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APPROXIMATION FOR A SUMMATION-INTEGRAL TYPE LINK OPERATORS

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ABSTRACT. The present article deals with the general family of summation-integral type operators. Here, we propose the Durrmeyer variant of the generalized Lupaş operators considered by Abel and Ivan (General Math. 15 (1) (2007) 21-34) and study local approximation, Voronovskaja type formula, global approximation, Lipchitz type space and weighted approximation results. Also, we discuss the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation.

1. INTRODUCTION

In 2007, Abel and Ivan [1] proposed a general sequence of linear positive operators with $c = c_n > \beta$ for certain constant $\beta > 0$ as

$$\mathcal{L}_n^c(f;x) = \sum_{k=0}^{\infty} w_{n,k}^c \; f\left(\frac{k}{n}\right), \;\forall x \in [0,\infty), \tag{1.1}$$

where $w_{n,k}^c = \left(\frac{c}{1+c}\right)^{ncx} \frac{(ncx)_k}{k!(1+c)^k}$, and the Pochhammer symbol $(u)_k$ is defined as $(u)_k = u(u+1)(u+2)\cdots(u+k-1)$. It was seen that the operators $\mathcal{L}_n^c$ reproduce the linear functions. Note that the operators $\mathcal{L}_n^c$ are well-defined for all sufficiently large n, for $n > \frac{A}{\log(1+c)}$, the infinite sum in (1.1) is convergent, provided that

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$|f(t)| \leq Ke^{At}, t > 0$. They have studied the order of approximation and the complete asymptotic expansion of the operators (1.1). Gupta [13] defined the Durmeyer type modification of the operators (1.1) and discussed the some direct results. Gupta and Malik [16] also established quantitative asymptotic theorem and direct results by means of Ditzian-Totik moduli of smoothness of these operators. Very recently, Gupta et al. [14] considered the Durrmeyer variant of the Baskakov operators involving inverse Pòlya-Eggenberger distribution and studied the local and global approximation properties of these operators. Many researchers have defined the Durrmeyer variant of different sequence of linear positive operators and discussed their approximation behaviour(cf. [2–8, 10, 12, 15, 17–21, 23]).

Inspired by the above work, we consider a Durrmeyer type modification of the operators defined by (1.1) as follows:
For $\gamma > 0$ and $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma), \text{ as } t \to \infty\}$, we define

$$\mathcal{G}_{n,\alpha}^c(f;x) = \sum_{k=0}^{\infty} w_{n,k}^c \int_0^\infty b_{n,k}^\alpha f(t)dt, \tag{1.2}$$

where $\alpha > 0$, $b_{n,k}^\alpha = \dfrac{\alpha}{B\left(k+1, \frac{n}{\alpha}\right)} \dfrac{(\alpha t)^k}{(1+\alpha t)^{\frac{n}{\alpha}+k+1}}$, $w_{n,k}^c$ is given in (1.1) and $B(k+1, n)$ is the beta function defined by $B(x,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$.
We observed that for $f \in C_\gamma[0,\infty)$, the integral in the right hand side of (1.2) exists for all $n > \gamma$, and hence $\mathcal{G}_{n,\alpha}^c$ is well-defined.

The goal of the present paper is to discuss some direct results for the operators (1.2) e.g. local approximation, Voronovskaja type formula, global approximation, weighted approximation and present the rate of approximation for absolutely continuous functions having a derivative equivalent with a function of bounded variation.

2. Preliminary Results

Let $e_i(t) = t^i, i = \overline{0,4}$.

Lemma 2.1 ([13]). *For the operators $\mathcal{L}_n^c(f;x)$, we have*

(i) $\mathcal{L}_n^c(e_0;x) = 1$;
(ii) $\mathcal{L}_n^c(e_1;x) = x$;
(iii) $\mathcal{L}_n^c(e_2;x) = x^2 + \frac{(1+c)x}{nc}$;
(iv) $\mathcal{L}_n^c(e_3;x) = x^3 + \frac{3(1+c)x^2}{nc} + \frac{(1+c)x}{n^2c}\left[\frac{2(1-c)}{c} + 3\right]$;
(v) $\mathcal{L}_n^c(e_4;x) = x^4 + \frac{6(1+c)x^3}{nc} + \frac{x^2}{n^2}\left[\frac{11(1-c^2)}{c^2} + \frac{18(1+c)}{c}\right] + \frac{x}{n^3}\left[\frac{6(1+c^3)}{c^3} + \frac{12(1-c^2)}{c^2} + \frac{7(1+c)}{c}\right]$.

Using the definition of Gamma function, we obtain

$$\int_0^\infty b_{n,k}^\alpha t^i dt = \frac{\alpha^{-i}\Gamma(k+i+1)\Gamma\left(\frac{n}{\alpha}-i\right)}{\Gamma(k+1)\Gamma\left(\frac{n}{\alpha}\right)}. \tag{2.1}$$

Lemma 2.2. *For the operators $\mathcal{G}_{n,\alpha}^c(f;x)$, we have*

(i) $\mathcal{G}_{n,\alpha}^c(e_0;x) = 1;$

(ii) $\mathcal{G}_{n,\alpha}^c(e_1;x) = \frac{nx+1}{(n-\alpha)};$

(iii) $\mathcal{G}_{n,\alpha}^c(e_2;x) = \frac{n^2x^2}{(n-2\alpha)(n-\alpha)} + \frac{(1+4c)nx}{c(n-2\alpha)(n-\alpha)} + \frac{2}{(n-2\alpha)(n-\alpha)};$

(iv) $\mathcal{G}_{n,\alpha}^c(e_3;x) = \frac{n^3x^3}{(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{3(1+3c)n^2x^2}{c(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{(2+9c(1+2c))nx}{c^2(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{6}{(n-3\alpha)(n-2\alpha)(n-\alpha)};$

(v) $\mathcal{G}_{n,\alpha}^c(e_4;x) = \frac{n^4x^4}{(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{2(3+8c)n^3x^3}{c(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{(11+24c(2+3c))n^2x^2}{c^2(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{2(3+4c(4+3c(3+4c)))nx}{c^3(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{24}{(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)}.$

Proof. The lemma follows easily using the relation (2.1) and Lemma 2.1. Hence the details are omitted. □

Lemma 2.3. *For $m = 0, 1, 2, \cdots$, the m^{th} order central moments of $\mathcal{G}_{n,\alpha}^c$ defined as*
$\Theta_{n,\alpha,m}^c(x) = \mathcal{G}_{n,\alpha}^c((t-x)^m;x)$ and Lemma 2.2, we have

(i) $\Theta_{n,\alpha,1}^c(x) = \frac{x\alpha+1}{(n-\alpha)};$

(ii) $\Theta_{n,\alpha,2}^c(x) = \frac{x^2\alpha(n+2\alpha)}{(n-2\alpha)(n-\alpha)} + \frac{x(n+2c(n+2\alpha))}{c(n-2\alpha)(n-\alpha)} + \frac{2}{(n-2\alpha)(n-\alpha)};$

(iii) $\Theta_{n,\alpha,4}^c(x) = \frac{x^4\alpha^2(3n^2+46n\alpha+24\alpha^2)}{(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{2x^3\alpha\left(3(1+2c)n^2+4(9+23c)n\alpha+48c\alpha^2\right)}{c(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{x^2\left(3n^2(1+2c)^2+4(8+36c+51c^2)n\alpha+144c^2\alpha^2\right)}{c^2(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{x\left((6+8c(4+9c(1+c)))n+96c^3\alpha\right)}{c^3(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)} + \frac{24}{(n-4\alpha)(n-3\alpha)(n-2\alpha)(n-\alpha)}.$

Remark 2.1. *Applying Lemma 2.3, we get*

$$\begin{aligned}
\lim_{n\to\infty} n\ \Theta_{n,\alpha,1}^c(x) &= x\alpha+1;\\
\lim_{n\to\infty} n\ \Theta_{n,\alpha,2}^c(x) &= x^2\alpha + \frac{(1+2c)x}{c};\\
\lim_{n\to\infty} n^2\ \Theta_{n,\alpha,4}^c(x) &= 3x^4\alpha^2 + \frac{6(1+2c)x^3\alpha}{c} + \frac{3(1+2c)^2x^2}{c^2}.
\end{aligned}$$

3. Direct Results

Theorem 3.1. *Let $f \in C_\gamma[0,\infty)$. Then $\lim_{n\to\infty} \mathcal{G}_{n,\alpha}^c(f;x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. In view of Lemma 2.2, we get

$$\lim_{n\to\infty} \mathcal{G}_{n,\alpha}^c(e_i;x) = x^i, i = 0,1,2\cdots,$$

uniformly in each compact subset of $[0,\infty)$. Applying Bohman-Korovkin Theorem, it follows that $\lim_{n\to\infty}\mathcal{G}^c_{n,\alpha}(f;x)=f(x)$, uniformly in each compact subset of $[0,\infty)$. ∎

3.1. **Voronovskaja type theorem.** In this section we prove Voronvoskaja type asymptotic theorem for the operators $\mathcal{G}^c_{n,\alpha}$.

Theorem 3.2. *Let $f\in C_\gamma[0,\infty)$. If f'' exists at a point $x\in[0,\infty)$, then we have*

$$\lim_{n\to\infty} n\left[\mathcal{G}^c_{n,\alpha}(f;x)-f(x)\right]=(x\alpha+1)f'(x)+\frac{1}{2}\left[x^2\alpha+\frac{(1+2c)x}{c}\right]f''(x).$$

Proof. Applying Taylor's expansion, we can write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\xi(t,x)(t-x)^2, \tag{3.1}$$

where $\lim_{t\to x}\xi(t,x)=0$. By using the linearity of the operator $\mathcal{G}^c_{n,\alpha}$, we get

$$\mathcal{G}^c_{n,\alpha}(f;x)-f(x)=\mathcal{G}^c_{n,\alpha}((t-x);x)f'(x)+\frac{1}{2}\mathcal{G}^c_{n,\alpha}((t-x)^2;x)f''(x)$$
$$+\,\mathcal{G}^c_{n,\alpha}(\xi(t,x)(t-x)^2;x).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$n\mathcal{G}^c_{n,\alpha}(\xi(t,x)(t-x)^2;x)\le\sqrt{\mathcal{G}^c_{n,\alpha}(\xi^2(t,x);x)}\sqrt{n^2\mathcal{G}^c_{n,\alpha}((t-x)^4;x)}.$$

In view of Theorem 3.1, $\lim_{n\to\infty}\mathcal{G}^c_{n,\alpha}(\xi^2(t,x);x)=\xi^2(x,x)=0$, since $\xi(t,x)\to 0$ as $t\to x$, and using Remark 2.1 for every $x\in[0,\infty)$, we obtain

$$\lim_{n\to\infty} n^2\mathcal{G}^c_{n,\alpha}\left((t-x)^4;x\right)=3x^4\alpha^2+\frac{6(1+2c)x^3\alpha}{c}+\frac{3(1+2c)^2x^2}{c^2}. \tag{3.2}$$

Hence,

$$\lim_{n\to\infty} n\mathcal{G}^c_{n,\alpha}(\xi(t,x)(t-x)^2;x)=0.$$

From Remark 2.1, we have

$$\lim_{n\to\infty} n\mathcal{G}^c_{n,\alpha}(t-x;x)=x\alpha+1,$$
$$\lim_{n\to\infty} n\mathcal{G}^c_{n,\alpha}\left((t-x)^2;x\right)=x^2\alpha+\frac{(1+2c)x}{c}. \tag{3.3}$$

Collecting the results from above the theorem is proved. ∎

3.2. **Local approximation.** Let $\tilde{C_B}[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm

$$||f||_{\tilde{C_B}[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|.$$

For $f\in\tilde{C_B}[0,\infty)$, the Steklov mean is defined as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+u+v)-f(x+2(u+v))\right]du\,dv. \tag{3.4}$$

By simple estimation, it is note that

a) $\|f_h - f\|_{\tilde{C_B}[0,\infty)} \le \omega_2(f,h)$.

b) $f_h', f_h'' \in \tilde{C_B}[0,\infty)$ and $\|f_h'\|_{\tilde{C_B}[0,\infty)} \le \frac{5}{h}\omega(f,h), \quad \|f_h''\|_{\tilde{C_B}[0,\infty)} \le \frac{9}{h^2}\omega_2(f,h)$,

where the second order modulus of smoothness is given by

$$\omega_2(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} |f(x+2u) - 2f(x+u+v) + f(x+2v)|,\ \delta > 0.$$

The usual modulus of continuity of $f \in \tilde{C_B}[0,\infty)$ is defined as

$$\omega(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} |f(x+u) - f(x+v)|.$$

Theorem 3.3. *Let $f \in \tilde{C_B}[0,\infty)$. Then for every $x \ge 0$, the following inequality holds*

$$\left|\mathcal{G}^c_{n,\alpha}(f;x) - f(x)\right| \le 5\omega\left(f, \sqrt{\Theta^c_{n,\alpha,2}(x)}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{\Theta^c_{n,\alpha,2}(x)}\right).$$

Proof. For $x \ge 0$, and using the Steklov mean f_h that is given by (3.4), we can write

$$\left|\mathcal{G}^c_{n,\alpha}(f;x) - f(x)\right| \le \mathcal{G}^c_{n,\alpha}(|f - f_h|;x) + |\mathcal{G}^c_{n,\alpha}(f_h - f_h(x);x)| + |f_h(x) - f(x)|. \quad (3.5)$$

For every $f \in \tilde{\mathrm{C}}_B[0,\infty)$ and (1.2), we get

$$\left|\mathcal{G}^c_{n,\alpha}(f;x)\right| \le ||f||_{\tilde{C_B}[0,\infty)}. \quad (3.6)$$

By property (a) of Steklov mean and (3.6), we obtain

$$\mathcal{G}^c_{n,\alpha}(|f - f_h|;x) \le \|\mathcal{G}^c_{n,\alpha}(f - f_h)\|_{\tilde{C_B}[0,\infty)} \le \|f - f_h\|_{\tilde{C_B}[0,\infty)} \le \omega_2(f,h).$$

Using Taylor's expansion and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{G}^c_{n,\alpha}(f_h - f_h(x);x)\right| &\le \|f_h'\|_{\tilde{C_B}[0,\infty)}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^2;x)} \\ &\quad + \frac{1}{2}\|f_h''\|_{\tilde{C_B}[0,\infty)}\mathcal{G}^c_{n,\alpha}\left((t-x)^2;x\right).\end{aligned}$$

By Lemma 2.3 and property (b) of Steklov mean, we have

$$\left|\mathcal{G}^c_{n,\alpha}(f_h - f_h(x);x)\right| \le \frac{5}{h}\omega(f,h)\sqrt{\Theta^c_{n,\alpha,2}(x)} + \frac{9}{2h^2}\omega_2(f,h)\Theta^c_{n,\alpha,2}(x).$$

Now, taking $h = \sqrt{\Theta^c_{n,\alpha,2}(x)}$, and substituting the values in (3.5), we get the desired results. □

Theorem 3.4. *For any $f \in \tilde{C^1_B}[0,\infty)$ and $x \in [0,\infty)$, we have*

$$\left|\mathcal{G}^c_{n,\alpha}(f;x) - f(x)\right| \le \left|\Theta^c_{n,\alpha,1}(x)\right| |f'(x)| + 2\sqrt{\Theta^c_{n,\alpha,2}(x)}\ \omega\left(f', \sqrt{\Theta^c_{n,\alpha,2}(x)}\ \right). \quad (3.7)$$

Proof. Let $f \in \tilde{C^1_B}[0,\infty)$. For any $t \in [0,\infty), x \in [0,\infty)$, we have

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (f'(u) - f'(x))\,du.$$

Applying $\mathcal{G}^c_{n,\alpha}(\cdot;x)$ on both sides of the above relation, we get

$$\mathcal{G}^c_{n,\alpha}(f(t) - f(x);x) = f'(x)\mathcal{G}^c_{n,\alpha}(t-x;x) + \mathcal{G}^c_{n,\alpha}\left(\int_x^t (f'(u) - f'(x))\,du;x\right).$$

Using the well known property of modulus of continuity

$$|f(t)-f(x)| \leq \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right), \delta>0,$$

we obtain

$$\left|\int_x^t (f'(u)-f'(x))\,du\right| \leq \omega(f',\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

Therefore, it follows

$$\begin{aligned}\left|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)\right| &\leq |f'(x)|\ |\mathcal{G}^c_{n,\alpha}(t-x;x)| \\ &\quad + \omega(f',\delta)\left\{\frac{1}{\delta}\mathcal{G}^c_{n,\alpha}((t-x)^2;x)+\mathcal{G}^c_{n,\alpha}(|t-x|;x)\right\}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)\right| &\leq |f'(x)|\ |\mathcal{G}^c_{n,\alpha}(t-x;x)| \\ &\quad + \omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^2;x)}+1\right\}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^2;x)}.\end{aligned}$$

Choosing $\delta=\sqrt{\Theta^c_{n,\alpha,2}(x)}$, the required result follows. □

Let $d_1 \geq 0, d_2 > 0$ be fixed. We define the following Lipschitz-type space (see [22]):

$$Lip_M^{(d_1,d_2)}(r) := \left\{f \in C[0,\infty) : |f(t)-f(x)| \leq M\frac{|t-x|^r}{(t+d_1x^2+d_2x)^{\frac{r}{2}}};\ x,t \in (0,\infty)\right\},$$

where $r \in (0,1]$.

Theorem 3.5. *Let $f \in Lip_M^{(d_1,d_2)}(r)$ and $r \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$|\ \mathcal{G}^c_{n,\alpha}(f;x)-f(x)\ | \leq M\left(\frac{\Theta^c_{n,\alpha,2}(x)}{d_1x^2+d_2x}\right)^{\frac{r}{2}}.$$

Proof. By the Hölder's inequality with $p=\frac{2}{r}, q=\frac{2}{2-r}$, we get

$$\begin{aligned}
\mid \mathcal{G}_{n,\alpha}^{c}(f;x)-f(x)\mid &\leq \sum_{k=0}^{\infty} w_{n,k}^{c}(x)\int_{0}^{\infty} b_{n,k}^{\alpha}(t)|f(t)-f(x)|dt\\
&\leq \sum_{k=0}^{\infty} w_{n,k}^{c}(x)\left(\int_{0}^{\infty} b_{n,k}^{\alpha}(t)|f(t)-f(x)|^{\frac{2}{r}}dt\right)^{\frac{r}{2}}\\
&\leq \left\{\sum_{k=0}^{\infty} w_{n,k}^{c}(x)\int_{0}^{\infty} b_{n,k}^{\alpha}(t)|f(t)-f(x)|^{\frac{2}{r}}dt\right\}^{\frac{r}{2}}\left(\sum_{k=0}^{\infty} w_{n,k}^{c}(x)\right)^{\frac{2-r}{2}}\\
&= \left\{\sum_{k=0}^{\infty} w_{n,k}^{c}(x)\int_{0}^{\infty} b_{n,k}^{\alpha}(t)|f(t)-f(x)|^{\frac{2}{r}}dt\right\}^{\frac{r}{2}}\\
&\leq M\left(\sum_{k=0}^{\infty} w_{n,k}^{c}(x)\int_{0}^{\infty} b_{n,k}^{\alpha}(t)\frac{(t-x)^2}{(t+d_1x^2+d_2x)}dt\right)^{\frac{r}{2}}\\
&\leq \frac{M}{(d_1x^2+d_2x)^{\frac{r}{2}}}\left(\sum_{k=0}^{\infty} w_{n,k}^{c}(x)\int_{0}^{\infty} b_{n,k}^{\alpha}(t)(t-x)^2dt\right)^{\frac{r}{2}}\\
&= \frac{M}{(d_1x^2+d_2x)^{\frac{r}{2}}}\left(\mathcal{G}_{n,\alpha}^{c}((t-x)^2;x)\right)^{\frac{r}{2}}\\
&= \frac{M}{(d_1x^2+d_2x)^{\frac{r}{2}}}(\Theta_{n,\alpha,2}^{c}(x))^{\frac{r}{2}}.
\end{aligned}$$

Thus, the proof is completed. □

3.3. **Global approximation.** In this section, the first and the second order Ditzian-Totik moduli of smoothness are defined as

$$\overline{\omega}_{\phi}(f,\delta)=\sup_{0<|h|\leq\delta}\ \sup_{x,x+h\phi(x)\in[0,\infty)}|f(x+\phi(x)h)-f(x)| \tag{3.8}$$

and

$$\omega_{2,\phi}(f,\sqrt{\delta})=\sup_{0<|h|\leq\sqrt{\delta}}\ \sup_{x,x\pm h\phi(x)\in[0,\infty)}\left|f(x+h\phi(x))-2f(x)+f(x-h\phi(x))\right|,$$

respectively and the corresponding K-functional is

$$K_{2,\phi}(f,\delta)=\inf\{||f-g||+\delta||\phi^2g''||:g\in W^2(\phi)\},\ \ \delta>0,$$

where $W^2(\phi)=\{g\in\tilde{C_B}[0,\infty):g'\in AC[0,\infty),\phi^2g''\in\tilde{C_B}[0,\infty)\}$ and $g'\in AC[0,\infty)$ means that g' is absolutely continuous on $[0,\infty)$. It is well known that (see [9]) $K_{2,\phi}(f,\delta)\sim\omega_{2,\phi}(f,\sqrt{\delta})$ which means that there exists an absolute constant $M>0$ such that

$$M^{-1}\omega_{2,\phi}(f,\sqrt{\delta})\leq K_{2,\phi}(f,\delta)\leq M\omega_{2,\phi}(f,\sqrt{\delta}). \tag{3.9}$$

In the following we will consider $\phi(x)=1+x^2$.

Theorem 3.6. *Let $f \in \tilde{C_B}[0,\infty)$ and $x \in (0,\infty)$. Then, there is an absolute constant $M>0$ such that*

$$|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)| \;\leq\; 4K_{2,\phi}\left(f,\frac{M}{2n}\right)+\overline{\omega}_\phi\left(f,\frac{\sqrt{M}}{n}\right),$$

for n sufficiently large.

Proof. Define

$$\mathcal{T}^c_{n,\alpha}(f;x)=\mathcal{G}^c_{n,\alpha}(f;x)-f\left(\frac{nx+1}{n-\alpha}\right)+f(x).$$

Let $g\in W^2(\phi)$. Applying Taylor's expansion, we may write

$$g(t)=g(x)+g'(x)(t-x)+\int_0^t g''(u)(t-u)du.$$

Operating the operators $\mathcal{T}^c_{n,\alpha}$ to the above equation, we get

$$\begin{aligned}
&|\mathcal{T}^c_{n,\alpha}(g;x)-g(x)|\\
&\leq \mathcal{G}^c_{n,\alpha}\left(\left|\int_x^t |t-u|\;|g''(u)|du\right|;x\right)+\left|\int_x^{\frac{nx+1}{n-\alpha}}\left|\frac{nx+1}{n-\alpha}-u\right||g''(u)|\,du\right|\\
&\leq \frac{\|\phi^2g''\|}{\phi^2(x)}\left[\mathcal{G}^c_{n,\alpha}\left((t-x)^2;x\right)+\left|\int_x^{\frac{nx+1}{n-\alpha}}\left|\frac{nx+1}{n-\alpha}-u\right|du\right|\right]\\
&\leq \frac{\|\phi^2g''\|}{\phi^2(x)}\left[\Theta^c_{n,\alpha,2}(x)+\left(\Theta^c_{n,\alpha,1}(x)\right)^2\right]
\end{aligned}$$

In view of Remark 2.1 it follows that there is a positive constant $M>0$ such that

$$\frac{\Theta^c_{n,\alpha,2}(x)}{\phi^2(x)}\leq\frac{M}{n},\;\frac{\left(\Theta^c_{n,\alpha,1}(x)\right)^2}{\phi^2(x)}\leq\frac{M}{n^2},\;\forall x\in[0,\infty).$$

Thus,

$$|\mathcal{T}^c_{n,\alpha}(g;x)-g(x)|\leq M\|\phi^2g''\|\left[\frac{1}{n}+\frac{1}{n^2}\right]\leq\frac{2M}{n}\|\phi^2g''\|.$$

Now,

$$\begin{aligned}
|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)| \;&\leq\; |\mathcal{T}^c_{n,\alpha}(f-g;x)|+|\mathcal{T}^c_{n,\alpha}(g;x)-g(x)|+|f(x)-g(x)|\\
&\quad+\left|f\left(\frac{nx+1}{n-\alpha}\right)-f(x)\right|\\
&\leq\; 4||f-g||+\frac{2M}{n}\|\phi^2g''\|+\left|f\left(\frac{nx+1}{n-\alpha}\right)-f(x)\right|. \quad (3.10)
\end{aligned}$$

Also, we obtain

$$\begin{aligned}\left|f\left(\frac{nx+1}{n-\alpha}\right)-f(x)\right| &= \left|f\left(x+\phi(x)\frac{\left(\frac{nx+1}{n-\alpha}\right)-x}{\phi(x)}\right)-f(x)\right| \\ &\leq \sup\left|f\left(x+\phi(x)\frac{\Theta^c_{n,\alpha,1}(x)}{\phi(x)}\right)-f(x)\right| \\ &\leq \overline{\omega}_\phi\left(f,\frac{\sqrt{M}}{n}\right). \qquad (3.11)\end{aligned}$$

Using (3.10) and (3.11), we have

$$|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)|\leq 4\left[||f-g||+\frac{M}{2n}\|\phi^2 g''\|\right]+\overline{\omega}_\phi\left(f,\frac{\sqrt{M}}{n}\right).$$

Now, applying (3.9), the theorem is completed. □

3.4. **Weighted approximation.** Let $D_\varrho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)|\leq M_f\varrho(x)$, where M_f is a positive constant depending only on f and $\varrho(x)=1+x^2$ is a weight function. Let $C_\varrho[0,\infty)$ be the space of all continuous functions in $D_\varrho[0,\infty)$ endowed with the norm

$$\|f\|_\varrho:=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\varrho(x)}$$

and

$$C^*_\varrho[0,\infty):=\left\{f\in C_\varrho[0,\infty):\lim_{x\to\infty}\frac{|f(x)|}{\varrho(x)}\quad\text{exists and is finite}\right\}.$$

Theorem 3.7. *Let $f\in C^*_\varrho[0,\infty)$. Then, we have*

$$\lim_{n\to\infty}\|\mathcal{G}^c_{n,\alpha}(f)-f\|_\varrho=0. \qquad (3.12)$$

Proof. In order to prove this result it is sufficient to verify the following three relations (see [11])

$$\lim_{n\to\infty}\|\mathcal{G}^c_{n,\alpha}(e_i;\cdot)-e_i\|_\varrho=0,\ \ i=0,1,2. \qquad (3.13)$$

Since $\mathcal{G}^c_{n,\alpha}(e_0;x)=1$, the condition in (3.13) holds true for $i=0$.
By Lemma 2.2, we have

$$\|\mathcal{G}^c_{n,\alpha}(e_1;\cdot)-e_1\|_\varrho=\sup_{x\geq0}\frac{1}{1+x^2}\left|\frac{nx+1}{n-\alpha}-x\right|$$

$$=\sup_{x\geq0}\left(\frac{x}{1+x^2}\right)\left|\frac{\alpha}{n-\alpha}\right|+\sup_{x\geq0}\left(\frac{1}{1+x^2}\right)\frac{1}{n-\alpha}\leq\left|\frac{\alpha+1}{n-\alpha}\right|. \qquad (3.14)$$

Thus, $\lim_{n\to\infty}\|\mathcal{G}^c_{n,\alpha}(e_1;\cdot)-e_1\|_\varrho=0$. Finally, we obtain

$$\|\mathcal{G}^c_{n,\alpha}(e_2;\cdot)-e_2\|_\varrho$$

$$\begin{aligned} &= \sup_{x\geq 0}\frac{1}{1+x^2}\left|\frac{n^2x^2}{(n-2\alpha)(n-\alpha)}+\frac{(1+4c)nx}{c(n-2\alpha)(n-\alpha)}+\frac{2}{(n-2\alpha)(n-\alpha)}-x^2\right| \\ &\leq \sup_{x\geq 0}\frac{x^2}{1+x^2}\frac{|3n\alpha-2\alpha|\alpha}{|(n-\alpha)(n-2\alpha)|}+\sup_{x\geq 0}\frac{x}{1+x^2}\frac{n(1+4c)}{|(n-\alpha)(n-2\alpha)|} \\ &\quad +\sup_{x\geq 0}\frac{1}{1+x^2}\frac{2}{|(n-\alpha)(n-2\alpha)|}, \end{aligned} \tag{3.15}$$

which implies that $\lim_{n\to\infty}\|\mathcal{G}^c_{n,\alpha}(e_2;\cdot)-e_2\|_\varrho=0$. □

Let $f\in C^*_\varrho[0,\infty)$. We will define the weighted modulus of continuity introduced by Yüksel and Ispir [24] as follows:

$$\Omega(f;\delta)=\sup_{x\in[0,\infty),0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Lemma 3.1 ([24]). *Let $f\in C^*_\varrho[0,\infty)$, then:*

i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*
ii) $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$;
iii) *for each $m\in\mathbb{N},\Omega(f,m\delta)\leq m\Omega(f;\delta)$;*
iv) *for each $\lambda\in[0,\infty),\Omega(f;\lambda\delta)\leq(1+\lambda)\Omega(f;\delta)$.*

Theorem 3.8. *Let $f\in C^*_\varrho[0,\infty)$. Then there exists $\tilde{n}\in\mathbb{N}$ and a positive constant $Q(\alpha,c)$ depending on α and c such that*

$$\sup_{x\in(0,\infty)}\frac{|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq Q(\alpha,c)\Omega\left(f;n^{-1/2}\right),\ \textit{for } n>\tilde{n}. \tag{3.16}$$

Proof. For $t>0,x\in(0,\infty)$ and $\delta>0$, by the definition of $\Omega(f;\delta)$ and Lemma 3.1, we may write

$$\begin{aligned} |f(t)-f(x)| &\leq (1+(x+|x-t|)^2)\Omega(f;|t-x|) \\ &\leq 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Since $\mathcal{G}^c_{n,\alpha}$ is positive and linear, we get

$$\begin{aligned} |\mathcal{G}^c_{n,\alpha}(f;x)-f(x)| &\leq 2(1+x^2)\Omega(f;\delta)\left\{1+\mathcal{G}^c_{n,\alpha}((t-x)^2;x)\right. \\ &\quad \left.+\mathcal{G}^c_{n,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right\}. \end{aligned} \tag{3.17}$$

In view of (3.3) it follows that there is $n_1\in\mathbb{N}$ such that

$$\mathcal{G}^c_{n,\alpha}((t-x)^2;x)\leq Q_1(\alpha,c)\frac{(1+x^2)}{n},\ \text{for } n>n_1, \tag{3.18}$$

where $Q_1(\alpha,c)$ is a positive constant depending on α and c. Using Cauchy-Schwarz inequality, we may write

$$\mathcal{G}^c_{n,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) \le \frac{1}{\delta}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^2;x)} + \frac{1}{\delta}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^4;x)}\sqrt{\mathcal{G}^c_{n,\alpha}((t-x)^2;x)}. \tag{3.19}$$

From the relation (3.2) it follows that there is $n_2 \in \mathbb{N}$ such that

$$\sqrt{\mathcal{G}^c_{n,\alpha}\left((t-x)^4;x\right)} \le Q_2(\alpha,c)\frac{(1+x^2)}{n}, \text{ for } n > n_2, \tag{3.20}$$

where $Q_2(\alpha,c)$ is a positive constant depending on α and c.

Let $\tilde{n} = \max\{n_1,n_2\}$. Collecting the estimates (3.17)-(3.20) and taking

$$Q(\alpha,c) = 2\left(1+Q_1(\alpha,c)+\sqrt{Q_1(\alpha,c)}+Q_2(\alpha,c)\sqrt{Q_1(\alpha,c)}\right), \ \delta = \frac{1}{\sqrt{n}},$$

for $n > \tilde{n}$, we get (3.16). □

3.5. **Rate of approximation of $\mathcal{G}^c_{n,\alpha}$ operators for functions with derivatives of bounded variation.** In this section we study the rate of convergence of functions with a derivative of bounded variation.

Let $DBV[0,\infty)$ be the class of all functions $f \in D_\varrho[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. The function $f \in DBV[0,\infty)$ has the following representation

$$f(x) = \int_0^x g(t) + f(0),$$

where g is a function of bounded variation on each finite subinterval of $[0,\infty)$.

In order to discuss the approximation of the operators $\mathcal{G}^c_{n,\alpha}$ for functions having a derivative of bounded variation, we rewrite the operators (1.2) as follows:

$$\mathcal{G}^c_{n,\alpha}(f;x) = \int_0^\infty \mathcal{S}^c_{n,\alpha}(x,t)f(t)dt, \tag{3.21}$$

$$\mathcal{S}^c_{n,\alpha}(x,t) = \sum_{k=0}^\infty w^c_{n,k}b^\alpha_{n,k}.$$

Lemma 3.2. *For all $x \in (0,\infty)$ and sufficiently large n, we have*

i) $\zeta^c_{n,\alpha}(x,t) = \int_0^t \mathcal{S}^c_{n,\alpha}(x,u)du \le \frac{Q(\alpha,c)}{(x-t)^2}\frac{1+x^2}{n}$, $0 \le t < x$,

ii) $1-\zeta^c_{n,\alpha}(x,z) = \int_t^\infty \mathcal{S}^c_{n,\alpha}(x,u)du \le \frac{Q(\alpha,c)}{(z-x)^2}\frac{1+x^2}{n}$, $z < t < \infty$,

where $Q(\alpha,c)$ is a positive constant depending on α and c.

Proof. For sufficiently large n it follows from (3.3) that

$$\mathcal{G}^c_{n,\alpha}((u-x)^2;x) < Q(\alpha,c)\frac{1+x^2}{n}. \tag{3.22}$$

Applying Lemma 2.3, we get

$$\zeta^c_{n,\alpha}(x,t)=\int_0^t \mathcal{S}^c_{n,\alpha}(x,u)du \leq \int_0^t \left(\frac{x-u}{x-t}\right)^2 \mathcal{S}^c_{n,\alpha}(x,u)du$$
$$\leq \frac{1}{(x-t)^2}\mathcal{G}^c_{n,\alpha}\left((u-x)^2;x\right) \leq \frac{Q(\alpha,c)}{(x-t)^2}\frac{1+x^2}{n}.$$

The proof of ii) is similar hence the details are omitted. □

Theorem 3.9. *Let $f \in DBV[0,\infty)$. Then, for every $x\in(0,\infty)$ and sufficiently large n, we have*

$$|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)| \leq \frac{(\alpha x+1)}{(n-\alpha)}\left|\frac{f'(x+)+f'(x-))}{2}\right|$$

$$+\sqrt{Q(\alpha,c)\frac{1+x^2}{n}}\left|\frac{f'(x+)-f'(x-))}{2}\right|$$
$$+Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right)+\left(4M_f+\frac{M_f+|f(x)|}{x^2}\right)Q(\alpha,c)\frac{1+x^2}{n}$$
$$+|f'(x+)|\sqrt{Q(\alpha,c)\frac{1+x^2}{n}}+Q(\alpha,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x+Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}} f'_x,$$

where $Q(\alpha,c)$ is a positive constant depending on α and c, $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\leq t<x,\\ 0, & t=x,\\ f'(t)-f'(x+), & x<t<\infty.\end{cases} \tag{3.23}$$

Proof. For any $f\in DBV[0,\infty)$, from (3.23) we can write

$$f'(u)=\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)sgn(u-x)$$
$$+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right), \tag{3.24}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x,\\ 0, & u\neq x.\end{cases}$$

Since $\mathcal{G}_{n,\alpha}^{c}(e_0;x)=1$, using (3.21), for every $x\in(0,\infty)$ we get

$$\mathcal{G}_{n,\alpha}^{c}(f;x)-f(x)=\int_0^\infty \mathcal{S}_{n,\alpha}^{c}(x,t)(f(t)-f(x))dt \tag{3.25}$$
$$=\int_0^\infty \mathcal{S}_{n,\alpha}^{c}(x,t)\int_x^t f'(u)dudt$$
$$=-\int_0^x\left(\int_t^x f'(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$+\int_x^\infty\left(\int_x^t f'(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt.$$

Denote

$$I_1:=\int_0^x\left(\int_t^x f'(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt,\quad I_2:=\int_x^\infty\left(\int_x^t f'(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt.$$

Since $\int_x^t \delta_x(u)du=0$, and using relation (3.24), we get

$$I_1=\int_0^x\left\{\int_t^x \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)\right.$$
$$\left.+\frac{1}{2}\left(f'(x+)-f'(x-)\right)sgn(u-x)du\right\}\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$

$$=\frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^x(x-t)\mathcal{S}_{n,\alpha}^{c}(x,t)dt+\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$-\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^x(x-t)\mathcal{S}_{n,\alpha}^{c}(x,t)dt. \tag{3.26}$$

In a similar manner we obtain

$$I_2=\int_x^\infty\left\{\int_x^t \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)\right.$$
$$\left.+\frac{1}{2}\left(f'(x+)-f'(x-)\right)sgn(u-x)du\right\}\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$=\frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_x^\infty(t-x)\mathcal{S}_{n,\alpha}^{c}(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_x^\infty(t-x)\mathcal{S}_{n,\alpha}^{c}(x,t)dt. \tag{3.27}$$

Combining the relations (3.25)-(3.27), we get

$$\mathcal{G}_{n,\alpha}^{c}(f;x)-f(x)=\frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^\infty(t-x)\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^\infty|t-x|\mathcal{S}_{n,\alpha}^{c}(x,t)dt$$
$$-\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt.$$

Therefore,

$$\begin{aligned}\left|\mathcal{G}^c_{n,\alpha}(f;x)-f(x)\right| &= \left|\frac{f'(x+)+f'(x-)}{2}\right|\left|\mathcal{G}^c_{n,\alpha}(t-x;x)\right|\\ &+\left|\frac{f'(x+)-f'(x-)}{2}\right|\mathcal{G}^c_{n,\alpha}(|t-x|;x)\\ &+\left|\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}^c_{n,\alpha}(x,t)dt\right|+\left|\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}^c_{n,\alpha}(x,t)dt\right|. \quad (3.28)\end{aligned}$$

Now, let

$$\mathcal{E}^c_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}^c_{n,\alpha}(x,t)dt,$$

and

$$\mathcal{F}^c_{n,\alpha}(f'_x,x)=\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}^c_{n,\alpha}(x,t)dt.$$

Our aim is reduced to calculate the estimates of the terms $\mathcal{E}^c_{n,\alpha}(f'_x,x)$ and $\mathcal{F}^c_{n,\alpha}(f'_x,x)$. From the definition of $\zeta^c_{n,\alpha}$ given in Lemma 3.2, using the integration by parts, we may write

$$\mathcal{E}^c_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_t^x f'_x(u)du\right)\frac{\partial}{\partial t}\zeta^c_{n,\alpha}(x,t)dt=\int_0^x f'_x(t)\zeta^c_{n,\alpha}(x,t)dt.$$

Thus,

$$\begin{aligned}\left|\mathcal{E}^c_{n,\alpha}(f'_x,x)\right| &\le \int_0^x |f'_x(t)|\zeta^c_{n,\alpha}(x,t)dt\\ &\le \int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\zeta^c_{n,\alpha}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\zeta^c_{n,\alpha}(x,t)dt.\end{aligned}$$

Since $f'_x(x)=0$ and $\zeta^c_{n,\alpha}(x,t)\le 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\zeta^c_{n,\alpha}(x,t)dt &= \int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)-f'_x(x)|\zeta^c_{n,\alpha}(x,t)dt\\ &\le \int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x f'_x dt\le \bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x.\end{aligned}$$

By applying Lemma 3.2 and considering $t=x-\frac{x}{u}$, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\zeta^c_{n,\alpha}(x,t)dt &\le Q(\alpha,c)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\frac{dt}{(x-t)^2}\\ &\le Q(\alpha,c)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_t^x f'_x\right)\frac{dt}{(x-t)^2}\\ &= Q(\alpha,c)\frac{1+x^2}{nx}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{u}}^x f'_x\right)du\le Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right).\end{aligned}$$

Therefore,

$$|\mathcal{E}_{n,\alpha}^{c}(f_x',x)| \leq Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right). \tag{3.29}$$

Also, using integration by parts in $\mathcal{F}_{n,\alpha}^{c}(f_x',x)$ and applying Lemma 3.2, we have

$$\begin{aligned}
|\mathcal{F}_{n,\alpha}^{c}(f_x',x) &\leq \left|\int_x^{2x}\left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt\right| \\
&\quad + \left|\int_{2x}^{\infty}\left(\int_x^t f_x'(u)du\right)\mathcal{S}_{n,\alpha}^{c}(x,t)dt\right| \\
&\leq \left|\int_x^{2x} f_x'(u)du\right|\left|1-\zeta_{n,\alpha}^{c}(x,2x)\right| + \int_x^{2x}|f_x'(t)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt \\
&\quad + \left|\int_{2x}^{\infty}(f(t)-f(x))\mathcal{S}_{n,\alpha}^{c}(x,t)dt\right| + |f'(x+)|\left|\int_{2x}^{\infty}(t-x)\mathcal{S}_{n,\alpha}^{c}(x,t)dt\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_x^{2x}|f_x'(t)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt = \int_x^{x+\frac{x}{\sqrt{n}}}|f_x'(t)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt \\
&+\int_{x+\frac{x}{\sqrt{n}}}^{2x}|f_x'(t)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt = J_1 + J_2 \text{ (say)}.
\end{aligned}\tag{3.30}$$

Since $f_x'(x)=0$ and $1-\zeta_{n,\alpha}^{c}\leq 1$, we get

$$\begin{aligned}
J_1 &= \int_x^{x+\frac{x}{\sqrt{n}}}|f_x'(t)-f_x'(x)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt \\
&\leq \int_x^{x+\frac{x}{\sqrt{n}}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right)dt = \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'.
\end{aligned}$$

Applying Lemma 3.2 and considering $t = x+\dfrac{x}{u}$, we obtain

$$\begin{aligned}
J_2 &\leq Q(\alpha,c)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}|f_x'(t)-f_x'(x)|dt \\
&\leq Q(\alpha,c)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\left(\bigvee_x^t f_x'\right)dt = Q(\alpha,c)\frac{1+x^2}{nx}\int_1^{\sqrt{n}}\bigvee_x^{x+\frac{x}{u}} f_x' du \\
&\leq Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\left(\bigvee_x^{x+\frac{x}{u}} f_x'\right)du \leq Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).
\end{aligned}$$

Putting the values of J_1 and J_2 in (3.30), we get

$$\int_x^{2x}|f_x'(t)|\left(1-\zeta_{n,\alpha}^{c}(x,t)\right)dt \leq \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x' + Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).$$

Therefore,

$$|\mathcal{F}^c_{n,\alpha}(f'_x,x)| \leq M_f\int_{2x}^{\infty}(t^2+1)\mathcal{S}^c_{n,\alpha}(x,t)dt+|f(x)|\int_{2x}^{\infty}\mathcal{S}^c_{n,\alpha}(x,t)dt$$
$$+|f'(x+)|\sqrt{Q(\alpha,c)\frac{1+x^2}{n}}+Q(\alpha,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}f'_x+Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}}f'_x\right). \tag{3.31}$$

Since $t\leq 2(t-x)$ and $x\leq t-x$ when $t\geq 2x$, we obtain

$$M_f\int_{2x}^{\infty}(t^2+1)\mathcal{S}^c_{n,\alpha}(x,t)dt+|f(x)|\int_{2x}^{\infty}\mathcal{S}^c_{n,\alpha}(x,t)dt$$
$$\leq (M_f+|f(x)|)\int_{2x}^{\infty}\mathcal{S}^c_{n,\alpha}(x,t)dt+4M_f\int_{2x}^{\infty}(t-x)^2\mathcal{S}^c_{n,\alpha}(x,t)dt$$
$$\leq \frac{M_f+|f(x)|}{x^2}\int_{0}^{\infty}(t-x)^2\mathcal{S}^c_{n,\alpha}(x,t)dt+4M_f\int_{0}^{\infty}(t-x)^2\mathcal{S}^c_{n,\alpha}(x,t)dt$$
$$\leq \left(4M_f+\frac{M_f+|f(x)|}{x^2}\right)Q(\alpha,c)\frac{1+x^2}{n}. \tag{3.32}$$

Using the inequality (3.32), it follows

$$|\mathcal{F}^c_{n,\alpha}(f'_x,x)| \leq \left(4M_f+\frac{M_f+|f(x)|}{x^2}\right)Q(\alpha,c)\frac{1+x^2}{n}$$
$$+|f'(x+)|\sqrt{Q(\alpha,c)\frac{1+x^2}{n}}+Q(\alpha,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}f'_x+Q(\alpha,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}}f'_x\right). \tag{3.33}$$

From (3.28), (3.29) and (3.33), we get the required result. □

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