi_\rho(x_\tau) + \inf_{c \in \mathcal{A}_L} V_c(x^{(i)}) + \rho \\
&= \Xi_\rho(x_\tau) + v(x^{(i)}) + \rho \\
&\leq 2\Xi_\rho(x_\tau) + v(x_\tau) + \rho
\end{aligned}$$

Now, by the definition of $v^r(x)$, we can find $c \in \mathcal{A}_L$ such that

$$v^r(x) + \rho \geq E \left[\int_0^\tau e^{-\beta t} \{h(x_t) + |c_t|^2\} dt \right] + e^{-\beta \tau} v(x_\tau).$$

Thus, using the formal Markov property [6], we have

$$\begin{aligned}
&v^r(x) + \rho \\
&\geq \sum_i E \left[\int_0^\tau e^{-\beta t} \{h(x_t) + |c_t|^2\} dt + e^{-\beta \tau} (V_{c^{(i)}}(x_\tau) - 2\Xi_\rho(x_\tau) - \rho) : x_\tau \in S_i \right] \\
&= E \left[\int_0^\tau e^{-\beta t} \{h(x_t^\tau) + |c_t^\tau|^2\} dt + \int_\tau^\infty e^{-\beta t} \{h(x_t^\tau) + |c_t^\tau|^2\} dt | \mathcal{F}_\tau \right] \\
&\quad - 2E[e^{-\beta \tau} \Xi_\rho(x_\tau)] - \rho \\
&\geq v(x) - 2\Xi_\rho(x) - \rho,
\end{aligned}$$

where x_t^τ is the response to c_t^τ with $x_0^\tau = x_\tau$. Letting $\rho \rightarrow 0$, we deduce $v^r(x) \geq v(x)$, which completes the proof.

Theorem 2.4. *We assume (3), (4), (7) and (11). Then v_L is a viscosity solution of (5). Furthermore, v_L converges locally uniformly to a viscosity solution $u \in C(\mathbf{R})$ of (6) satisfying (4), (7) as $L \rightarrow \infty$.*

Proof. We note that (13) gives $E[\int_0^h |x_t|^2 dt] \leq e^{\beta_0 h} h f_2(x)$ for $h > 0$, and

$$\begin{aligned} & E \left[\sup_{0 \leq s \leq h} |x_s - x|^2 \right] \\ & \leq 3^2 \left(E \left[\left(\int_0^h |Ax_t| dt \right)^2 + \left(\int_0^h |c_t| dt \right)^2 + \left(\sup_{0 \leq s \leq h} \left| \int_0^s \sigma x_t dw_t \right| \right)^2 \right] \right) \\ & \leq 3^2 \left(|A|^2 h E \left[\int_0^h |x_t|^2 dt \right] + h^2 L^2 + C E \left[\int_0^h |x_t|^2 dt \right] \right). \end{aligned}$$

for constant $C > 0$. Hence we have

$$\lim_{h \rightarrow 0} \sup_{c \in \mathcal{A}_L} E \left[\sup_{0 \leq s \leq h} |x_s - x|^2 \right] = 0.$$

Thus we can apply a standard result of viscosity solutions [[4], Thm. 3.1, p. 220] to obtain the viscosity property of v_L , taking into account the uniform continuity of h on each compact interval. Since $v_L(x)$ is non-increasing, we can define $u(x)$ by $u(x) = \lim_{L \rightarrow \infty} v_L(x)$. By Theorem 2.3, it is clear that u satisfies (4), (7). Thus by Dini's theorem, we can observe the locally uniform convergence and the viscosity property of u [4]. The proof is complete.

3. CLASSICAL SOLUTIONS

We here study the smoothness of the viscosity solution u of (5).

Proposition 3.1. *We assume (3), (4), (7) and (11). Further we assume that $h(x) : \text{convex}$, then $v_L(x)$ and $u(x)$ are convex.*

Proof. For any $\epsilon > 0$, there exist $c, \hat{c} \in \mathcal{A}_L$ such that

$$\begin{aligned} E \left[\int_0^\infty e^{-\beta t} \{h(x_t) + |c_t|^2\} dt \right] & < v_L(x) + \epsilon, \\ E \left[\int_0^\infty e^{-\beta t} \{h(\hat{x}_t) + |\hat{c}_t|^2\} dt \right] & < v_L(\hat{x}) + \epsilon, \end{aligned}$$

where

$$\begin{aligned} dx_t &= [Ax_t + c_t]dt + \sigma x_t dw_t, \quad x_0 = x \in \mathbf{R}, \\ d\hat{x}_t &= [A\hat{x}_t + \hat{c}_t]dt + \sigma \hat{x}_t dw_t, \quad \hat{x}_0 = \hat{x} \in \mathbf{R}. \end{aligned}$$

We set

$$\begin{aligned}\tilde{c}_t &= \xi c_t + (1 - \xi)\hat{c}_t, \\ \tilde{x}_t &= \xi x_t + (1 - \xi)\hat{x}_t, \\ \tilde{x}_0 &= \xi x + (1 - \xi)\hat{x} \equiv \tilde{x},\end{aligned}$$

for $0 < \xi < 1$. Clearly,

$$d\tilde{x}_t = [A\tilde{x}_t + \tilde{c}_t]dt + \sigma\tilde{x}_tdw_t.$$

Hence, by convexity

$$\begin{aligned}v_L(\tilde{x}) &\leq E\left[\int_0^\infty e^{-\beta t}\{h(\tilde{x}_t) + |\tilde{c}_t|^2\}dt\right] \\ &\leq \xi E\left[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt\right] \\ &\quad + (1 - \xi)E\left[\int_0^\infty e^{-\beta t}\{h(\hat{x}_t) + |\hat{c}_t|^2\}dt\right] \\ &\leq \xi(v_L(x) + \epsilon) + (1 - \xi)(v_L(\hat{x}) + \epsilon).\end{aligned}$$

Letting $\epsilon \longrightarrow 0$, we get

$$v_L(\tilde{x}) = v_L(\xi x + (1 - \xi)\hat{x}) \leq \xi v_L(x) + (1 - \xi)v_L(\hat{x}),$$

which completes the convexity of $v_L(x)$. From the definition of $v_L(x)$, for each positive integer L , we have $0 \leq v_{L+1}(x) \leq v_L(x)$, $x \in \mathbf{R}$. Since $v_L(x)$ is non-increasing, we can define $u(x)$ by $u(x) = \lim_{L \rightarrow \infty} v_L(x)$. Hence we see that $u(x)$ is also convex.

Theorem 3.2. *We assume (3), (4), (7) and (11). Then we have*

$$u \in C^2(\mathbf{R} \setminus \{0\}). \quad (22)$$

Proof. Step 1: By the convexity of u we recall a classical result of Alexandrov [6] to see that Lebesgue measure of $\mathbf{R} \setminus \mathcal{D} \cup \{0\} = 0$, where $\mathcal{D} = \left\{x \in \mathbf{R} : u \text{ is twice differentiable at } x\right\}$. By the definition of twice-differentiability, we have $(u'(x), u''(x)) \in \mathbf{J}^{+2}u(x) \cap \mathbf{J}^{-2}u(x)$ for all $x \in \mathcal{D}$, and hence

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + A x u' - \frac{(u')^2}{4} + h(x) = 0, \quad \forall x \in \mathcal{D}.$$

Let $d^+u(x)$ and $d^-u(x)$ denote the right- and left-hand derivatives respectively. For all $x \in (\mathbf{R} \setminus \{0\})$, define $r^\pm(x)$ by

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 r^\pm(x) + A x d^\pm u(x) - \frac{(d^\pm u(x))^2}{4} + h(x) = 0. \quad (23)$$

Since $d^+u = d^-u = u'$ on \mathcal{D} , we have $r^+ = r^- = u''$ a.e. By definition, $d^+u(x)$ is right continuous, and so is $r^+(x)$. Hence it is easy to see that

$$\begin{aligned} u(y) - u(x) &= \int_x^y d^+u(s)ds \\ d^+u(s) - d^+u(x) &= \int_x^s r^+(t)dt, \quad s > x. \end{aligned}$$

Thus we get

$$\begin{aligned} R(u; y) &:= \left\{ u(y) - u(x) - d^+u(x)(y - x) - \frac{1}{2}r^+(x)|y - x|^2 \right\} / |y - x|^2 \\ &= \int_x^y \left(d^+u(s) - d^+u(x) - r^+(x)(s - x) \right) ds / |y - x|^2 \quad (24) \\ &= \int_x^y \left\{ \int_x^s \left(r^+(t) - r^+(x) \right) dt \right\} ds / |y - x|^2 \longrightarrow 0 \text{ as } y \downarrow x. \end{aligned}$$

Step 2: We claim that $u(x)$ is differentiable at $x \in \mathbf{R} \setminus \mathcal{D} \cup \{0\} = 0$.

It is well known in [2] that $\delta u(x) = \left[d^+u(x), d^-u(x) \right]$, for all $x \in (\mathbf{R} \setminus \{0\})$, where $\delta u(x)$ is the generalized gradient of u at x . Suppose $d^+u(x) > d^-u(x)$. We set

$$\begin{aligned} \hat{p} &= \xi d^+u(x) + (1 - \xi) d^-u(x) \\ \hat{r} &= \xi r^+(x) + (1 - \xi) r^-(x), \quad 0 < \xi < 1. \end{aligned}$$

If $\liminf_{y \rightarrow x} R(u; y) < 0$, then we can find a sequence $y_m \rightarrow x$ such that $\lim_{m \rightarrow \infty} R(u; y_m) < 0$. By (24), we may consider that $y_m \leq y_{m+1} < x$ for every m , taking a subsequence if necessary. Hence

$$\lim_{m \rightarrow \infty} \frac{u(y_m) - u(x) - d^+u(x)(y_m - x)}{|y_m - x|} \leq 0,$$

this leads to $d^+u(x) \leq d^-u(x)$, which is a contradiction. Thus we have $(d^+u(x), r^+(x)) \in J^{2,-}u(x)$ and similarly, $(d^-u(x), r^-(x)) \in J^{2,-}u(x)$. By the convexity of $J^{2,-}u(x)$, we get $(\hat{p}, \hat{r}) \in J^{2,-}u(x)$. Now we note that

$$(\hat{p})^2 < \xi(d^+u(x))^2 + (1 - \xi)(d^-u(x))^2,$$

and hence by (23)

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 \hat{r} + Ax\hat{p} - \frac{(\hat{p})^2}{4} + h(x) > 0.$$

On the other hand, by the definition of viscosity solution

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axp - \frac{p^2}{4} + h(x) \leq 0 \quad \forall (p, q) \in J^{2,-}u(x),$$

which is a contradiction. Therefore we deduce that $\delta u(x)$ is a singleton, and so u is differentiable at x [2].

Step 3: We claim that u' is continuous on $(\mathbf{R} \setminus \{0\})$. Let $x_m \rightarrow x$ and $p_m = u'(x_m) \rightarrow p$. Then we have by convexity $u(y) \geq u(x) + p(y - x)$, for all y . Hence we see that $p \in D^-u(x)$, where

$$D^-u(x) = \{p \in \mathbf{R} : \liminf_{y \rightarrow x} \{u(y) - u(x) - p(y - x)\}/|y - x| \geq 0\}.$$

Since $\delta u(x) = D^-u(x)$ and $\delta u(x)$ is a singleton, we deduce $p = u'(x)$ [[2], prop.4.7,p.66]. **Step 4:** We set $w = u'$. Since

$$-\beta w(x_m) + \frac{1}{2}\sigma^2 x_m^2 w'(x_m) + Ax_m w(x_m) - \frac{(w(x_m))^2}{4} + h(x_m) = 0,$$

where $x_m \in \mathbf{D}$, the sequence $\{w'(x_m)\}$ converges uniquely as $x_m \rightarrow x \in \mathbf{R} \setminus \mathcal{D} \cup \{0\}$, and w is Lipschitz near x by monotonicity. Hence, we have a well-known result in nonsmooth analysis that $\delta w(x)$ coincides with the convex hull of the set

$$\mathbf{D}^*w(x) = \left\{ q \in \mathbf{R} : q = \lim_{m \rightarrow \infty} w'(x_m), \quad x_m \in \mathcal{D} \rightarrow x \right\}.$$

Then

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axw(x) - \frac{(w'(x))^2}{4} + h(x) = 0 \quad \forall q \in \delta w(x).$$

Hence we observe that $\delta w(x)$ is a singleton, and then $w(x)$ is differentiable at x . The continuity of $w'(x)$ follows immediately. Thus we conclude that $w \in C^1(\mathbf{R} \setminus \{0\})$ and $(\mathbf{R} \setminus \mathcal{D} \cup \{0\})$ is empty. The proof is complete.

Theorem 3.3. *We assume (3), (4), (7) and (11). Further we assume that*

$$h(x)/x^2 \rightarrow \hat{h} \in \mathbf{R}_+ \text{ as } x \rightarrow 0. \quad (25)$$

Then we have

$$u \in C^1(\mathbf{R}) \cap C^2(\mathbf{R} \setminus \{0\}). \quad (26)$$

In addition, if $\hat{h} = 0$, then

$$u \in C^2(\mathbf{R}). \quad (27)$$

Proof. We first observe that v_L is a viscosity solution of the boundary value problem:

$$\begin{aligned} V'' + G(x, V, V') &= 0 \quad \text{in } (a, b) \\ V(a) &= v_L(a), \quad V(b) = v_L(b), \end{aligned} \quad (28)$$

for any interval $[a, b] \subset \mathbf{R} \setminus \{0\}$ where

$$G(x, V, V') = 2\{-\alpha V + AxV' + \min_{|r| \leq L}(|r|^2 + rV') + h(x)\}/\sigma^2 x^2 = 0.$$

Standard elliptic regularity theory Fleming and Soner [[6], Thm. 4.1] and the uniqueness of viscosity solutions Crandall, Ishii and Lions [4] yield that v_L is smooth in (a, b) . Thus

$$\begin{aligned} |\min_{|r| \leq L}(|r|^2 + rv_L')| &\leq |\min_{r \in \mathbf{R}}(|r|^2 + rv_L')| = (|v_L'|/2)^2 \\ &\leq \{(|v_L'|/2)^2 + 1\}. \end{aligned}$$

By the Theorem 3.2, we have $u \in C^2(\mathbf{R} \setminus \{0\})$.

To prove (26), it suffices to show that u has the following property:

$$u'(x) = o(1) \text{ as } x \rightarrow 0. \quad (29)$$

By (25), there exists $\lambda > 0$, for any $\varepsilon > 0$ such that $h(x) \leq (\hat{h} + \varepsilon)x^2$ for $|x| < \lambda$, and hence, by (4)

$$h(x) \leq (\hat{h} + \varepsilon)x^2 + C(1/\lambda^n + 1)|x|^n, \quad \forall x \in \mathbf{R}. \quad (30)$$

Note that $u(x) \leq E[\int_0^\infty e^{-\beta t} h(x_t^0) dt]$. Then we have by (13)

$$u'(x) = 0(x^2) \text{ as } x \rightarrow 0. \quad (31)$$

Now, by convexity

$$u(y) \geq u(x) + u'(x)(y - x), \quad x \neq 0.$$

Substituting $y = 2x$, and $y = 0$ we get $u(2x) \geq u(x) + u'(x)x$ and $u(x) - u'(x)x \leq u(0) = 0$ by (31). Hence

$$\frac{u(2x)}{x^2} \geq \frac{u'(x)}{x} \geq \frac{u(x)}{x^2}, \quad (32)$$

which implies (29).

Finally, suppose $\hat{h} = 0$. Then, by virtue of (30), we have $u(x) = o(x^2)$ as $x \rightarrow 0$. Moreover, by (32), $u'(x) = o(x)$ as $x \rightarrow 0$. Dividing (5) by x^2 and passing to the limit, we get $u''(0) = 0$, which implies (27).

4. AN APPLICATION TO QUADRATIC CONTROL THEORY

We shall study the quadratic control problem (1) over the class \mathcal{A}_{ad} of admissible controls, subject to (2), where

$\mathcal{A}_{ad} = \{c = (c_t) \in \mathcal{A} : \lim_{T \rightarrow \infty} E[e^{-\beta T} |x_T|^n] = 0 \text{ for the response } x_t \text{ to } c_t\}$. We consider the stochastic differential equation

$$dx_t^* = [Ax_t^* - u'(x_t^*)/2]dt + \sigma x_t^* dw_t, \quad x_0^* = x. \quad (33)$$

Theorem 4.1. *We assume (3), (4), (7), (11) and (25). Then the optimal control c_t^* is given by*

$$c_t^* = -u'(x_t^*)/2. \quad (34)$$

Proof. Since u' is continuous, (33) admits a weak solution x_t^* up to explosion time $\sigma = \inf\{t : |x_t^*| = \infty\}$. Taking into account $xu'(x) \geq 0$, we can show $(x_t^*)^2 \leq (x_t^0)^2$ by the comparison theorem. Hence $\sigma = \infty$. By the monotonicity of $u'(x)$, the uniqueness of (33) holds. Thus we conclude that (33) has a unique strong solution (x_t^*) .

It follows from (14) that

$$E[e^{-\beta T}(1 + |x_T^*|^n)] \leq e^{-(\beta-\beta_0)T} E[e^{-\beta_0 T} f_n(x_T^0)] \longrightarrow 0 \quad \text{as } T \rightarrow \infty,$$

where x_t^0 is a unique solution of (18). So $(c_t^*) \in \mathcal{A}_{ad}$. Since u satisfies (4), we see by (32) and (13) that

$$\begin{aligned} E\left[\int_0^T e^{-2\beta t} (x_t^* u'(x_t^*))^2 dt\right] &\leq E\left[\int_0^T e^{-2\beta t} u(2x_t^*)^2 dt\right] \\ &\leq CE\left[\int_0^T e^{-2\beta t} (1 + |x_t^*|^{2n}) dt\right] \\ &\leq CE\left[\int_0^T e^{-2\beta t} f_{2n}(x_t^0) dt\right] < \infty, \end{aligned}$$

and hence $\int_0^t e^{-\beta s} \sigma x_s^* u'(x_s^*) dw_s$ is a martingale. Then we apply Ito's formula for convex functions [7,p.219] to obtain

$$\begin{aligned} &E[e^{-\beta T} u(x_T^*)] \\ &= u(x) + E\left[\int_0^T e^{-\beta t} \left(-\beta u + Axu' + c_t^* u' + \frac{1}{2} \sigma^2 x^2 u''\right) \Big|_{x=x_t^*} dt\right] \\ &= u(x) - E\left[\int_0^T e^{-\beta t} \{h(x_t^*) + |c_t^*|^2\} dt\right]. \end{aligned}$$

Passing to the limit, we have $J(c^*) = u(x)$. By the same calculation as above, we can see that

$$E[e^{-\beta T \wedge \tau_n} u(x_{T \wedge \tau_n})] \geq u(x) - E\left[\int_0^{T \wedge \tau_n} e^{-\beta t} \{h(x_t) + |c_t|^2\} dt\right],$$

where $\{\tau_n\}$ is a sequence of localizing stopping times for the local martingale. Letting $\tau_n \longrightarrow \infty$ and then $T \longrightarrow \infty$, we obtain $u(x) \leq J(c)$ for all $c \in \mathcal{A}_{ad}$. The proof is complete.

General stochastic control problem: we can further study a stochastic control problem for linear degenerate systems to minimize the

discounted expected cost:

$$J(c) = E\left[\int_0^\infty e^{-\beta t} \{h(x_t) + |c_t|^n\} dt\right]$$

over $c \in \mathcal{A}$ subject to the degenerate stochastic differential equation (2) and a continuous function f on \mathbf{R} such that (4) and (7), in addition

$$k_0|x|^n - k_1 \leq h(x)$$

for some constants $k_0, k_1 > 0$ and for a fixed integer $n \geq 2$.

Acknowledgements: I would like to express my sincere gratitude to Professor A.B.M. Abdus Sobhan Miah for his advice, support and encouragement.

REFERENCES

- [1] T. M. Apostol. Mathematical Analysis, Addison-Wesley, (1974).
- [2] Bardi, M. and Capuzzo-Dolcetta, I. Optimal Control and Viscosity Solutions of Hamilton-Jacobi- Bellman Equations, Birkhäuser, Boston, (1997).
- [3] Bensoussan, A. Stochastic Control by Functional Analysis Methods, North-Holland, Amsterdam, (1982).
- [4] Crandall, M.G., Ishii, H. and Lions, P.L. User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. 27, (1982), 1-67.
- [5] Da Prato, G. Direct solution of a Riccati equation arising in stochastic control theory. Appl. Math. Optim. 11, (1984), 191-208.
- [6] Fleming, W.H. and Soner, H.M. Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New York, (1993).
- [7] S. Koike and M. Morimoto. On variational inequalities for leavable bounded-velocity control, Appl. Math. Optim., 18, (2003), pp.1-20.
- [8] J.L. Menaldi and M. Robin. On some cheap control problems for diffusions process, Trans. Amer. Math. Soc. vol. 278, (1983), pp. 771-802.
- [9] Nisio, M. Stochastic Control Theory, ISI Lecture Notes 9, MacMillan, (1981).

DEPARTMENT OF STATISTICS, SHAHJALAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, SYLHET-3114, BANGLADESH.

E-mail address: baten_math@yahoo.com

Received February 18, 2005