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EISENHART PROBLEM TO SUBMANIFOLDS IN NON-FLAT REAL SPACE FORM

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Abstract. We apply the Eisenhart problem to study the geometric properties of submanifold M of non-flat real space form. It is shown that M is a hypersphere S^3 when σ is parallel. When σ is either semi-parallel or recurrent, then M is either an extrinsic sphere and normal flat or mean curvature vector is parallel in the normal space, respectively.

1. INTRODUCTION

Riemannian invariants play the most fundamental role in Riemannian geometry. They provide the intrinsic characteristics of Riemannian manifolds. Moreover, they affect the behavior of Riemannian manifolds in general. Classically, among Riemannian curvature invariants, people have studied Sectional, Ricci and Scalar curvatures intensively since Riemann.

The study of submanifold was initiated by Darboux and Nash. In 1971, Chern et al. studied submanifold with parallelism of the second fundamental form. Minimal surfaces of an Euclidean m-space E^m and minimal surfaces of hypersphers of E^m are surfaces with parallel mean curvature vector of E^m, i.e. $\nabla H = 0$. In 1973 Chen [8] and in 1974 Yau [18] proved that if M is a surface with parallel mean curvature vector of a real space form $R^m(c)$, then M is one of the surfaces: (1) A minimal surface in $R^m(c)$. (2) A minimal surface in a small hypersphere of $R^m(c)$. (3) A surface with constant mean curvature in a 3-sphere of $R^m(c)$. This shows that the study of surfaces with parallel mean curvature vector in $R^m(c)$ is reduced to that of minimal surfaces except the case (3). In 2004, Turkay [17]

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classified the surfaces immersed in E^5 satisfying the condition $R^{\perp}(X,Y)\cdot H = 0$. Then in 2011, Kadri et al. [12] studied H-recurrent surfaces in Euclidean space E^m. The authors Yano, Kon, Chen, Djoric, Mihai, Kim, Tripathi, Shaikh, Hasan Shahid, Özgur, De and Bagewadi [7, 2, 5, 9, 1, 3] studied submanifold in different structures of manifold.

In 1923 [11], Eisenhart proved that if a positive definite Riemannian manifold (M,g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925 [13], Levy obtained necessary and sufficient conditions for the existence of such tensors. Since then, many others investigated the Eisenhart problem of finding symmetric and skew-symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity, Sharma [14] investigated Eisenhart problem on non-flat real and complex space forms, in 1989.

In this paper, we study submanifolds of Real space form whose second fundamental forms are parallel, semiparallel, recurrent using the following results.

Theorem 1.1 ([14])**.** *A symmetric parallel second order covariant tensor h in a non-flat real space form of dimension $n > 2$ is a scalar multiple of the metric tensor i.e.,*

$$h(X,W) = \frac{tr.h}{n} g(X,W), \tag{1.1}$$

where X, W are vector fields.

Theorem 1.2. *([8, 18]). Let M be a smooth surface in m-dimensional real space form $R^m(c)$ of constant sectional curvature c. If H is parallel in the normal bundle, then M is one of the following surfaces;*

(1) *M is minimal surface in $R^m(c)$.*
(2) *M is minimal surface in a small hypersphere of $R^m(c)$.*
(3) *M is a surface with constant mean curvature $\|H\|$ in S^3 of $R^m(c)$.*

Theorem 1.3 ([12])**.** *Let M be a smooth submanifold in E^n. If M satisfies the H-recurrent condition $\nabla_X H = \lambda(X)H$, then M is $R^{\perp}$-parallel, where λ is a 1-form.*

Theorem 1.4 ([17])**.** *If M is a surface satisfying $R^{\perp}(X,Y)\cdot H = 0$, then M is either minimal or totally umbilical or normally flat i.e., $R^{\perp} = 0$.*

Theorem 1.5 ([10])**.** *Every semi-parallel surface is H-parallel.*

2. PRELIMINARIES

Let $R^m(c)$ denote an m-dimensional real space form of constant sectional curvature c. The Riemannian curvature tensor $\widetilde{R}$ of $R^m(c)$ satisfies

$$\widetilde{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.1}$$

Assume that $\phi : M \longrightarrow R^m(c)$ is an immersion of an n-dimensional Riemannian manifold M into $R^m(c)$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections on M

and $R^m(c)$, respectively. The Gauss and Weingarten formulas are given by [16]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for all vector X, Y tangent to M and normal vector field N on M, where ∇ is the Riemannian connection on M determined by the induced metric g, σ is a symmetric covariant tensor of order 2, $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$ of M and A_N is the shape operator which is related to σ by $g(\sigma(X,Y),N) = g(A_N X, Y)$. The Gauss equation is given by

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - A_{\sigma(Y,Z)}X + A_{\sigma(X,Z)}Y, \tag{2.2}$$

where Z, W are vector fields tangent to M. The first covariant derivative of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.3}$$

where $\widetilde{\nabla}$ is called the Wander-Bortolotti connection of M [6]. We denote by $\nabla^p T$ the covariant differential of the pth order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannain manifold (M,g) with the Levi-Civita connection. According to [15, 16], the tensor T is said to be recurrent if the following condition holds on M

$$(\nabla T)(X_1, ...X_k; X)T(Y_1, ..., Y_k) = (\nabla T)(Y_1, ...Y_k; X)T(X_1, ...X_k), \tag{2.4}$$

where $X, X_1, Y_1, ...X_k, Y_k \in TM$. From (2.4) it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique 1-form ϕ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(log \parallel T \parallel) \tag{2.5}$$

holds on U, where $\parallel T \parallel$ denotes the norm of T, $\parallel T \parallel^2 = g(T,T)$.
The mean curvature vector field is defined by

$$H = \frac{tr.\sigma}{n}. \tag{2.6}$$

Definition 2.1. A submanifold M is said to be totally umbilical if we have

$$\sigma(X,Y) = Hg(X,Y). \tag{2.7}$$

Definition 2.2. A submanifold M is said to be totally geodesic if $\sigma(X,Y) = 0$ for each $X, Y \in TM$ and is minimal if $H = 0$ on M.

A totally umbilical submanifold M in a non-flat real space form is called an extrinsic sphere if its mean curvature vector field is a nonzero parallel vector field. 1-dimensional extrinsic spheres in Riemannian manifolds are called circles. Totally umbilical submanifolds of dimension ≥ 2 in real space forms are either totally geodesic submanifolds or extrinsic spheres.

3. PARALLEL AND SEMI-PARALLEL SUBMANIFOLDS IN A NON-FLAT REAL SPACE FORMS

Let σ be parallel, i.e., $(\widetilde{\nabla}_X\sigma)(Y,Z)=0$.
Using (2.3) in the above equation implies

$$\nabla^{\perp}_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ)=0.$$

Since σ is symmetric, covariant tensor of order 2 we have by virtue (1.1)

$$\nabla^{\perp}_X\{\frac{tr.\sigma}{n}g(Y,Z)\}-\frac{tr.\sigma}{n}g(\nabla_XY,Z)-\frac{tr.\sigma}{n}g(Y,\nabla_XZ)=0. \tag{3.1}$$

Using equations (2.6) in (3.1) we get

$$\nabla^{\perp}_X\{Hg(Y,Z)\}-Hg(\nabla_XY,Z)-Hg(Y,\nabla_XZ)=0.$$

This implies

$$(\nabla^{\perp}_XH)g(Y,Z)+H\nabla^{\perp}_Xg(Y,Z)=HXg(Y,Z).$$

Putting an orthonormal basis over Y and Z in the above equation implies

$$(\nabla^{\perp}_XH)n=0.$$

The above equation becomes

$$\nabla^{\perp}_XH=0.$$

This implies H is parallel in normal bundle. Then we have the following result from Theorem 1.2.

Theorem 3.1. *Let M be a submanifold of a non flat real space form. If σ is parallel, then M is one of the following surfaces:*

(1) *M is minimal surface in $R^m(c)$.*
(2) *M is minimal surface in a small hypersphere of $R^m(c)$.*
(3) *M is a surface with constant mean curvature $\|H\|$ in S^3 of $R^m(c)$.*

Let $\tilde{R}$ and σ satisfy the equation $\tilde{R}\cdot\sigma=0$, i.e., M be semiparallel.
The above equation implies

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)=0. \tag{3.2}$$

Using equations (2.1) and Gauss equation (2.2) in (3.2) and setting $U=V=e_i$ we obtain

$$R^{\perp}(X,Y)H=0. \tag{3.3}$$

This implies that H is either constant or zero or $R^{\perp}=0$, hence we state the following from Theorem (1.4).

Theorem 3.2. *Let M be a submanifold of a non-flat real space form. If σ is semiparallel, then M is either totally umbilical or minimal or normal flat.*

Then M must be either part of an extrinsic sphere or a plane which is totally umbilical [4, 6]. Hence we state the following:

Corollary 3.3. *Let M be a connected and compact submanifold of a non-flat real space form, then σ is semi-parallel if and only if M is either an extrinsic sphere or a plane.*

We state the following from Theorem (1.5).

Theorem 3.4. *Let M be a submanifold of a non flat real space form. If σ is semi-parallel, then M is H-parallel.*

4. RECURRENT SUBMANIFOLDS IN A NON-FLAT REAL SPACE FORM

Suppose σ is recurrent, then from (2.5) we get

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \phi(X)\sigma(Y,Z). \tag{4.1}$$

Using (2.3) in the above equation implies

$$\nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ)) = \phi(X)\sigma(Y,Z). \tag{4.2}$$

Setting $Y = Z = e_i$ in (4.2), then we obtain

$$(\nabla^{\perp}_X H) = -\phi(X)H. \tag{4.3}$$

We state the following:

Theorem 4.1. *Let M be a submanifold of a non-flat real space form. If σ is recurrent, then the mean curvature vector is recurrent in the normal space.*

We state the following result from Theorem 1.3.

Corollary 4.2. *Let M be a submanifold of a non-flat real space form. If σ is recurrent, then M is $R^{\perp}$-parallel.*

Hence Theorem 1.4 follows.

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