Journal de Théorie des Nombres
de Bordeaux **19** (2007), 357–372

On Robin's criterion for the Riemann hypothesis

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Résumé. Le critère de Robin spécifie que l'hypothèse de Riemann (RH) est vraie si et seulement si l'inégalité de Robin $\sigma(n) := \sum_{d|n} d < e^\gamma n \log\log n$ est vérifiée pour $n \geq 5041$, avec γ la constante d'Euler(-Mascheroni). Nous montrons par des méthodes élémentaires que si $n \geq 37$ ne satisfait pas au critère de Robin il doit être pair et il n'est ni sans facteur carré ni non divisible exactement par un premier. Utilisant une borne de Rosser et Schoenfeld, nous montrons, en outre, que n doit être divisible par une puissance cinquième > 1. Comme corollaire, nous obtenons que RH est vraie ssi chaque entier naturel divisible par une puissance cinquième > 1 vérifie l'inégalité de Robin.

Abstract. Robin's criterion states that the Riemann Hypothesis (RH) is true if and only if Robin's inequality $\sigma(n) := \sum_{d|n} d < e^\gamma n \log\log n$ is satisfied for $n \geq 5041$, where γ denotes the Euler(-Mascheroni) constant. We show by elementary methods that if $n \geq 37$ does not satisfy Robin's criterion it must be even and is neither squarefree nor squarefull. Using a bound of Rosser and Schoenfeld we show, moreover, that n must be divisible by a fifth power > 1. As consequence we obtain that RH holds true iff every natural number divisible by a fifth power > 1 satisfies Robin's inequality.

1. Introduction

Let $\mathcal{R}$ be the set of integers $n \geq 1$ satisfying $\sigma(n) < e^\gamma n \log\log n$. This inequality we will call *Robin's inequality.* Note that it can be rewritten as

$$\sum_{d|n} \frac{1}{d} < e^\gamma \log\log n.$$

Ramanujan [8] (in his original version of his paper on highly composite integers, only part of which, due to paper shortage, was published, for the shortened version see [7, pp. 78-128]) proved that if RH holds then every sufficiently large integer is in $\mathcal{R}$. Robin [9] proved that if RH holds, then

Manuscrit reçu le 19 juin 2006.

actually every integer $n \geq 5041$ is in $\mathcal{R}$. He also showed that if RH is false, then there are infinitely many integers that are not in $\mathcal{R}$. Put $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040\}$. The set $\mathcal{A}$ consists of the integers $n \leq 5040$ that do not satisfy Robin's inequality. Note that none of the integers in $\mathcal{A}$ is divisible by a 5th power of a prime.

In this paper we are interested in establishing the inclusion of various infinite subsets of the natural numbers in $\mathcal{R}$. We will prove in this direction:

Theorem 1.1. *Put* $\mathcal{B} = \{2, 3, 5, 6, 10, 30\}$. *Every squarefree integer that is not in* $\mathcal{B}$ *is an element of* $\mathcal{R}$.

A similar result for the odd integers will be established:

Theorem 1.2. *Any odd positive integer* n *distinct from* $1, 3, 5$ *and* 9 *is in* $\mathcal{R}$.

On combining Robin's result with the above theorems one finds:

Theorem 1.3. *The RH is true if and only if for all even non-squarefree integers* ≥ 5044 *Robin's inequality is satisfied.*

It is an easy exercise to show that the even non-squarefree integers have density $\frac{1}{2} - \frac{2}{\pi^2} = 0.2973\cdots$ (cf. Tenenbaum [11, p. 46]). Thus, to wit, this paper gives at least half a proof of RH !

Somewhat remarkably perhaps these two results will be proved using only very elementary methods. The deepest input will be Lemma 2.1 below which only requires pre-Prime Number Theorem elementary methods for its proof (in Tenenbaum's [11] introductory book on analytic number theory it is already derived within the first 18 pages).

Using a bound of Rosser and Schoenfeld (Lemma 3.1 below), which ultimately relies on some explicit knowledge regarding the first so many zeros of the Riemann zeta-function, one can prove some further results:

Theorem 1.4. *The only squarefull integers not in* $\mathcal{R}$ *are* 1, 4, 8, 9, 16 *and* 36.

We recall that an integer n is said to be squarefull if for every prime divisor p of n we have $p^2 | n$. An integer n is called t-free if $p^t \nmid m$ for every prime number p. (Thus saying a number is squarefree is the same as saying that it is 2-free.)

Theorem 1.5. *All* 5*-free integers greater than* 5040 *satisfy Robin's inequality.*

Together with the observation that all exceptions ≤ 5040 to Robin's inequality are 5-free and Robin's criterion, this result implies the following alternative variant of Robin's criterion.

Theorem 1.6. *The RH holds iff for all integers n divisible by the fifth power of some prime we have $\sigma(n) < e^{\gamma} n \log\log n$.*

The latter result has the charm of not involving a finite range of integers that has to be excluded (the range $n \le 5040$ in Robin's criterion). We note that a result in this spirit has been earlier established by Lagarias [5] who, using Robin's work, showed that the RH is equivalent with the inequality

$$\sigma(n) \le h(n) + e^{h(n)} \log(h(n)),$$

where $h(n) = \sum_{k=1}^{n} 1/k$ is the harmonic sum.

2. Proof of Theorem 1 and Theorem 2

Our proof of Theorem 1.1 requires the following lemmata.

Lemma 2.1.

(1) *For $x \ge 2$ we have*

$$\sum_{p \le x} \frac{1}{p} = \log\log x + B + O(\frac{1}{\log x}),$$

where the implicit constant in Landau's O-symbol does not exceed $2(1 + \log 4) < 5$ *and*

$$B = \gamma + \sum_{p} \left(\log(1 - \frac{1}{p}) + \frac{1}{p} \right) = 0.2614972128 \cdots$$

denotes the (Meissel-)Mertens constant.

(2) *For $x \ge 5$ we have*

$$\sum_{p \le x} \frac{1}{p} \le \log\log x + \gamma.$$

Proof. 1) This result can be proved with very elementary methods. It is derived from scratch in the book of Tenenbaum [11], p. 16. At p. 18 the constant B is determined.
2) One checks that the inequality holds true for all primes p satisfying $5 \le p \le 3673337$. On noting that

$$B + \frac{2(1 + \log 4)}{\log 3673337} < \gamma,$$

the result then follows from part 1. □

Remark. More information on the (Meissel-)Mertens constant can be found e.g. in the book of Finch [4, §2.2].

Remark. Using deeper methods from (computational) prime number theory Lemma 2.1 can be considerably sharpened, see e.g. [10], but the point we want to make here is that the estimate given in part 2, which is the estimate we need in the sequel, is a rather elementary estimate.

We point out that 15 is in $\mathcal{R}$.

Lemma 2.2. *If r is in $\mathcal{A}$ and $q \geq 7$ is a prime, then rq is in $\mathcal{R}$, except when $q = 7$ and $r = 12,\ 120$ or 360.*

Corollary 2.1. *If r is in $\mathcal{B}$ and $q \geq 7$ is a prime, then rq is in $\mathcal{R}$.*

Proof. of Lemma 2.2. One verifies the result in case $q = 7$. Suppose that r is in $\mathcal{A}$. Direct computation shows that $11r$ is in $\mathcal{R}$. From this we obtain for $q \geq 11$ that

$$\frac{\sigma(rq)}{rq} = (1+\frac{1}{q})\frac{\sigma(r)}{r} \leq \frac{12\sigma(r)}{11r} = \frac{\sigma(11r)}{11r} < e^{\gamma}\log\log(11r) \leq e^{\gamma}\log\log(qr).$$

□

Proof. of Theorem 1.1. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n. Put $\omega(n) = m$. The assertion is easily provable for those integers with $m = 1$ (the primes that is). Suppose it is true for $m-1$, with $m \geq 2$ and let us consider the assertion for those squarefree n with $\omega(n) = m$. So let $n = q_1 \cdots q_m$ be a squarefree number that is not in $\mathcal{B}$ and assume w.l.o.g. that $q_1 < \cdots < q_m$. We consider two cases:

Case 1: $q_m \geq \log(q_1 \cdots q_m) = \log n$.

If $q_1 \cdots q_{m-1}$ is in $\mathcal{B}$, then if q_m is not in $\mathcal{B}$, $n = q_1 \ldots q_{m-1}q_m$ is in $\mathcal{R}$ (by the corollary to Lemma 2.2) and we are done, and if q_m is in $\mathcal{B}$, the only possibility is $n = 15$ which is in $\mathcal{R}$ and we are also done.

If $q_1 \cdots q_{m-1}$ is not in $\mathcal{B}$, by the induction hypothesis we have

$$(q_1+1)\cdots(q_{m-1}+1) < e^{\gamma} q_1 \cdots q_{m-1} \log\log(q_1 \cdots q_{m-1}),$$

and hence

$$(q_1+1)\cdots(q_{m-1}+1)(q_m+1) < e^{\gamma} q_1 \cdots q_{m-1}(q_m+1) \log\log(q_1 \cdots q_{m-1}).$$

We want to show that

$$\begin{aligned} &e^{\gamma} q_1 \cdots q_{m-1}(q_m+1)\log\log(q_1 \cdots q_{m-1}) \\ &\leq e^{\gamma} q_1 \cdots q_{m-1} q_m \log\log(q_1 \cdots q_{m-1}q_m) = e^{\gamma} n \log\log n. \end{aligned} \tag{2.1}$$

Indeed (2.1) is equivalent with

$$q_m \log\log(q_1 \cdots q_{m-1}q_m) \geq (q_m+1)\log\log(q_1 \cdots q_{m-1}),$$

or alternatively

(2.2)

$$\frac{q_m(\log\log(q_1\cdots q_{m-1}q_m)-\log\log(q_1\cdots q_{m-1}))}{\log q_m}\ge\frac{\log\log(q_1\cdots q_{m-1})}{\log q_m}.$$

Suppose that $0<a<b$. Note that we have

$$\frac{\log b-\log a}{b-a}=\frac{1}{b-a}\int_a^b\frac{dt}{t}>\frac{1}{b}. \tag{2.3}$$

Using this inequality we infer that (2.2) (and thus (2.1)) is certainly satisfied if the next inequality is satisfied:

$$\frac{q_m}{\log(q_1\cdots q_m)}\ge\frac{\log\log(q_1\cdots q_{m-1})}{\log q_m}.$$

Note that our assumption that $q_m\ge\log(q_1\cdots q_m)$ implies that the latter inequality is indeed satisfied.

Case 2: $q_m<\log(q_1\cdots q_m)=\log n$.

Note that the above inequality implies $q_m\ge 7$ since $\log 2<2,\ \log 6<3$ and $\log 30<5$. It is easy to see that $\sigma(n)<e^\gamma n\log\log n$ is equivalent with

$$\sum_{j=1}^{m}(\log(q_j+1)-\log q_j)<\gamma+\log\log\log(q_1\cdots q_m). \tag{2.4}$$

Note that

$$\log(q_1+1)-\log q_1=\int_{q_1}^{q_1+1}\frac{dt}{t}<\frac{1}{q_1}.$$

In order to prove (2.4) it is thus enough to prove that

$$\frac{1}{q_1}+\cdots+\frac{1}{q_m}\le\sum_{p\le q_m}\frac{1}{p}\le\gamma+\log\log\log(q_1\cdots q_m). \tag{2.5}$$

Since $q_m\ge 7$ we have by part 2 of Lemma 2.1 and the assumption $q_m<\log(q_1\cdots q_m)$ that

$$\sum_{p\le q_m}\frac{1}{p}\le\gamma+\log\log q_m<\gamma+\log\log\log(q_1\cdots q_m),$$

and hence (2.5) is indeed satisfied. □

Theorem 1.2 will be derived from the following stronger result.

Theorem 2.1. *For all odd integers except* 1, 3, 5, 9 *and* 15 *we have*

$$\frac{n}{\varphi(n)}<e^\gamma\log\log n, \tag{2.6}$$

where $\varphi(n)$ *denotes Euler's totient function.*

To see that this is a stronger result, let $n = \prod_{i=1}^{k} p_i^{e_i}$ be the prime factorisation of n and note that for $n \geq 2$ we have

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{k} \frac{1 - p_i^{-e_i - 1}}{1 - p_i^{-1}} < \prod_{i=1}^{k} \frac{1}{1 - p_i^{-1}} = \frac{n}{\varphi(n)}. \tag{2.7}$$

We let $\mathcal{N}$ ($\mathcal{N}$ in acknowledgement of the contributions of J.-L. Nicolas to this subject) denote the set of integers $n \geq 1$ satisfying (2.6). Our proofs of Theorems 1.2 and 2.1 use the next lemma.

Lemma 2.3. *Put* $\mathcal{S} = \{3^a \cdot 5^b \cdot q^c : q \geq 7 \text{ is prime}, \ a, b, c \geq 0\}$. *All elements from* $\mathcal{S}$ *except* 1, 3, 5 *and* 9 *are in* $\mathcal{R}$. *All elements from* $\mathcal{S}$ *except* 1, 3, 5, 9 *and* 15 *are in* $\mathcal{N}$.

Proof. If n is in $\mathcal{S}$ and $n \geq 31$ we have

$$\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)} \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{q}{q-1} \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} < e^\gamma \log\log n.$$

Using this observation the proof is easily completed. □

Remark. Let y be any integer. Suppose that we have an infinite set of integers all having no prime factors $> y$. Then $\sigma(n)/n$ and $n/\varphi(n)$ are bounded above on this set, whereas $\log\log n$ tends to infinity. Thus only finitely many of those integers will not be in $\mathcal{R}$, respectively $\mathcal{N}$. It is a finite computation to find them all (cf. the proof of Lemma 2.3).

Proof of Theorem 2.1. As before we let $m = \omega(n)$. If $m \leq 1$ then, by Lemma 2.3, n is in $\mathcal{N}$, except when $n = 1, 3, 5$ or 9. So we may assume $m \geq 2$. Let $\kappa(n) = \prod_{p|n} p$ denote the squarefree kernel of n. Since $n/\varphi(n) = \kappa(n)/\varphi(\kappa(n))$ it follows that if r is a squarefree number satisfying (2.6), then all integers n with $\kappa(n) = r$ satisfy (2.6) as well. Thus we consider first the case where $n = q_1 \cdots q_m$ is an odd squarefree integer with $q_1 < \cdots < q_m$. In this case n is in $\mathcal{N}$ iff

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^\gamma \log\log n.$$

Note that

$$\frac{q_i}{q_i - 1} \leq \frac{3}{2} \text{ and } \frac{q_i}{q_i - 1} < \frac{q_{i-1} + 1}{q_{i-1}},$$

and hence

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{3}{2} \prod_{i=1}^{m-1} \frac{q_i + 1}{q_i} = \frac{\sigma(n_1)}{n_1},$$

where $n_1 = 2n/q_m < n$. Thus, $n/\varphi(n) < \sigma(n_1)/n_1$. If n_1 is in $\mathcal{R}$, then invoking Theorem 1.1 we find

$$\frac{n}{\varphi(n)} < \frac{\sigma(n_1)}{n_1} < e^\gamma \log\log n_1 < e^\gamma \log\log n,$$

and we are done.

If n_1 is not in $\mathcal{R}$, then by Theorem 1.1 it follows that n must be in $\mathcal{S}$. The proof is now completed on invoking Lemma 2.3. □

Proof of Theorem 1.2. One checks that $1, 3, 5$ and 9 are not in $\mathcal{R}$, but 15 is in $\mathcal{R}$. The result now follows by Theorem 2.1 and inequality (2.7). □

2.1. Theorem 2.1 put into perspective. Since the proof of Theorem 2.1 can be carried out with such simple means, one might expect it can be extended to quite a large class of even integers. However, even a superficial inspection of the literature on $n/\varphi(n)$ shows this expectation to be wrong.

Rosser and Schoenfeld [10] showed in 1962 that

$$\frac{n}{\varphi(n)} \le e^{\gamma}\log\log n + \frac{5}{2\log\log n},$$

with one exception: $n = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$. They raised the question of whether there are infinitely many n for which

$$\frac{n}{\varphi(n)} > e^{\gamma}\log\log n, \tag{2.8}$$

which was answered in the affirmative by J.-L. Nicolas [6]. More precisely, let $N_k = 2\cdot 3\cdots\cdot p_k$ be the product of the first k primes, then if the RH holds true (2.8) is satisfied with $n = N_k$ for every $k \ge 1$. On the other hand, if RH is false, then there are infinitely many k for which (2.8) is satisfied with $n = N_k$ and there are infinitely many k for which (2.8) is not satisfied with $n = N_k$. Thus the approach we have taken to prove Theorem 1.2, namely to derive it from the stronger result Theorem 2.1, is not going to work for even integers.

3. Proof of Theorem 1.4

The proof of Theorem 1.4 is an immediate consequence of the following stronger result.

Theorem 3.1. *The only squarefull integers* $n \ge 2$ *not in* $\mathcal{N}$ *are* 4, 8, 9, 16, 36, 72, 108, 144, 216, 900, 1800, 2700, 3600, 44100 *and* 88200.

Its proof requires the following two lemmas.

Lemma 3.1. [10]. *For* $x > 1$ *we have*

$$\prod_{p\le x}\frac{p}{p-1} \le e^{\gamma}(\log x + \frac{1}{\log x}).$$

Lemma 3.2. *Let* $p_1 = 2,\ p_2 = 3, \ldots$ *denote the consecutive primes. If*

$$\prod_{i=1}^{m}\frac{p_i}{p_i-1} \ge e^{\gamma}\log(2\log(p_1\cdots p_m)),$$

then $m \leq 4$.

Proof. Suppose that $m \geq 26$ (i.e. $p_m \geq 101$). It then follows by Theorem 10 of [10], which states that $\theta(x) := \sum_{p \leq x} \log p > 0.84x$ for $x \geq 101$, that $\log(p_1 \cdots p_m) = \theta(p_m) > 0.84 p_m$. We find that

$$\log(2\log(p_1 \cdots p_m)) > \log p_m + \log 1.68 \geq \log p_m + \frac{1}{\log p_m},$$

and so, by Lemma 3.1, that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq e^{\gamma} \left(\log p_m + \frac{1}{\log p_m} \right) < e^{\gamma} \log(2\log(p_1 \cdots p_m)).$$

The proof is then completed on checking the inequality directly for the remaining values of m. □

Proof of Theorem 3.1. Suppose that n is squarefull and $n/\varphi(n) \geq e^{\gamma} \log\log n$. Put $\omega(n) = m$. Then

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \geq \frac{n}{\varphi(n)} \geq e^{\gamma} \log\log n \geq e^{\gamma} \log(2\log(p_1 \cdots p_m)).$$

By Lemma 3.2 it follows that $m \leq 4$. In particular we must have

$$2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{35}{8} \geq e^{\gamma} \log\log n,$$

whence $n \leq \exp(\exp(e^{-\gamma} 35/8)) \leq 116144$. On numerically checking the inequality for the squarefull integers ≤ 116144, the proof is then completed. □

Remark. The squarefull integers ≤ 116144 are easily produced on noting that they can be uniquely written as $a^2 b^3$, with a a positive integer and b squarefree.

4. On the ratio $\sigma(n)/(n \log\log n)$ as n ranges over various sets of integers

We have proved that Robin's inequality holds for large enough odd numbers, squarefree and squarefull numbers. A natural question to ask is how large the ratio $f_1(n) := \sigma(n)/(n \log\log n)$ can be when we restrict n to these sets of integers. We will consider the same question for the ratio $f_2(n) := n/(\varphi(n) \log\log n)$. Our results in this direction are summarized in the following result:

Theorem 4.1. *We have*

(1) $\limsup_{n \to \infty} f_1(n) = e^{\gamma}$, (2) $\limsup_{\substack{n \to \infty \\ n \text{ is squarefree}}} f_1(n) = \frac{6e^{\gamma}}{\pi^2}$, (3) $\limsup_{\substack{n \to \infty \\ n \text{ is odd}}} f_1(n) = \frac{e^{\gamma}}{2}$,

and, moreover,

$$(4)\ \limsup_{n\to\infty} f_2(n) = e^\gamma,\ (5)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefree}}} f_2(n) = e^\gamma,\ (6)\ \limsup_{\substack{n\to\infty \\ n \text{ is odd}}} f_2(n) = \frac{e^\gamma}{2}.$$

Furthermore,

$$(7)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefull}}} f_1(n) = e^\gamma,\ (8)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefull}}} f_2(n) = e^\gamma.$$

(The fact that the corresponding lim infs are all zero is immediate on letting n run over the primes.)

Part 4 of Theorem 4.1 was proved by Landau in 1909, see e.g. [1, Theorem 13.14], and the remaining parts can be proved in a similar way. Gronwall in 1913 established part 1. Our proof makes use of a lemma involving t-free integers (Lemma 4.1), which is easily proved on invoking a celebrated result due to Mertens (1874) asserting that

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} \sim e^\gamma \log x, \quad x\to\infty. \tag{4.1}$$

Lemma 4.1. *Let $t \ge 2$ be a fixed integer. We have*

$$(1)\ \limsup_{\substack{n\to\infty \\ t-\text{free integers}}} f_1(n) = \frac{e^\gamma}{\zeta(t)},\ (2)\ \limsup_{\substack{n\to\infty \\ \text{odd } t-\text{free integers}}} f_1(n) = \frac{e^\gamma}{2\zeta(t)(1-2^{-t})}.$$

Proof. 1) Let us consider separately the prime divisors of n that are larger than $\log n$. Let us say there are r of them. Then $(\log n)^r < n$ and thus $r < \log n/\log\log n$. Moreover, for $p > \log n$ we have

$$\frac{1-p^{-t}}{1-p^{-1}} < \frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}.$$

Thus,

$$\prod_{\substack{p\mid n \\ p>\log n}} \frac{1-p^{-t}}{1-p^{-1}} < \left(\frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}\right)^{\frac{\log n}{\log\log n}}.$$

Let p_k denote the largest prime factor of n. We obtain

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{k} \frac{1-p_i^{-e_i-1}}{1-p_i^{-1}} \le \prod_{i=1}^{k} \frac{1-p_i^{-t}}{1-p_i^{-1}}$$

$$< \left(\frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}\right)^{\frac{\log n}{\log\log n}} \prod_{p\le\log n} \frac{1-p^{-t}}{1-p^{-1}}, \tag{4.2}$$

where in the derivation of the first inequality we used that $e_i < t$ by assumption. Note that the factor before the final product is of size

$1 + O((\log\log n)^{-1})$ and thus tends to 1 as n tends to infinity. On invoking (4.1) and noting that $\prod_{p\le \log n}(1-p^{-t}) \sim \zeta(t)^{-1}$, it follows that the $\limsup \le e^{\gamma}/\zeta(t)$.

In order to prove the $\ge$ part of the assertion, take $n = \prod_{p\le x} p^{t-1}$. Note that n is t-free. On invoking (4.1) we infer that

$$\frac{\sigma(n)}{n} = \prod_{p\le x} \frac{1-p^{-t}}{1-p^{-1}} \sim \frac{e^{\gamma}}{\zeta(t)} \log x.$$

Note that $\log n = t\sum_{p\le x}\log p = t\theta(x)$, where $\theta(x)$ denotes the Chebyshev theta function. By an equivalent form of the Prime Number Theorem we have $\theta(x) \sim x$ and hence $\log\log n = (1+o_t(1))\log x$. It follows that for the particular sequence of infinitely many n values under consideration we have

$$\frac{\sigma(n)}{n\log\log n} = \frac{e^{\gamma}}{\zeta(t)}\Big(1+o_t(1)\Big).$$

Thus, in particular, for a given $\epsilon > 0$ there are infinitely many n such that

$$\frac{\sigma(n)}{n\log\log n} > \frac{e^{\gamma}}{\zeta(t)}(1-\epsilon).$$

2) Can be proved very similarly to part 1. Namely, the third product in (4.2) will extend over the primes $2 < p \le \log n$ and for the $\ge$ part we consider the integers n of the form $n = \prod_{2<p\le x} p^{t-1}$. □

Remark. Robin [9] has shown that if RH is false, then there are infinitely many integers n not in $\mathcal{R}$. As n ranges over these numbers, then by part 1 of Lemma 4.1 we must have $\max\{e_i\} \to \infty$, where $n = \prod_{i=1}^{k} p_i^{e_i}$.

Proof of Theorem 4.1.
1) Follows from part 1 of Lemma 4.1 on letting t tend to infinity. A direct proof (similar to that of Lemma 4.1) can also be given, see e.g. [3]. This result was proved first by Gronwall in 1913.
2) Follows from part 1 of Lemma 4.1 with $t = 2$.
3) Follows on letting t tend to infinity in part 2 of Lemma 4.1.
4) Landau (1909).
5) Since $f_2(n) \le f_2(\kappa(n))$, part 5 is a consequence of part 4.
6) A consequence of part 4 and the fact that for odd integers n and $a \ge 1$ we have $f_2(2^a n) = 2f_2(n)(1 + O((\log n \log\log n)^{-1}))$.
7) Consider numbers of the form $n = \prod_{p\le x} p^{t-1}$ and let t tend to infinity. These are squarefull for $t \ge 3$ and using them the $\ge$ part of the assertion follows. The $\le$ part follows of course from part 3.
8) It is enough here to consider the squarefull numbers of the form $n = \prod_{p\le x} p^2$. □

5. Reduction to Hardy-Ramanujan integers

Recall that $p_1, p_2, \ldots$ denote the consecutive primes. An integer of the form $\prod_{i=1}^{s} p_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ we will call an *Hardy-Ramanujan integer*. We name them after Hardy and Ramanujan who in a paper entitled 'A problem in the analytic theory of numbers' (Proc. London Math. Soc. **16** (1917), 112–132) investigated them. See also [7, pp. 241–261], where this paper is retitled 'Asymptotic formulae for the distribution of integers of various types'.

Proposition 5.1. *If Robin's inequality holds for all Hardy-Ramanujan integers* $5041 \leq n \leq x$*, then it holds for all integers* $5041 \leq n \leq x$*. Asymptotically there are* $\exp((1+o(1))2\pi\sqrt{\log x/3\log\log x})$ *Hardy-Ramanujan numbers* $\leq x$*.*

Hardy and Ramanujan proved the asymptotic assertion above. The proof of the first part requires a few lemmas.

Lemma 5.1. *For* $e > f > 0$*, the function*

$$g_{e,f} : x \to \frac{1-x^{-e}}{1-x^{-f}}$$

is strictly decreasing on $(1, +\infty]$*.*

Proof. For $x > 1$, we have

$$g'_{e,f}(x) = \frac{ex^f - fx^e + f - e}{x^{e+f+1}(1-x^{-f})^2}.$$

Let us consider the function $h_{e,f} : x \to ex^f - fx^e + f - e$. For $x > 1$, we have $h'_{e,f}(x) = efx^f\left(1 - x^{e-f}\right) < 0$. Consequently $h_{e,f}$ is decreasing on $(1, +\infty]$ and since $h_{e,f}(1) = 0$, we deduce that $h_{e,f}(x) < 0$ for $x > 1$ and so $g_{e,f}(x)$ is strictly decreasing on $(1, +\infty]$. □

Remark. In case f divides e, then

$$\frac{1-x^{-e}}{1-x^{-f}} = 1 + \frac{1}{x^f} + \frac{1}{x^{2f}} + \cdots + \frac{1}{x^e},$$

and the result is obvious.

Lemma 5.2. *If* $q > p$ *are primes and* $f > e$*, then*

$$\frac{\sigma\left(p^f q^e\right)}{p^f q^e} > \frac{\sigma\left(p^e q^f\right)}{p^e q^f}. \tag{5.1}$$

Proof. Note that the inequality (5.1) is equivalent with

$$(1-p^{-1-f})(1-p^{-1-e})^{-1} > (1-q^{-1-f})(1-q^{-1-e})^{-1}.$$

It follows by Lemma 5.1 that the latter inequality is satisfied. □

Let $n = \prod_{i=1}^{s} {q_i}^{e_i}$ be a factorisation of n, where we ordered the primes q_i in such a way that $e_1 \geq e_2 \geq e_3 \geq \cdots$ We say that $\bar{e} = (e_1, \ldots, e_s)$ is the exponent pattern of the integer n. Note that $\Omega(n) = e_1 + \ldots + e_s$, where $\Omega(n)$ denotes the total number of prime divisors of n. Note that $\prod_{i=1}^{s} {p_i}^{e_i}$ is the minimal number having exponent pattern $\bar{e}$. We denote this (Hardy-Ramanujan) number by $m(\bar{e})$.

Lemma 5.3. *We have*

$$\max\left\{\frac{\sigma(n)}{n} \mid n \text{ has factorisation pattern } \bar{e}\right\} = \frac{\sigma(m(\bar{e}))}{m(\bar{e})}.$$

Proof. Since clearly $\sigma(p^e)/p^e > \sigma(q^e)/q^e$ if $p < q$, the maximum is assumed on integers $n = \prod_{i=1}^{s} {p_i}^{f_i}$ having factorisation pattern $\bar{e}$. Suppose that n is any number of this form for which the maximum is assumed, then by Lemma 5.2 it follows that $f_1 \geq f_2 \geq \cdots \geq f_s$ and so $n = m(\bar{e})$. □

Lemma 5.4. *Let $\bar{e}$ denote the factorisation pattern of n.*

(1) *If $\sigma(n)/n \geq e^{\gamma} \log\log n$, then $\sigma(m(\bar{e}))/m(\bar{e}) \geq e^{\gamma} \log\log m(\bar{e})$.*
(2) *If $\sigma(m(\bar{e}))/m(\bar{e}) < e^{\gamma} \log\log m(\bar{e})$, then $\sigma(n)/n < e^{\gamma} \log\log n$ for every integer n having exponent pattern $\bar{e}$.*

Proof. A direct consequence of the fact that $m(\bar{e})$ is the smallest number having exponent pattern $\bar{e}$ and Lemma 5.3. □

Lemma 5.5. *Let $\bar{e}$ denote the factorisation pattern of n. If $n \geq 5041$ and $m(\bar{e}) \leq 5040$, then n is in $\mathcal{R}$.*

Proof. Suppose that $m(\bar{e}) \leq 5040$. Since $\max\{\omega(r) : r \leq 5040\} = 5$ we must have $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4} p_5^{e_5}$ and so

$$\frac{\sigma(n)}{n} \leq \prod_{p \leq 11} \frac{1 - p^{-5}}{1 - p^{-1}} = 4.6411\cdots$$

Assume that $n \notin \mathcal{R}$ and $n \geq 5041$. We infer that

$$4.6411\cdots = \prod_{p \leq 11} \frac{1 - p^{-5}}{1 - p^{-1}} \geq e^{\gamma} \log\log n,$$

whence $\log n \leq 13.55$. A MAPLE computation now shows that $n \in \mathcal{R}$, contradicting our assumption that $n \notin \mathcal{R}$. □

On invoking the second part of Lemma 5.4 and Lemma 5.5, the proof of Proposition 5.1 is completed.

6. The proof of Theorem 1.5

Our proof of Theorem 1.5 makes use of lemmas 6.1, 6.2 and 6.3.

Lemma 6.1. *Let $t \geq 2$ be fixed. Suppose that there exists a t-free integer exceeding* 5040 *that does not satisfy Robin's inequality. Let n be the smallest such integer. Then $P(n) < \log n$, where $P(n)$ denotes the largest prime factor of n.*

Proof. Write $n = r \cdot q$ with $P(n) = q$ and note that r is t-free. The minimality assumption on n implies that either $r \leq 5040$ and does not satisfy Robin's inequality or that r is in $\mathcal{R}$. First assume we are in the former case. Since 720 is the largest integer a in $\mathcal{A}$ with $P(a) \leq 5$ and $5 \cdot 720 \leq 5040$, it follows that $q \geq 7$. By Lemma 2.3 we then infer, using the assumption that $n > 5040$, that $n = qr$ is in $\mathcal{R}$; a contradiction. Thus we may assume that r is in $\mathcal{R}$ and therefore $r \geq 7$. We will now show that this together with the assumption $q \geq \log n$ leads to a contradiction, whence the result follows.

So assume that $q \geq \log n$. This implies that $q \log q \geq \log n \log\log n > \log n \log\log r$ and hence

$$\frac{q}{\log n} > \frac{\log\log r}{\log q}.$$

This implies that

$$\frac{q(\log\log n - \log\log r)}{\log q} > \frac{\log\log r}{\log q}, \tag{6.1}$$

where we used that

$$\frac{\log\log n - \log\log r}{\log q} = \frac{1}{\log n - \log r}\int_{\log r}^{\log n}\frac{dt}{t} > \frac{1}{\log n}.$$

Inequality (6.1) is equivalent with $(1 + 1/q)\log\log r < \log\log n$. Now we infer that

(6.2)

$$\frac{\sigma(n)}{n} = \frac{\sigma(qr)}{qr} \leq \left(1 + \frac{1}{q}\right)\frac{\sigma(r)}{r} < \left(1 + \frac{1}{q}\right)e^{\gamma}\log\log r < e^{\gamma}\log\log n,$$

where we used that σ is submultiplicative (that is $\sigma(qr) \leq \sigma(q)\sigma(r)$). The inequality (6.2) contradicts our assumption that $n \notin \mathcal{R}$. □

Lemma 6.2. *All 5-free Hardy-Ramanujan integers $n > 5040$ with $P(n) \leq 73$ satisfy Robin's inequality.*

Proof. There are 12649 5-free Hardy-Ramanujan integers n with $P(n) \leq 73$, that are easily produced using MAPLE. A further MAPLE computation learns that all integers exceeding 5040 amongst these (12614 in total) are in $\mathcal{R}$. □

Remark. On noting that $\prod_{p\le 73} p^4 < \prod_{p\le 20000} p$ and invoking Robin's result [9, p. 204] that an integer $n \notin \mathcal{R}$ with $n > 5040$ satisfies $n \ge \prod_{p\le 20000} p$, an alternative proof of Lemma 6.2 is obtained.

Lemma 6.3. *For $x \ge 3$ and $t \ge 2$ we have that*

$$\sum_{p\le x} \log\left(\frac{1-p^{-t}}{1-p^{-1}}\right) \le -\log\zeta(t) + \frac{t}{(t-1)}x^{1-t} + \gamma + \log\log x + \log\left(1+\frac{1}{\log^2 x}\right).$$

The proof of this lemma on its turn rests on the lemma below.

Lemma 6.4. *Put $R_t(x) = \prod_{p>x}(1-p^{-t})^{-1}$. For $x \ge 3$ and $t \ge 2$ we have that $\log(R_t(x)) \le tx^{1-t}/(t-1)$.*

Proof. We have

$$R_t(x) = -\sum_{p>x}\log\left(1-\frac{1}{p^t}\right) = \sum_{p>x}\sum_{m=1}^{\infty}\frac{1}{mp^{tm}} \le \sum_{p>x}\sum_{m=1}^{\infty}\frac{1}{(p^m)^t}$$
$$\le \sum_{n>x}\frac{1}{n^t} \le \frac{1}{x^t} + \sum_{n>x+1}\frac{1}{n^t} \le \frac{1}{x^{t-1}} + \int_x^{\infty}\frac{du}{u^t} = \frac{t}{t-1}x^{1-t}.$$

□

Proof of Lemma 6.3. On noting that $\prod_{p\le x}(1-p^{-t}) = R_t(x)/\zeta(t)$ and invoking Lemma 6.4 we obtain

$$\sum_{p\le x}\log\left(1-\frac{1}{p^t}\right) = -\log\zeta(t) + \log(R_t(x)) \le -\log\zeta(t) + \frac{t}{t-1}x^{1-t}.$$

On combining this estimate with Lemma 3.1, the estimate then follows. □

Lemma 6.5. *Let m be a 5-free integer such that $P(m) < \log m$ and m does not satisfy Robin's inequality. Then $P(m) \le 73$.*

Proof. Put $t = 5$. Write $P_t(x) = \prod_{p\le x}(1-p^{-t})/(1-p^{-1})$. Put $z = \log m$. The assumptions on m imply that $\sigma(m)/m \le P_t(z)$. This inequality in combination with Lemma 6.3 yields

$$\log\left(\frac{\sigma(m)}{m}\right) \le -\log\zeta(t) + \frac{t}{(t-1)z^{t-1}} + \gamma + \log\log z + \log\left(1+\frac{1}{\log^2 z}\right).$$

Once

$$-\log\zeta(t) + \frac{t}{t-1}z^{1-t} + \gamma + \log\log z + \log\left(1+\frac{1}{\log^2 z}\right) < \gamma + \log\log z,$$

Robin's inequality is satisfied. We infer that once we have found a $z_0 \geq 3$ such that

$$\frac{t}{t-1} z_0^{1-t} + \log\left(1 + \frac{1}{\log^2 z_0}\right) - \log\zeta(t) < 0,$$

then Robin's inequality will be satisfied in case $z \geq z_0$. One finds that $z_0 = 196$ will do. It follows that $z < 196$ and hence $\sigma(m)/m < P_5(193) = 9.18883221\ldots$. Note that if $e^\gamma \log\log m \geq P_5(193)$, then Robin's inequality is satisfied. So we conclude that $\log m \leq \exp(P_5(193)e^{-\gamma}) = 174.017694\ldots$. Since 173 is the largest prime < 175 we know that m must satisfy $\sigma(m)/m < P_5(173) = 8.992602079\ldots$. We now proceed as before, but with $P_5(193)$ replaced by $P_5(173)$. Indeed, this 'cascading down' can be repeated several times before we cannot reduce further. This is at the point where we have reached the conclusion that $z = \log m \leq 73$. Then we cannot reduce further since $\exp(P_5(73)e^{-\gamma}) > 73$. □

Proof of Theorem 1.5. By contradiction. So suppose a 5-free integer exceeding 5040 exists that does not satisfy Robin's inequality. We let n be the smallest such integer. By Lemma 6.1 it follows that $P(n) < \log n$, whence by Lemma 6.5 we infer that $P(n) \leq 73$. We will now show that n is a Hardy-Ramanujan number. On invoking Lemma 6.2 the proof is then completed.

It thus remains to establish that n is a Hardy-Ramanujan number. Let $\bar{e}$ denote the factorisation pattern of n. Note that $m(\bar{e})$ is 5-free and that $m(\bar{e}) \leq n$. By the minimality of n and part 1 of Lemma 5.4 it follows that we cannot have that $5041 \leq m(\bar{e}) < n$ and so either $m(\bar{e}) = n$, in which case we are done as $m(\bar{e})$ is a Hardy-Ramanujan number, or $m(\bar{e}) \leq 5040$, which by Lemma 5.5 leads us to conclude that $n \in \mathcal{R}$, contradicting our assumption that $n \notin \mathcal{R}$. □

By the method above we have not been able to replace 5-free by 6-free in Theorem 1.5 (this turns out to require a substantial computational effort). Recently J.-L. Nicolas kindly informed the authors of an approach (rather different from the one followed here and being less self-contained) that might lead to a serious improvement of the 5-free.

7. Acknowledgement

We thank J.-C. Lagarias for pointing out reference [2]. Furthermore, E. Bach, P. Dusart, O. Ramaré and, especially, J.-L. Nicolas, for their remarks. Keith Briggs we thank for his willingness to do large scale computations on our behalf. In the end, however, it turned out that only modest computations are needed in order to establish Theorem 6 (the main result of this paper). Also thanks are due to Masako Soyogi for pointing out an error in the proof of Proposition 1 in an earlier version.

We especially like to thank the referee for his meticulous proofreading and many helpful remarks. This led to a substantial improvement of the quality of our paper.

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