Khayyam J. Math. 2 (2016), no. 2, 168–176
DOI: 10.22034/kjm.2016.41250

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

GEODESIC FLOWS ON THE QUOTIENT OF THE UPPER HALF PLANE OVER THE HECKE GROUP

SANAZ LAMEI

Communicated by F.H. Ghane

Abstract. The Hecke group G_α is a family of discrete sub-groups of $PSL(2, \mathbb{R})$. The quotient space of the action of G_α on the upper half plane gives a Riemann surface. The geodesic flows on this surface are ergodic. Here, by constructing a phase space for the geodesic flows hitting an appropriate cross section, we find the arithmetic code of these flows and show that their code space is a topological Markov chain.

1. Introduction and preliminaries

Let $\mathcal{H} = \{z = x + i\,y : \, y > 0\}$ be the upper half plane with hyperbolic metric $ds = \frac{|dz|}{y}$. With this metric, the geodesics on $\mathcal{H}$ are of the form $x = a$ or they are semicircles with center on x-axis [6].

By identifying the transformation $\frac{az+b}{cz+d}$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the group of orientation preserving isometries on $\mathcal{H}$ is the group

$$PSL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1, \;\; a, b, c, d \in \mathbb{R} \right\}.$$

The discrete subgroups of $PSL(2, \mathbb{R})$ are called the Fuchsian groups and the action of them on $\mathcal{H}$ is discontinuous. This means that if $G \leq PSL(2, \mathbb{R})$ is a Fuchsian group, then for any $z = x + iy \in \mathcal{H}$, the orbit $G(z) = \{g(z), \; g \in G\}$ has no accumulation point in $\mathcal{H}$. A Fuchsian group is called of the first kind if its limit set $\Lambda(G)$ equals $\mathbb{R} \cup \{\infty\}$.

Let $T_\alpha(z) = z + \alpha$ and $S(z) = -\frac{1}{z}$. Hecke proved that $G_\alpha = \langle T_\alpha, S \rangle$ is a Fuchsian subgroup of $PSL(2, \mathbb{R})$ if and only if $\alpha = 2\cos\frac{\pi}{q}$, for some integer

Date: Received: 18 September 2016; Revised: 7 December; Accepted: 18 December 2016.
2010 *Mathematics Subject Classification.* Primary 37D40; Secondary 20H05.
Key words and phrases. Hecke group, geodesic flow, arithmetic coding.

$q \geq 3$ [4]. Under this condition, G_α is called the *Hecke group*. When $q = 3$, then $\alpha = 1$ and G_1 is called the modular group. The action of the Hecke group G_α on $\mathcal{H}$ gives a Riemann surface denoted by $\mathcal{M}_\alpha = \mathcal{H}/G_\alpha$. This surface is called the *Hecke surface* and topologically is a punctured sphere with two elliptic fixed points of orders 2 and q. Any oriented geodesic γ on $\mathcal{M}_\alpha$ lifts to infinitely many oriented geodesics on $\mathcal{H}$. All of these geodesics are G_α-equivalent; this means that if $\gamma_1, \gamma_2 \in \mathcal{H}$ are two lifts of $\gamma \in \mathcal{M}_\alpha$, then there exists $g \in G_\alpha$ such that $\gamma_1 = g\gamma_2$.

Let $T^1\mathcal{H} = \cup_{z\in\mathcal{H}} T^1_z\mathcal{H}$ and $T^1\mathcal{M}_\alpha = \cup_{x\in\mathcal{M}_\alpha} T^1_x\mathcal{M}_\alpha$ be the unit tangent bundles of $\mathcal{H}$ and $\mathcal{M}_\alpha$, respectively. The projection map $\pi : T^1\mathcal{H} \to T^1\mathcal{M}_\alpha$ projects the vectors tangent to equivalent geodesics on $\mathcal{H}$ to the vectors tangent to the corresponding geodesic on $\mathcal{M}_\alpha$.

The geodesic flow $\varphi^t : T^1\mathcal{H} \to T^1\mathcal{H}$ is defined as $\varphi^t(v) = w$, where $v, w \in T^1\mathcal{H}$ are tangent to a geodesic and hyperbolic distance of their base points equals t. Also, $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

In Section 2, we give the Rosen algorithm to compute the α-minus continued fraction of real numbers [7] with a little adjustment via a function H_α. Then in Section 3 using the notion of reduced geodesics and the orbit of the point $\frac{2}{\alpha}$ under H_α, we will introduce an appropriate phase space for the cross section of geodesic flows on a Hecke surface. In Section 4, we give a coding which is a useful tool to verify the dynamical properties of geodesic flows on this surface. The obtained code space is topologically Markov chain.

2. α-minus continued fraction

There are different ways to find the α-minus continued fraction (α-MCF) expansion for a given $x \in \mathbb{R}$, but when the algorithm of finding the coefficients of the fraction is given by a function, then x can be uniquely expressed by an α-MCF expansion. Here we define such a function as follows.

For $x \in \mathbb{R}$, let

$$H_\alpha(x) = \begin{cases} \frac{-1}{x - \lfloor \frac{x}{\alpha} + \frac{1}{2} \rfloor \alpha} & \text{if } x \in \mathbb{R} - \{\alpha\mathbb{Z}\} \\ 0 & \text{if } x \in \alpha\mathbb{Z}, \end{cases}$$

where $\lfloor . \rfloor$ is the floor function. This function is simply the composition of a reflection function $\frac{-1}{x}$ and a translation function $x - \lfloor \frac{x}{\alpha} + \frac{1}{2} \rfloor \alpha$, which are similar to the action of the elements of G_α with generators $S(z) = \frac{-1}{z}$ and $T_\alpha(z) = z + \alpha$. Note that if $\frac{2k-1}{2}\alpha \leq x < \frac{2k+1}{2}\alpha$, then $\lfloor \frac{x}{\alpha} + \frac{1}{2} \rfloor = k$. Here we give an algorithm to find α-MCF expansion of $x \in \mathbb{R}$.

Definition 2.1. Let $x = x_0 \in \mathbb{R}$ and $a_0 = \lfloor \frac{x}{\alpha} + \frac{1}{2} \rfloor$. Define

$$x_j = \frac{-1}{x - a_{j-1}\alpha} \quad \text{and} \quad a_j = \lfloor \frac{x_j}{\alpha} + \frac{1}{2} \rfloor.$$

If $x_j \in \alpha\mathbb{Z}$, then the algorithm stops. For $x \in \mathbb{R}$, we obtain a finite or infinite sequence of non-zero integers $a_0, a_1, a_2, \cdots$. So, x can be expressed by

$$x = [a_0, a_1, \cdots]_\alpha = \alpha a_0 - \cfrac{1}{\alpha a_1 - \cfrac{1}{\alpha a_2 - \cfrac{1}{\ddots}}}. \tag{2.1}$$

Definition 2.2. Define $c_\alpha^+ : \mathbb{R} \to \{\mathbb{Z} - \{0\}\}^{\mathbb{N}}$ to be the forward code map as $c_\alpha^+(x) = [a_0, a_1, \cdots]_\alpha$ whose entries are the coefficients of the α-MCF expansion of x obtained in (2.1).

Remark 2.3. 1) If $x = [a_0, a_1, \cdots, a_n]_\alpha$, then $x = \alpha a_0 - \cfrac{1}{\alpha a_1 - \cfrac{1}{\ddots - \cfrac{1}{\alpha a_n}}}$ as in (2.1). Using the generators of G_α, we have $x = T_\alpha^{a_0} S T_\alpha^{a_1} S \cdots S T_\alpha^{a_n}(0)$. If $x = [a_0, a_1, \cdots, a_n, \cdots]_\alpha$ has an infinite α-MCF expansion, then $x = \lim_{n\to\infty} T_\alpha^{a_0} S T_\alpha^{a_1} S \cdots S T_\alpha^{a_n}(0)$ which is a limit point for G_α.

2) Two points $x = x = [a_0, a_1, \cdots]_\alpha$ and $y = x = [b_0, b_1, \cdots]_\alpha$ are equivalent if they have the same tail i.e., there exist $i_0, j_0 \geq 1$ such that for all $k \geq 0$, $a_{i_0+k} = b_{j_0+k}$.

Let $\mathcal{A} = \mathbb{Z} - \{0\}$ be the set of alphabets. Any finite sequence of non-zero integers $a_i, \cdots, a_{i+n}$ is called a *block*. Let

$$\Sigma_\alpha^+ = \{(a_0, a_1, \cdots)_\alpha : \ x = [a_0, a_1, \cdots]_\alpha, x \in \mathbb{R} - \alpha\mathbb{Z}\} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

Define the shift map $\sigma : \Sigma_\alpha^+ \to \Sigma_\alpha^+$ as $\sigma(a_i) = a_{i+1}$, $i \in \mathbb{N}$. The pair $(\Sigma_\alpha^+, \sigma)$ is called symbolic dynamics.

Remark 2.4. The forward code map is a conjugacy between the maps H_α and σ. That is, $c^+(H_\alpha(x)) = \sigma(c^+(x))$.

3. Phase space

Recall that $\mathcal{M}_\alpha = \mathcal{H}/G_\alpha$. A fundamental region $\mathcal{D}_\alpha$ corresponds to $\mathcal{M}_\alpha$ or equivalently to the group G_α. For any side s_i of $\mathcal{D}_\alpha$, there exist a generator $g \in G_\alpha$ and a side s_j of $\mathcal{D}_\alpha$ such that $s_j = g(s_i)$ or $s_j = g^{-1}(s_i)$. The closure of the region $\mathcal{D}_\alpha$ for the Hecke group G_α is the region

$$\overline{\mathcal{D}_\alpha} = \{z \in \mathcal{H} : \ |z| \leq 1, \ |x| \leq \frac{\alpha}{2}\}.$$

Since G_α is a discrete group, it acts discontinuously on $\mathcal{H}$; that is for any $g_i \in G_\alpha$ and $z \in \mathcal{H}$, $\lim_{n\to\infty} g_1 \circ g_2 \circ \cdots \circ g_n(z)$ does not have any accumulation point inside $\mathcal{H}$. Therefore, $\bigcup_{g \in G_\alpha} g\mathcal{D}_\alpha$ will tile $\mathcal{H}$. Let $\mathcal{N}_\alpha$ be the net of the sides of the regions $g\mathcal{D}_\alpha$, for all $g \in G_\alpha$.

Let $\tilde{\gamma}$ be a geodesic on $\mathcal{M}_\alpha$. This geodesic lifts to infinitely many geodesics $\gamma_i = (w_i, u_i) \in \mathcal{H}$ such that for any two geodesics γ_1 and γ_2, there exists $g \in G_\alpha$ such that $\gamma_1 = g\gamma_2$. Let v be a vector tangent to γ in the direction of the geodesic with base point at $p \in \gamma$. Let $g(v)$ be the vector with base point $g(p) \in g(\gamma)$ and tangent to γ in the direction of the geodesic. The vectors $g(v)$ for all $g \in G_\alpha$ are equivalent and project to just one vector on $\mathcal{M}_\alpha$. We use this property to find an appropriate cross section for the set of geodesic flows.

A natural cross section for geodesic flows usually called the geometric cross section is the set of all vectors with base point on the sides of $\mathcal{D}_\alpha$ and pointing inward $\mathcal{D}_\alpha$. These vectors have the property that they are tangent to the geodesic $\gamma = (w,\, u)$ with $|u| \le 1$ and $|w| \ge 1$.

Since the geodesic flow moves the unit vectors tangent to the geodesics with speed one, we can deal with the geodesic itself instead of the geodesic flow.

Any geodesic $\gamma = (w,\, u)$ corresponds to a unique point $(w,\, u)$ in the wu-plane. A simple calculation shows that the corresponding geodesics on wu-plane form a curvilinear region [1], but since we aim to find the arithmetic codes of the geodesics in which the code of w is independent from the code for u, we should choose a rectangular region. Therefore, we will pick an equivalent fundamental region $\mathcal{R}_\alpha$ instead of $\mathcal{D}_\alpha$ by considering the vector $g(v)$ instead of v for an appropriate $g \in G_\alpha$ with base point on the boundary of $g(\mathcal{D}_\alpha)$. The region $\mathcal{R}_\alpha$ has the property that the set of geodesics meeting it, forms a rectangular phase space in uw-plane denoted by $\mathbb{T}_\alpha$. Such a region $\mathcal{R}_\alpha$ is not unique.

Definition 3.1. A geodesic $\gamma = (w,\, u) \in \mathcal{H}$ with $|u| \le 1$ and $|w| \ge 1$ is called a *reduced* geodesic, if it meets $\mathcal{R}_\alpha$.

Let $\mathbb{T}_\alpha$ be the set of all reduced geodesics $\gamma = (w,\, u)$ for a fixed $\mathcal{R}_\alpha$. Clearly, $\mathcal{R}_\alpha$ being a fundamental region, means that if $\gamma = (w,\, u) \in \mathbb{T}_\alpha$, then $S\gamma$ or $T_\alpha^k \gamma \notin \mathbb{T}_\alpha$, for $k \in \mathbb{Z} - \{0\}$. Since the geodesic flows on $\mathcal{M}_\alpha$ is ergodic [5], it suffices to find the rectangular phase space $\mathbb{T}_\alpha$ such that for any $\gamma = (w,\, u) \in \mathbb{T}_\alpha$, $ST_\alpha^{-k}(\gamma) \in \mathbb{T}_\alpha$, for some $k \in \mathbb{Z} - \{0\}$.

For $\alpha = 2\cos\frac{\pi}{q}$, let

$$h_q = \begin{cases} \frac{q-2}{2} & \text{if } q \text{ is even} \\ \frac{q-3}{2} & \text{if } q \text{ is odd.} \end{cases}$$

The orbits of the points $\pm\frac{\alpha}{2}$ play an important rule to find the boundaries of the phase space. In the following lemma, we obtain the orbit of these points and later in Definitions 3.3 and 3.4, we use them to construct the phase space.

Lemma 3.2. *The point $\frac{2}{\alpha}$ has the finite α-MCF expansion*

$$\frac{2}{\alpha} = \begin{cases} [\underbrace{-1, \cdots, -1}_{h_q - 1 \text{ times}}]_\alpha & \textit{if } q \textit{ is even} \\ [\underbrace{-1, \cdots, -1}_{h_q - 1 \text{ times}}, 2, \underbrace{-1, \cdots, -1}_{h_q - 1 \text{ times}}]_\alpha & \textit{if } q \textit{ is odd.} \end{cases}$$

Proof. First, let $q = 2p$. From $U_\alpha^q = (ST_\alpha)^q = \mathrm{Id}$, we have $U_\alpha^p(x) = (U_\alpha^{-1})^p(x)$. Therefore,

$$-\cfrac{1}{\alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots - \frac{1}{x+\alpha}}}} = -\alpha - \cfrac{1}{-\alpha - \cfrac{1}{\ddots - \alpha - \frac{1}{x}}}.$$

This implies that

$$\cfrac{1}{\alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots - \frac{1}{x+\alpha}}}} = \alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots \alpha - \frac{1}{-x}}}. \tag{3.1}$$

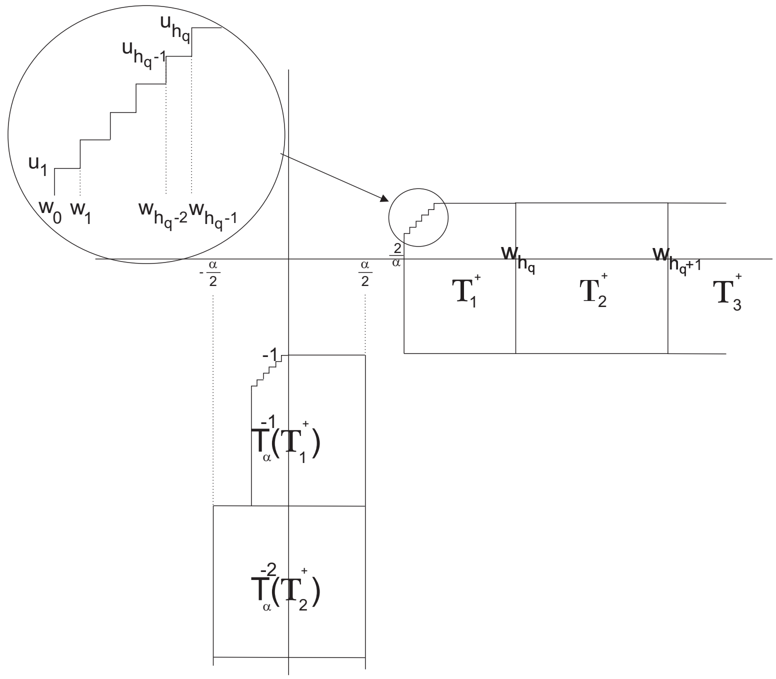

FIGURE 1. The phase space $\mathbb{T}_\alpha^+$ together with $\mathbb{S}_\alpha^+$ for even q.

To obtain the orbit of $x = \frac{-\alpha}{2}$, let $A = T_\alpha U_\alpha^{p-1}(x)$. Then (3.1) becomes $\frac{1}{A} = \alpha - \frac{1}{A}$. So, $A = \frac{2}{\alpha}$ and we are done.

For an odd q, a similar argument or using the relations in [7], gives the result. □

Definition 3.3. For q even, set

$$\mathcal{A}_\alpha = \{w_0 = \frac{2}{\alpha}, w_1 = H_\alpha(w_0), \cdots, w_{h_q-1} = H_\alpha^{h_q-1}(w_0), w_{h_q} = \frac{3}{2}\alpha\}.$$

Define the step function

$$f_q(x) = \begin{cases} u_i = \frac{-1}{u_{i+1}} + \alpha & \text{if } 1 \le i \le h_q - 1 \text{ , and } \ w_{i-1} \le x \le w_i \\ u_{h_q} = \alpha - 1 & \text{if } w \ge w_{h_q} \ . \end{cases}$$

See Figure 1.

For q odd and $h_q = \frac{q-3}{2}$, let

$$\mathcal{A}_\alpha = \{w_0 = \frac{2}{\alpha},\ w_1 = H_\alpha(w_0),\ \cdots, w_{q-3} = H_\alpha^{q-3}(w_0),\ w_{q-2} = \frac{3}{2}\alpha, \}.$$

Also, set $u_1 = ST_\alpha^{-2}(u_{q-2})$, $u_i = ST_\alpha^{-1}(u_{i-1})$, $2 \le i \le h_q+1$, $u_{h_q+2} = ST_\alpha^{-2}(u_{h_q+1})$ and $u_j = ST_\alpha^{-1}(u_{j-1})$ for $h_q + 3 \le j \ge q - 2$. See Figure 2. In this case define the step function as

$$f_q(x) = \begin{cases} u_i & \text{if } i \in \{1, \cdots, q-2\} - \{h_q + 1\}, \\ & \qquad w_{i-1} \le x \le w_{i+h_q (\mod q-2)} \\ u_{h_q+1} & \text{if } w \ge w_{h_q} \ . \end{cases}$$

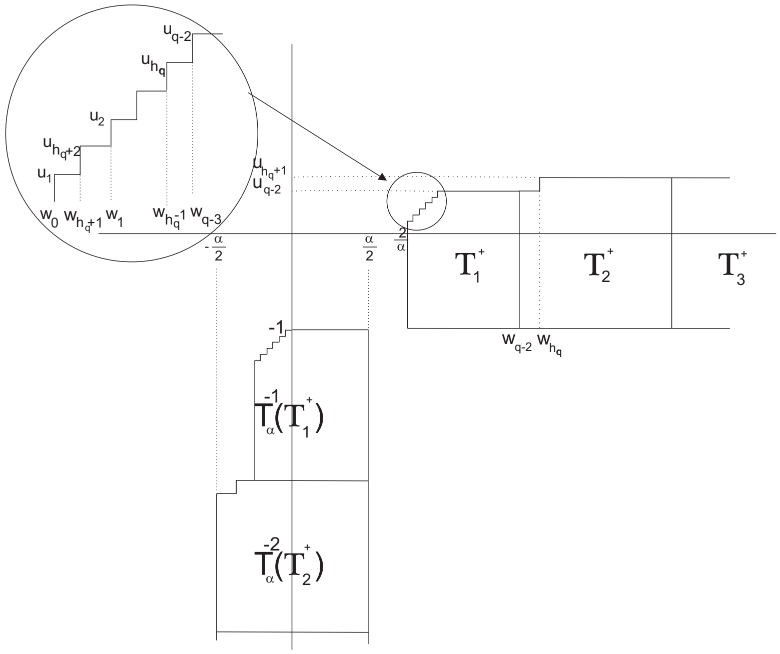

FIGURE 2. The phase space $\mathbb{T}_{\alpha}^{+}$ together with $\mathbb{S}_{\alpha}^{+}$ for odd q.

Definition 3.4. For even q, let $\mathbf{T}_1^+$ be the region consisted of the union of rectangles R_i whose vertical and horizontal sides are on $x = w_{i-1}$, $x = w_i$, $y = -1$ and $y = u_i$, $1 \le i \le h_q$ See Figure 1. Let $\mathbf{T}_k^+$, $k \ge 2$ be the square with vertices on $\left(\frac{2k-1}{2}\alpha,\, \alpha - 1\right)$ and $\left(\frac{2k+1}{2}\alpha,\, -1\right)$.

For an odd q, let $\mathbf{T_1^+} = \cup_{i=1}^{q-3} R_i \cup R_{q-2}$, where R_i's are rectangles with sides on $w = w_{i-1}$, $w = w_{i+h_q(\mod q-2)}$, $u = u_{h_q+1} - \alpha$, $u = u_i$, $i \in \{1, \cdots, q-2\} - \{h_q+1\}$ and the sides of R_{q-2} are on $w = w_{q-3}$, $w = w_{q-2}$, $u = u_{h_q+1} - \alpha$ and $u = u_{q-2}$. Let $\mathbf{T}_2^+$ be the union of two rectangles one of them has the vertices at $(\frac{3\alpha}{2},\, u_{q-2})$ and $(w_{h_q},\, u_{h_q+1} - \alpha)$ and the other one has vertices on $(w_{h_q},\, u_{h_q+1})$ and $(\frac{2k+1}{2}\alpha,\, u_{h_q+1} - \alpha)$. For $k \ge 3$, $\mathbf{T}_k^+$ is a square with vertices at $(\frac{2k-1}{2}\alpha,\, u_{h_q+1})$, $(\frac{2k+1}{2}\alpha,\, u_{h_q+1} - \alpha)$.

For any region A$^+$ in the wu-plain, let A$^- = -$A$^+$ and $\mathbf{T}_k^+ = -\mathbf{T}_{-k}^-$ for $k \ge 1$. For both cases q even or odd, let $\mathbb{T}_\alpha^+ = \cup_{k=1}^{\infty} \mathbf{T}_k^+$, $\mathbb{T}_\alpha = \mathbb{T}_\alpha^- \cup \mathbb{T}_\alpha^+$, $\mathbf{S}_k^+ = T^{-k}\mathbf{T}_k^+$, $\mathbb{S}_\alpha^+ = \cup_{k \ge 1} \mathbf{S}_k^+$ and $\mathbb{S}_\alpha = \mathbb{S}_\alpha^+ \cup \mathbb{S}_\alpha^-$. The notion $\mathbf{T}_i^\pm$ means $\mathbf{T}_i^+$ or $\mathbf{T}_i^-$ for $i > 0$ or $i < 0$, respectively.

For $k \neq 0$, define the function T_R on $\mathbb{T}_\alpha \cup \mathbb{S}_\alpha$ as

$$T_R(w,\, u) = \begin{cases} T_\alpha^{-k}(w,\, u) = (w - k\alpha,\, u - k\alpha), & \text{on } \mathbf{T}_k^\pm \\ S(w,\, u) = \left(\frac{-1}{w},\, \frac{-1}{u}\right), & \text{on } \mathbb{S}_\alpha. \end{cases}$$

In the next theorem, we want to show that the region $\mathbb{T}_\alpha$ introduced in Definition (3.4) is a phase space.

Theorem 3.5. *The function T_R is invariant on $\mathbb{T}_\alpha \cup \mathbb{S}_\alpha$ i.e., $T_R(\mathbb{T}_\alpha \cup \mathbb{S}_\alpha) = \mathbb{T}_\alpha \cup \mathbb{S}_\alpha$.*

Proof. The boundaries of $\mathbb{T}_\alpha$ and $\mathbb{S}_\alpha$ are consisted of segments parallel to the x and y axis and T_R maps them to another segments parallel to x and y axis, respectively. To prove the theorem it suffices to show that the set of horizontal boundary sides and the set of vertical boundary sides are closed under T_R.

By our construction, $\mathbb{S}_\alpha = \cup_{\substack{k=-\infty \\ k\neq 0}} T_\alpha^{-k}\mathbf{T}_k^{\pm}$. It suffices to show that $\mathbb{T}_\alpha = \cup_{\substack{k=-\infty \\ k\neq 0}} ST_\alpha^{-k}\mathbf{T}_k^{\pm}$. The regions $\mathbf{T}_k^{+}$ and $\mathbf{T}_{-k}^{-}$ are symmetric. Hence without loss of generality, let $k \geq 1$.

First, let q be even. For $k = 1$, $\mathbf{T}_1^{+}$ has vertical boundaries on $w = w_0$, $w = w_1, \cdots, w_{h_q-1}$ and $w_{h_q} = \frac{3\alpha}{2}$. By Lemma 3.2 and the choice of w_i's by Definition 3.3, we know that $ST_\alpha^{-1}w_i = w_{i+1}$, $0 \leq i \leq h_q - 2$, $T_\alpha^{-1}w_{h_q-1} = 0$ and $ST_\alpha^{-(2+k)}w_{h_q+k} = w_0$ for $k \geq 0$. In fact the union of the right vertical sides of $T_\alpha^{-i}\mathbf{T}_i^{+}$ for $i \geq 2$ and the union of the left vertical sides of $T_\alpha^{-i}\mathbf{T}_i^{+}$ for $i \geq 1$ form the left and right vertical sides of $\mathbb{S}^{-}$, respectively. The reflection of the left and right vertical sides of $\mathbb{S}^{-}$ under S will map to the positive parts of left and right vertical sides of $\mathbf{T}_1^{+}$ and $\mathbf{T}_{-1}^{-}$, respectively.

On the other hand, since the height of $\mathbf{T}_i^{+}$ for $i \geq 2$ equals α, one gets $ST_\alpha^{-1}(u_{h_q}) = -(u_{h_q} - \alpha)$. This leads to $u_{h_q} = \alpha - 1$, because $|u| < 1$ and $1 \leq \alpha \leq 2$. According to the relations between w_i's, u_i's should be found such that $ST_\alpha^{-1}(u_i) = u_{i+1}$ for $1 \leq i \leq h_q - 1$. This is equivalent to say $u_i = \frac{-1}{u_{i+1}} + \alpha$, $1 \leq i \leq h_q - 1$. Denote by Q_i, the ith quadrant of the wu-plane. All of the above arguments show that $S(\mathbb{S}_\alpha^{+} \cap Q_3) = \mathbb{T}_\alpha^{+} \cap Q_1$ and $S(\mathbb{S}_\alpha^{+} \cap Q_4) = \mathbb{T}_\alpha^{-} \cap Q_2$.

Now let q be odd. Again Lemma 3.2 and the construction of w_i's in Definition 3.3 show that $ST_\alpha^{-1}w_i = w_{i+1}$, $i \in \{1, \cdots, q-3\} - \{h_q\}$, $ST_\alpha^{-2}w_{h_q} = w_{h_q+1}$, $T_\alpha^{-1}w_{q-2} = 0$. We need to show that there are values u_i, $1 \leq i \leq q-2$ satisfying the rules between w_i's. We prove the existence of such u_i's by induction on q. According to Figure 2 for any odd q,

$$(ST_\alpha^{-1})^{h_q} ST_\alpha^{-2} u_{h_q+1} = -(u_{h_q+1} - \alpha). \tag{3.2}$$

The value $q = 3$ gives a rectangular region for $\mathbb{T}_\alpha$. Let $q = 5$. Then (3.2) gives $\frac{-1}{A-\alpha} - \alpha = \frac{1}{B}$, where $A = B = u_{h_q+1} - \alpha$ (For further use denote $u_{h_q+1} - \alpha$ with different symbols A and B in different side of the equation). Now

$$(A\alpha - \alpha^2 + 1)B + A - \alpha = 0. \tag{3.3}$$

By letting $x = \alpha$, $y = z - \alpha x$, $z = 1$ and $w = -x$, equation (3.3) can be written as

$$(Ax + y)B + zA + w = 0. \tag{3.4}$$

Since $A = B$, we have $A^2x + (y+z)A + w = 0$. This equation has solution for A and consequently for u_{h_q+1}, because $w = -x$.

Now, suppose for an arbitrary odd q, the equation (3.4) holds for appropriate x_q, y_q, z_q and w_q. Consider the integer $q+2$. Then (3.2) and (3.4) gives $(\frac{-1}{A-\alpha}x_q + y_q)B + z_q(\frac{-1}{A-\alpha}) + w_q = 0$ which is equal to

$$((z_q - \alpha x_q)A - x_q - (z_q - \alpha x_q)\alpha)B + Aw_q - z_q + x_q\alpha = 0.$$

Now for $q+2$, we have $x_{q+2} = z_q - \alpha x_q = y_q$, $y_{q+2} = -x_q - x_{q+2}\alpha = z_{q+2} - \alpha x_{q+2}$, $z_{q+2} = -x_q$ and $w_{q+2} = -x_{q+2}$. Again letting $A = B$, A will have solution since $w_{q+2} = -x_{q+2}$. Now, Finding u_{h_q} from (3.2), and letting $ST_\alpha^{-1}(u_i) = u_{i+1}$, $1 \le i \le h_q$, $ST_\alpha^{-2}(u_{h_q+1}) = u_{h_q+2}$ and $ST_\alpha^{-1}(u_i) = u_{i+1}$, $h_q + 2 \le i \le q-3$, we will find the remaining u_i's.

By an argument as in the previous case, we have $T_R(\mathbb{T}_\alpha \cup \mathbb{S}_\alpha) = \mathbb{T}_\alpha \cup \mathbb{S}_\alpha$. $\square$

Corollary 3.6. *Let $\alpha = 2\cos\frac{\pi}{q}$. Then the reduced region is a rectangular region with $\frac{q-2}{2}$ or $q-2$ steps for q being even or odd, respectively. The vertices of the region is given by Definitions 3.3 and 3.4.*

4. Coding

In this section, we will verify the dynamical properties of geodesics on the Hecke surface via the coding of the geodesics.

For a point $(w,\, u) \in \mathbf{T}_k^\pm$, define $c^+ : \pi_1(\mathbb{T}_\alpha) \to \mathcal{A}_\alpha$ by $w \mapsto k$ and $c^- : \pi_2(\mathbb{T}_\alpha) \to \mathcal{A}_\alpha$ by $u \mapsto k'$, where $T_\alpha^{k'} S(w,\, u) \in \mathbf{T}_{k'}^\pm$.

Since we are interested in bi-infinite sequences, we do not consider the points $\alpha\mathbb{Q}$. These points are exactly the orbit of the point 0 under the elements of G_α. In other words, we do not consider the countable set of points $G_\alpha(0)$ which has the Lebesgue measure zero.

For $(w_0,\, u_0) \in \mathbb{T}_\alpha$, $w_0, u_0 \notin \alpha\mathbb{Q}$, let $C^+ : \pi_1(\mathbb{T}_\alpha) \to \mathcal{A}^\mathbb{N}$ be defined by $w_0 \mapsto (n_0,\, n_1,\, n_2, \cdots)_\alpha$, where $n_i = c^+(\pi_1(\mathbf{T}_{n_i}^\pm))$ with $(w_{n_i},\, u_{n_i}) = ST^{-(n_i-1)}(w_{n_{i-1}},\, u_{n_{i-1}}) \in \mathbf{T}_{n_i}^\pm$, $i \ge 0$. Similarly, let $C^- : \pi_2(\mathbb{T}_\alpha) \to \mathcal{A}^\mathbb{N}$ by $u_0 \mapsto (n_{-1},\, n_{-2}, \cdots)_\alpha$, where $n_i = c^-(\pi_1(\mathbf{T}_{n_{-i}}^\pm))$ with $(w_{n_{-i}},\, u_{n_{-i}}) = ST^{n_i}(w_{n_{-i+1}},\, u_{n_{-i+1}}) \in \mathbf{T}_{n_{-i}}^\pm$. Now define the code function $C : \mathbb{T}_\alpha \to \mathcal{A}^\mathbb{Z}$ as $C(w,\, u) = C^-(u) \times C^+(w)$. That is $C(w,\, u) = (\cdots, n_{-2},\, n_{-1},\, n_0,\, n_1,\, n_2, \cdots)_\alpha$ where $C^+(w) = (n_0,\, n_1,\, n_2, \cdots)_\alpha$ and $C^-(u) = (n_{-2},\, n_{-1}, \cdots)_\alpha$

Let $\Sigma_\alpha = \{x = (\cdots, n_{-2},\, n_{-1},\, n_0,\, n_1,\, n_2, \cdots)_\alpha : x = C(w,\, u),\ (w,\, u) \in \mathbb{T}_\alpha\}$. Finite sequeces of elements in Σ_α are called words or admissible blocks. Other blocks of $\mathcal{A}^\mathbb{Z}$ where do not appear in elements of Σ_α and have minimal length are called forbidden blocks and are denoted by $\mathcal{F}_\alpha$. If all of the blocks in $\mathcal{F}_\alpha$ have length less than a number $M+1$, then Σ_α is called a countable M-step Markov chain.

Remark 4.1. 1) For $(w,\, u) \in \mathbb{T}_\alpha$, the following diagram commutes

$$\begin{array}{ccc} \mathbb{T}_\alpha \cup \mathbb{S}_\alpha & \xrightarrow{T_R} & \mathbb{T}_\alpha \cup \mathbb{S}_\alpha \\ C\downarrow & & \downarrow C \\ \Sigma_\alpha & \xrightarrow{\sigma} & \Sigma_\alpha. \end{array}$$

2) Since for any $k \in \mathbb{Z} - \{0\}$, $ST^{-k}(\pi_1(\mathbf{T}_k^\pm)) = \cup \pi_1 \mathbf{T}_i^\pm$, $i \in \mathbb{Z} - \{0\}$, the intervals $\{\pi_1(\mathbf{T}_i^\pm) : i \in \mathbb{Z} - \{0\}\}$ form a Markov partition.

Theorem 4.2. *Σ_α is a countable h_q-step and $(q-2)$-step Markov chain for q even and odd, respectively.*

Proof. Let q be even. For $|i| \ge 2$, $ST_\alpha^{-i}\mathbf{T}_i^\pm \cap \mathbf{T}_j^\pm \ne \emptyset$, $j \in \mathbb{Z} - \{0\}$. If $|i| = 1$, then $(ST_\alpha^{-i})^k(\mathbf{T}_i^\pm \cap \mathbf{T}_j^\pm) \ne \emptyset$ only for $1 \le k \le h_q$. This shows that for q even, $\mathcal{F}_\alpha = \{[\underbrace{1, 1, \cdots, 1}_{h_q \text{ times}}, m],\ [\underbrace{-1, -1, \cdots, -1}_{h_q \text{ times}}, -m],\ m \in \mathbb{N}\}$.

Let q be odd. Then $\mathcal{F}_q = \{[\underbrace{1,1,\cdots,1}_{h_q \text{ times}}, 2\ \underbrace{1,1,\cdots,1}_{h_q \text{ times}}, m],\ [\underbrace{-1,-1,\cdots,-1}_{h_q \text{ times}},\ -2\ \underbrace{-1,-1,\cdots,-1}_{h_q \text{ times}}, -m],\ m\in\mathbb{N}\}$.

Therefore, in both cases the length of forbidden blocks is finite. Thus Σ_α is a countable h_q-step or $(q-2)$-step Markov chain for q even or odd, respectively. □

Remark 4.3. 1) In Theorem 3.5, T_R is an invariant function on $\mathbb{T}_\alpha \cup \mathbb{S}_\alpha$ for real numbers α such that G_α is a Hecke group. In [2] and [3], Ahmadi Dastjerdi and the author showed that the reduced region $\mathbb{T}_\alpha$ satisfying $T_R(\mathbb{T}_\alpha \cup \mathbb{S}_\alpha) = \mathbb{T}_\alpha \cup \mathbb{S}_\alpha$ is a rectangle if and only if $\alpha \in \{1,\ \frac{2\sqrt{3}}{3},\ \sqrt{2},\ 2\}$, but $\alpha = \frac{2\sqrt{3}}{3}$ and $\alpha = 2$ do not generate a Hecke group. Also, for $\alpha = 1$, there are three types of rectangle regions $\mathbb{T}_\alpha$ where each of which induced a special type of α-MCF expansion.

2) If α does not correspond to a Hecke group, M_α is not defined but as in the first part of this remark, $\mathbb{T}_\alpha$ may exist. Therefore, $(\mathbb{T}_\alpha \cup \mathbb{S}_\alpha,\ T_R)$ can be considered as an abstract dynamical system not necessarily realizing geodesics on M_α. There is not known result showing that if there exist real numbers other than $\{\frac{2\sqrt{3}}{3},\ 2,\ 2\cos\frac{\pi}{q}\}$ defining the region $\mathbb{T}_\alpha$ satisfying $T_R(\mathbb{T}_\alpha \cup \mathbb{S}_\alpha) = \mathbb{T}_\alpha \cup \mathbb{S}_\alpha$.

Such abstract systems give a wide variety of dynamical systems whose code space Σ_α is defined by an infinite alphabet set. In all cases in this paper and in [2] and [3], Σ_α is an M-step Markov chain for an appropriate M, but this may not be the case for Σ_α if α is other than $\{\frac{2\sqrt{3}}{3},\ 2,\ 2\cos\frac{\pi}{q}\}$.

References

1. R. Adler and L. Flatto, *Cross Section Map for Geodesic Flow on the Modular Surface*, Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Oxford University Press, Oxford, 1991.
2. D. Ahmadi Dastjerdi and S. Lamei, *Geodesic flow on the quotient space of the action of $< z+2, -1/z >$ on the upper half plane*, An. St. Univ. Ovidius Constanta, **20** (2012), no. 3, 37-50.
3. D. Ahmadi Dastjerdi and S. Lamei, *On the arithmetic codings on the Hecke triangle surface*, Journal of Advanced Research in Dynamical and Control Systems, **7** (2015), no.2, 98-113.
4. E. Hecke, *ber die Bestimmung Direchletscher Reihen durch ihre Funktionalgleichungen*, Mathematische Annalen, **112**, (1935), 664-699.
5. E. Hoph, *Ergodentheories*, Abh. Sächs. Akad. Wiss. Leipzi, **91** (1939), 261.
6. S. Katok and I. Ugarcovici, *Symbolic dynamics for the modular surface and beyond*, Bull. Amer. Math. Soc., **47** (2007), 87-132.
7. D. Rosen, *A class of continued fractions associated with certain properly discontinuous groups*, Duke Math. J., **21** (1954), 549-563.
8. H. Weber, *Lehrbuch der Algebra*, I, 2nd edition, Viehweg, Braunschweig, 1898

Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: lamei@guilan.ac.ir