Khayyam J. Math. 3 (2017), no. 1, 25–32
DOI: 10.22034/kjm.2017.44920

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

CERTAIN PROPERTIES OF A SUBCLASS OF UNIVALENT FUNCTIONS WITH FINITELY MANY FIXED COEFFICIENTS

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Communicated by J. Brzdęk

Abstract. In this paper a new class of analytic, univalent and normalized functions with finitely many fixed coefficients is defined. Properties like coefficient condition, radii of starlikeness and convexity, extreme points and integral operators applied to functions in the class are investigated.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions f defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Such a function has the Taylor series expansion about the origin as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \Delta. \tag{1.1}$$

We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions that are univalent. Goodman [2, 3] defined and studied the subclass of uniformly starlike and uniformly convex functions. Murugusundaramoorthy et al. [4] extended the study of the above subclass by fixing the second coefficient. In recent times, researchers [1, 5] have defined new subclasses of $\mathcal{S}$ by fixing a finite number of coefficients of functions. In this paper, we consider the subclass $\mathcal{SD}(\alpha)$ of $\mathcal{S}$ by fixing finitely many coefficients and properties of the functions in this subclass are examined.

Date: Received: 11 January 2017; Revised: 30 March 2017; Accepted: 10 April 2017.

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2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, univalent function, fixed coefficient, extreme point.

$\mathcal{T}$ denotes the subclass of $\mathcal{S}$ consisting of functions with negative coefficients. Thus if $f \in \mathcal{T}$ then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.2}$$

Definition 1.1 ([6]). A function $f \in \mathcal{S}$ is in the class $\mathcal{SD}(\alpha)$ if it satisfies the analytic criteria

$$Re\left\{\frac{f(z)}{z}\right\} \geq \alpha\left|f'(z) - \frac{f(z)}{z}\right|, \alpha \geq 0. \tag{1.3}$$

The intersection of the classes $\mathcal{T}$ and $\mathcal{SD}(\alpha)$ is denoted by $\mathcal{TSD}(\alpha)$. We now state a necessary and sufficient condition for the functions in $\mathcal{S}$ to be in $\mathcal{TSD}(\alpha)$.

Theorem 1.2. *A function of the form* (1.2) *is in the class* $\mathcal{TSD}(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}[1 + \alpha(n-1)]|a_n| \leq 1, \;\; \alpha \geq 0. \tag{1.4}$$

Proof. Assume that f of the form (1.2) satisfies (1.4). Then
$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$
$\geq 1 - \left|\frac{f(z)}{z} - 1\right| - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$
$= 1 - \sum_{n=2}^{\infty}|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n| \geq 0$. Hence $f \in \mathcal{TSD}(\alpha)$.

Conversely,
$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right| > 0.$

which implies $Re\{1 - \sum_{n=2}^{\infty}|a_n|z^{n-1}\} - \alpha|\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}| > 0$

Letting z to take real values and as $|z| \to 1$, we get

$1 - \sum_{n=2}^{\infty}|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n| \geq 0.$

which implies $\sum_{n=2}^{\infty}[1 + \alpha(n-1)]|a_n| \leq 1.$ □

Corollary 1.3. *For* $f \in \mathcal{TSD}(\alpha)$

$$a_n \leq \frac{1}{1 + \alpha(n-1)}, n \geq 2. \tag{1.5}$$

We now introduce the subclass $\mathcal{TSD}(\alpha, p_k)$ of $\mathcal{TSD}(\alpha)$. This class consists of all those functions in $\mathcal{TSD}(\alpha)$ which are of the form

$$f(z) = z - \sum_{i=2}^{k}\frac{p_i}{1 + \alpha(i-1)}z^i - \sum_{n=k+1}^{\infty} a_n z^n. \tag{1.6}$$

Several interesting properties of the functions in this class are proved in the subsequent sections.

2. Coefficient Estimates

We now prove the coefficient estimate for functions in the class $\mathcal{TSD}(\alpha, p_k)$.

Theorem 2.1. *A function of the form (1.6) is in the class $\mathcal{TSD}(\alpha, p_k)$ if and only if*

$$\sum_{n=k+1}^{\infty}[1+\alpha(n-1)]a_n \leq 1-\sum_{i=2}^{k}p_i, \tag{2.1}$$

where $\alpha \geq 0,\ 0 \leq p_i \leq 1$ and $0 \leq \sum_{i=2}^{k} p_i \leq 1$. The result is sharp.

Proof. By (1.5),

$$a_i = \frac{p_i}{1+\alpha(i-1)},\ i=2,3,...,k,\ 0 \leq p_i \leq 1,\ 0 \leq \sum_{i=2}^{k}p_i \leq 1. \tag{2.2}$$

which implies $\sum_{i=2}^{k} p_i + \sum_{n=k+1}^{\infty}[1+\alpha(n-1)]a_n \leq 1$.

Conversely,

$Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$\geq 1 - \left|\frac{f(z)}{z} - 1\right| - \alpha\left|f'(z) - \frac{f(z)}{z}\right|$

$= 1 - \sum_{n=2}^{\infty}|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n|$

$= 1 - \sum_{i=2}^{k}[1+\alpha(i-1)]|a_i| - \sum_{n=k+1}^{\infty}[1+\alpha(n-1)]|a_n|$

$= 1 - \sum_{i=2}^{k}p_i - \sum_{n=k+1}^{\infty}[1+\alpha(n-1)]|a_n|$, by (2.2)

≥ 0, by (2.1).

Thus $f \in \mathcal{TSD}(\alpha, p_k)$. The sharpness of the result follows by taking

$$f(z) = z - \sum_{i=2}^{k}\frac{p_i}{1+\alpha(i-1)}z^i - \frac{\left(1-\sum_{i=2}^{k}p_i\right)}{1+\alpha(n-1)}z^n,\ n \geq 1. \tag{2.3}$$

□

The following corollary is a consequence of Theorem 2.1.

Corollary 2.2. *If f is in the class $\mathcal{TSD}(\alpha, p_k)$, then*

$$a_n \leq \frac{1-\sum_{i=2}^{k}p_i}{1+\alpha(n-1)},\ n \geq k+1. \tag{2.4}$$

The result is sharp for the functions f given by (2.3).

3. Closure Theorems

Theorem 3.1. *The class $\mathcal{TSD}(\alpha, p_k)$ is convex.*

Proof. Let f, g be two functions in $\mathcal{TSD}(\alpha, p_k)$. Then

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(n-1)} z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

$$g(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(n-1)} z^i - \sum_{n=k+1}^{\infty} b_n z^n,$$

where $0 \le p_i \le 1, 0 \le \sum_{i=2}^{k} p_i \le 1$.

Define $h(z) = \lambda f(z)+(1-\lambda)g(z)$. Then $h(z) = z-\sum_{i=2}^{\infty} \frac{p_i}{1+\alpha(i-1)} z^i - \sum_{n=k+1}^{\infty}[\lambda a_n + (1-\lambda)b_n]z^n$.

Now,

$\sum_{n=k+1}^{\infty}[1+\alpha(n-1)][\lambda a_n + (1-\lambda)b_n]$

$= \lambda \sum_{n=k+1}^{\infty}[1+\alpha(n-1)]a_n + (1-\lambda)\sum_{n=k+1}^{\infty}[1+\alpha(n-1)]b_n$

$\le \lambda(1-\sum_{i=2}^{k} p_i) + (1-\lambda)(1-\sum_{i=2}^{k} p_i)$

$= 1 - \sum_{i=2}^{k} p_i$

which implies $h(z) \in \mathcal{TSD}(\alpha, p_k)$. □

Theorem 3.2. *Let*

$$f_k(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(n-1)} z^i \tag{3.1}$$

and

$$f_n(z) = z - \sum_{i=2}^{k} \frac{p_i}{[1+\alpha(i-1)]} z^i - \frac{\left(1-\sum_{i=2}^{k} p_i\right)}{1+\alpha(n-1)} z^n, n \ge k+1. \tag{3.2}$$

Then $f \in \mathcal{TSD}(\alpha, p_k)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z), \tag{3.3}$$

where $\lambda_n \ge 0,\ (n \ge k)$ and $\sum_{n=k}^{\infty} \lambda_n = 1$.

Proof. Suppose $f \in \mathcal{T}$ can be expressed in the form (3.3). Then

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{1+\alpha(i-1)} z^i - \sum_{n=k+1}^{\infty} \frac{\lambda_n[1-\sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} z^n. \tag{3.4}$$

Now, $\sum_{n=k+1}^{\infty}[1+\alpha(n-1)]\frac{\lambda_n[1-\sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} = \sum_{n=k+1}^{\infty} \lambda_n[1-\sum_{i=2}^{k} p_i]$

$= [1 - \sum_{i=2}^{k} p_i] \sum_{n=k+1}^{\infty} \lambda_n$

$= [1 - \sum_{i=2}^{k} p_i](1 - \lambda_k)$

$\leq 1 - \sum_{i=2}^{k} p_i$

which implies $f \in \mathcal{TSD}(\alpha, p_k)$.

Conversely, for $n \geq k+1$, set

$$\lambda_n = \frac{[1 + \alpha(n-1)]a_n}{1 - \sum_{i=2}^{\infty} p_i}, n \geq k+1. \tag{3.5}$$

and

$$\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n. \tag{3.6}$$

Then f can be represented as $f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z)$. □

Corollary 3.3. *The extreme points of the class $\mathcal{TSD}(\alpha, p_k)$ are the functions $f_n, (n \geq k)$ given by (3.1) and (3.2).*

4. Integral Operator

The Alexander Operator for the functions in the class $\mathcal{S}$ is defined as

$$\mathcal{I}(f) = \int_0^z \frac{f(t)}{t} dt. \tag{4.1}$$

This operator maps the class of starlike functions onto the class of convex functions. The effect of this operator on the functions in the class $\mathcal{TSD}(\alpha, p_k)$ is given in the following theorem.

Theorem 4.1. *Let f defined by (1.5) be in the class $\mathcal{TSD}(\alpha, p_k)$. Then $\mathcal{I}(f)$ belongs to the class $\mathcal{TSD}(\alpha, q_k)$ where $q_k = \frac{p_k}{k}$.*

Proof. We have

$$\mathcal{I}(f) = z - \sum_{i=2}^{k} \frac{q_i}{1 + \alpha(i-1)} z^i - \sum_{n=k+1}^{\infty} \frac{a_n}{n} z^n. \tag{4.2}$$

Now,

$\sum_{n=k+1}^{\infty} [1 + \alpha(n-1)] \frac{a_n}{n}$

$\leq \frac{1}{k+1} \sum_{n=k+1}^{\infty} [1 + \alpha(n-1)] a_n$

$\leq \frac{1}{k+1}(1 - \sum_{i=2}^{k} p_i)$

$= \frac{1}{k+1} - \sum_{i=2}^{k} \frac{p_i}{k+1}$

$< 1 - \sum_{i=2}^{k} \frac{p_i}{i}$

which implies $\mathcal{I}(f) \in \mathcal{TSD}(\alpha, q_k)$. □

5. Radius of Starlikeness and Convexity

In this section, we derive the radii results for the function in the class $\mathcal{TSD}(\alpha, p_k)$ to be starlike or convex of order β.

Theorem 5.1. *The function given by (1.5) in the class $\mathcal{TSD}(\alpha, p_k)$ is starlike of order β $(0 \leq \beta \leq 1)$ in the disk $|z| < r_1$ where r_1 is the largest value which satisfies*

$$\sum_{i=2}^{\infty}\left[\frac{(2-i)-\beta)}{1+\alpha(i-1)}\right]p_i r^{i-1} + \frac{((2-n)-\beta)[1-\sum_{i=2}^{k}p_i]}{1+\alpha(n-1)}r^{n-1} \leq \beta. \tag{5.1}$$

Proof. $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{i=2}^{k}[1-i]\frac{p_i}{1+\alpha(i-1)}r^{i-1} - \sum_{n=k+1}^{\infty}(n-1)a_n r^{n-1}}{1-\sum_{i=2}^{k}\frac{p_i}{1+\alpha(i-1)}r^{i-1} - \sum_{n=k+1}^{\infty}a_n r^{n-1}}$

which is less than or equal to $1-\beta$ for $|z| \leq r$ if and only if

$$\sum_{i=2}^{k}\frac{(2-i)-\beta}{1+\alpha(i-1)}p_i r^{i-1} + \sum_{n=k+1}^{\infty}((2-n)-\beta)a_n r^{n-1} \leq 1-\beta. \tag{5.2}$$

By Corollary 2.2, we may set

$$a_n = \frac{[1-\sum_{i=2}^{k}p_i]}{1+\alpha(n-1)}\lambda_n, \ \ n \geq k+1, \tag{5.3}$$

where $\lambda_n \ \geq 0 \ \ (n \geq k+1), \ \sum_{n=k+1}^{\infty}\lambda_n \leq 1$.
For each fixed r, choosing an integer $n_0 = n_0(r)$ for which $\frac{((2-n)-\beta)r^{n-1}}{1+\alpha(n-1)}$ is a maximum, we obtain

$$\sum_{n=k+1}^{\infty}(n-\beta)a_n r^{n-1} \leq \frac{((2-n_0)-\beta)[1-\sum_{i=2}^{k}p_i]}{1+\alpha(n_0-1)}r^{n_0-1}. \tag{5.4}$$

Hence f is starlike of order β in $|z| \leq r_1$ provided

$$\sum_{i=2}^{k}\frac{(2-i)-\beta}{1+\alpha(i-1)}p_i r^{i-1} + \frac{((2-n_0)-\beta)[1-\sum_{i=2}^{k}p_i]}{1+\alpha(n_0-1)}r^{n_0-1} \leq 1-\beta. \tag{5.5}$$

We find the value of r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{k}\frac{(2-i)-\beta}{1+\alpha(i-1)}p_i r_0^{i-1} + \frac{(2-n_0)-\beta)[1-\sum_{i=2}^{k}p_i]}{1+\alpha(n_0-1)}r_0^{n_0-1} = 1-\beta \tag{5.6}$$

which is the radius of starlikeness of order β for functions in the class $\mathcal{TSD}(\alpha, p_k)$. □

In the following theorem we obtain the radius of convexity for functions in the class $\mathcal{TSD}(\alpha, p_k)$.

Theorem 5.2. *The function given by* (1.5) *in the class* $\mathcal{TSD}(\alpha, p_k)$ *is convex of order* β $(0 \leq \beta \leq 1)$ *in the disk* $|z| < r_2$ *where* r_2 *is the largest value which satisfies*

$$\sum_{i=2}^{\infty}\left[\frac{i(i-\beta)}{1+\alpha(i-1)}\right]p_i r^{i-1} + \frac{n(n-\beta)[1-\sum_{i=2}^{k} p_i]}{1+\alpha(n-1)} r^{n-1} \leq \beta. \tag{5.7}$$

Proof. $\left|\frac{zf''(z)}{f(z)}\right| \leq \frac{\sum_{i=2}^{k}\frac{i(i-1)p_i}{1+\alpha(i-1)}r^{i-1}+\sum_{n=k+1}^{\infty} n(n-1)a_n r^{n-1}}{1-\sum_{i=2} k\frac{ip_i}{1+\alpha(i-1)}-\sum_{n=k+1}^{\infty} na_n r^{n-1}}$
which is less than or equal to $1-\beta$ for $|z| < r$ if and only if

$$\sum_{i=2}^{\infty}\frac{i(i-\beta)}{1+\alpha(i-1)} r^{i-1} + \sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq 1-\beta. \tag{5.8}$$

Using Corollary 2.2 and for each fixed r, choosing an integer $n_0 = n_0(r)$ for which $\frac{n_0(n_0-\beta)r^{n-1}}{1+\alpha(n_0-1)}$ is a maximum, we get

$$\sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r_0^{n-1}. \tag{5.9}$$

Hence f is convex of order β in $|z| < r_2$ provided

$$\sum_{i=2}^{\infty}\frac{i(i-\beta)}{1+\alpha(i-1)} r^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r^{n-1} \leq 1-\beta. \tag{5.10}$$

We find the value of r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{\infty}\frac{i(i-\beta)}{1+\alpha(i-1)} r_0^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{1+\alpha(n_0-1)} r_0^{n-1} \leq 1-\beta \tag{5.11}$$

which is the radius of convexity of order β for functions in the class $\mathcal{TSD}(\alpha, p_k)$. □

Acknowledgement. We thank the referees of this article for their valuable comments .

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