Journal de Théorie des Nombres
de Bordeaux ***9*** *(1997), 41–50*

Some Ω-results related to the fourth power moment of the Riemann Zeta-function and to the additive divisor problem

par JERZY KACZOROWSKI* ET BOGDAN SZYDŁO*

RÉSUMÉ. Nous donnons des estimations de type Ω relatives au terme reste dans la formule asymptotique pour le moment d'ordre quatre de la fonction zeta de Riemann et au terme reste dans le problème des diviseurs.

ABSTRACT. We improve the "two-sided" omega results in the fourth power mean problem for the Riemann zeta-function and in the additive divisor problem.

1. Introduction

Let $E_2(x)$ denote the remainder term in the asymptotic formula for the fourth power mean of the Riemann zeta-function:

$$\int_0^x |\zeta(\frac{1}{2}+it)|^4\,dt = xP(\log x) + E_2(x),$$

where P is a certain polynomial of degree four. Further, let $E(x,k)$ denote the remainder term in the additive divisor problem:

$$\sum_{n\le x} d(n)d(n+k) = xP_k(\log x) + E(x,k),$$

where k is a fixed positive integer, $d(n)$ stands for the number of positive divisors of n, and P_k is a certain quadratic polynomial.

In this note $E(x)$ will always denote $E_2(x)$ or $E(x,k)$.

Recent results on oscillations of these remainders can be written as follows

$$E(x) = \Omega_{\pm}(x^{1/2}) \qquad (x\to\infty). \tag{1}$$

They are due to Y. Motohashi [10] in the case of $E(x)=E_2(x)$, and to the second author [11] in the case of $E(x)=E(x,k)$.

*Supported by KBN Grant nr. 2 PO3A 028 09.
Manuscrit reçu le 13 mars 1996.

The results stated in (1) have in fact been obtained in an analogous way. First of all, the meromorphic continuation of the Mellin transforms

$$Z(s) = \int_1^\infty E(x)x^{-s-1}\,dx \qquad (\Re s > 1)$$

is needed. In the context of the fourth power mean of the Riemann zeta-function it was obtained by Y. Motohashi [10] and in the context of the additive divisor problem for the first time by L.A. Tahtadjan and A.I. Vinogradov [12] (cf. also [5] for some revision of [12]). The methods adopted to achieve this goal made essential use of the spectral theory of the hyperbolic Laplacian. It turns out that the only singularities of $Z(s)$ in the half plane $\Re s \geq 1/2$ are simple poles at $s = 1/2 + i\kappa$, where $1/4 + \kappa^2$ are certain eigenvalues of the hyperbolic Laplacian. It is well known that $1/4 + \kappa^2 > 0$. In particular, $Z(s)$ is regular at $s = 1/2$. The relevant "nonvanishing" of the residues:

$$r = \mathrm{Res}_{s=1/2+i\kappa} Z(s) \neq 0$$

for infinitely many $1/2 + i\kappa$, was obtained by Y. Motohashi [8],[9], cf. also [11]. This is crucial for a successful application of a certain general result of E. Landau [7] (cf. also [1]). The issue is the two-sided omega result (1).

In this note we indicate a possibility of further improvements of (1), cf. Theorems 1 and 2 below.

Firstly, we remark that quite detailed information on the size of $E(x)$ is now available, see [2], [3], [4] and [9]. Consequently, the mentioned above theorem of E. Landau may possibly be replaced by a more appropriate means, something like a recent generalization of it by the first author [6], Theorem 12. It states that if $f : [1, \infty) \to \mathbb{R}$ is a function such that

$$f(x) = O(x^\lambda) \tag{2}$$

for every $\lambda > \theta$, and which Mellin transform

$$F(s) = \int_1^\infty f(x)x^{-s-1}\,dx$$

is regular at $s = \theta$ and is *not* regular in any half-plane $\sigma > \theta - \eta$ with $\eta > 0$, then the Lebesgue measure of both sets

$$\{1 \leq x \leq T : f(x) > 0\} \quad \text{and} \quad \{1 \leq x \leq T : f(x) < 0\}$$

is $\Omega(T^{1-\varepsilon})$ for every $\varepsilon > 0$ as T tends to infinity. The first difficulty we are faced with when applying this result to errors like $E(x)$ is that we are unable to verify (2), though it conjecturally holds in this case. Actually, the following known estimates ([3], [2] resp.)

$$E_2(x) \ll x^{2/3} \log^c x \quad , \quad E(x,k) \ll x^{2/3+\varepsilon}$$

for every positive ε, where c is an effectively computable positive constant, coupled with a slight modification of the just mentioned Theorem 12 from [6] together with analytic properties of $Z(s)$ could produce the following estimates

$$\mu\{1 \le x \le T : E(x) > x^{1/2-\varepsilon}\} = \Omega(T^{5/6-\varepsilon}),$$
$$\mu\{1 \le x \le T : E(x) < -x^{1/2-\varepsilon}\} = \Omega(T^{5/6-\varepsilon}),$$

in which μ denotes the Lebesgue measure on the real axis and ε denotes an arbitrary positive real number.

We prove more than this.

THEOREM 1. *There exist constants $c > 0$ and $r_0 > 0$ such that for every $0 < a < r_0$ we have as $T \to \infty$*

$$\mu\{1 \le x \le T : E_2(x) > a\sqrt{x}\} = \Omega(T/\log^c T),$$
$$\mu\{1 \le x \le T : E_2(x) < -a\sqrt{x}\} = \Omega(T/\log^c T).$$

THEOREM 2. *For every fixed positive integer k there exists a constant $r(k) > 0$ such that for every $0 < a < r(k)$ and $\varepsilon > 0$ we have as $T \to \infty$*

$$\mu\{1 \le x \le T : E(x,k) > a\sqrt{x}\} = \Omega(T^{1-\varepsilon}),$$
$$\mu\{1 \le x \le T : E(x,k) < -a\sqrt{x}\} = \Omega(T^{1-\varepsilon}).$$

The main idea leading to these results is that instead of using individual estimates of type (2) we rather resort to estimates in mean, cf. Section 2, formula (5). This is crucial since A. Ivić and Y. Motohashi [4] proved recently the excellent inequalities

$$\int_0^U E_2(x)^2\,dx \ll U^2 \log^c U \tag{3}$$

with an effectively computable constant $c > 0$, and

$$\int_0^U E(x,k)^2\,dx \ll_{\varepsilon,k} U^{2+\varepsilon} \tag{4}$$

for every positive ε. We prove that (3) and (4) are sharp.

THEOREM 3. *We have as $T \to \infty$*

$$\int_T^{2T} E^2(x)\,dx = \Omega(T^2).$$

A. Ivić and Y. Motohashi conjectured in [3] that

$$\int_0^T E_2^2(x)\,dx = AT^2 + F_2(T), \quad F_2(T) = o(T^2)$$

as $T \to \infty$, and also that $F_2(T) = O(T^{3/2+\varepsilon})$ for every positive ε. The above Theorem 3 supports to some extent this conjecture. Finally, let us remark that Theorem 3 has the ordinary Ω-results

$$E(x) = \Omega(x^{1/2})$$

as corollaries. These were proved earlier than (1), cf. [3], [9].

Acknowledgement. We thank Professor Aleksandar Ivić for valuable remarks on the first version of this work.

2. A Laudau-type theorem for Mellin transforms

We formulate now a general theorem which plays the principal role in this note. Let θ be a fixed real number and $f(x)$ be real for $x \geq 1$. Suppose that for $U \geq 1$ we have

$$\int_U^{2U} f^2(x)\,dx \leq U^{2\theta+1}\eta(U), \tag{5}$$

where $\eta : [1, \infty) \to \mathbb{R}$ is a certain positive and nondecreasing continuous function such that for every $\varepsilon > 0$

$$\eta(x) = O_\varepsilon(x^\varepsilon) \tag{6}$$

as $x \to \infty$. Then for every $\sigma > \theta$ we have

$$\int_1^\infty |f(x)|x^{-\sigma-1}\,dx < \infty.$$

Indeed, from (5) and (6) it follows for $\sigma > \theta$ and $\delta > 0$

$$\begin{aligned}
\int_1^\infty |f(x)|x^{-\sigma-1}\,dx &= \int_1^\infty |f(x)|x^{-\sigma-1/2+\delta/2}x^{-(1+\delta)/2}\,dx \\
&\le \left(\int_1^\infty f^2(x)x^{-1-2\sigma+\delta}\,dx\right)^{1/2}\left(\int_1^\infty x^{-1-\delta}\,dx\right)^{1/2} \\
&\ll_\delta \left(\sum_{k=0}^\infty 2^{-k(1+2\sigma-\delta)}\int_{2^k}^{2^{k+1}} f^2(x)\,dx\right)^{1/2} \\
&\le \left(\sum_{k=0}^\infty 2^{-k(1+2\sigma-\delta)}2^{k(2\theta+1)}\eta(2^k)\right)^{1/2} \\
&\ll_\varepsilon \left(\sum_{k=0}^\infty 2^{-k(2(\sigma-\theta)-\delta-\varepsilon)}\right)^{1/2} < \infty,
\end{aligned}$$

if $\delta > 0$ and $\varepsilon > 0$ are chosen so that $\delta + \varepsilon < 2(\sigma - \theta)$.

Hence the Mellin transform

$$F(s) = \int_1^\infty f(x)x^{-s-1}\,dx$$

is holomorphic in the half-plane $\Re s = \sigma > \theta$. Moreover, suppose that for real numbers r and $t_0 \neq 0$ we have

$$\limsup_{\sigma\to\theta+}(\sigma-\theta)|F(\sigma+it_0)| \ge r. \tag{7}$$

Under these assumptions we have the following result.

THEOREM 4. *For every* $a < r + \liminf_{\sigma\to\theta+}(\sigma-\theta)F(\sigma)$ *and for every* $b > -r + \limsup_{\sigma\to\theta+}(\sigma-\theta)F(\sigma)$ *we have as* $T \to \infty$

$$\mu\{1 \le x \le T : f(x) > ax^\theta\} = \Omega(T/\eta(T)), \tag{8}$$

$$\mu\{1 \le x \le T : f(x) < bx^\theta\} = \Omega(T/\eta(T)), \tag{9}$$

where $\mu(A)$ *denotes the Lebesgue measure of a set* A *of real numbers.*

Let us observe that the above theorem is in fact a tauberian-type result. Indeed, we do not need any information on F to the left of θ. In particular, we do not assume that F has analytic continuation to any region larger than the half-plane of absolute convergence. Applying this theorem to $E(x)$ requires less information on the analytic character of $Z(s)$ than is in fact available. In particular, we make no use of its meromorphic continuation on the half-plane $\sigma < 1/2$. However, we require that $Z(s)$ behaves in a rather special, "singular" way when s approaches to the vertical $\sigma = 1/2$ from right. So, it is sufficient and natural for us to employ the fact that $Z(s)$ has a simple pole (with a non-zero residue) on the line $\sigma = 1/2$, but no singularity at $s = 1/2$, cf. section 1.

3. Proofs

Theorems 1 and 2 follow from Theorem 4 by taking $f(x) = E_2(x)$ or $f(x) = E(x,k)$. Indeed, the mean square estimates (3) and (4) and the analytic properties of Mellin transforms $Z(s)$ (cf. section 1) show that all assumptions of Theorem 4 are then satisfied; we have $\theta = 1/2$, $r = \sup_{\kappa\in\mathbb{K}} |\mathrm{Res}_{s=1/2+i\kappa} Z(s)| > 0$, and $\lim_{\sigma\to\theta^+}(\sigma-\theta)Z(\sigma) = 0$.

In order to deduce Theorem 3 from Theorem 4 let us put

$$\eta(U) := \sup_{1\le T\le U} \frac{1}{T^2}\int_T^{2T} E^2(x)\,dx$$

and suppose the contrary, i.e. that

$$\int_T^{2T} E^2(x)\,dx = o(T^2) \tag{10}$$

holds. Then $\eta(U)$ is bounded as $U\to\infty$. From Theorem 4 we have for a certain sequence $T\to\infty$ and $c > 0$

$$\mu\{T \le x \le 2T : E(x) > \frac{r}{2}x^{1/2}\} \ge cT$$

with $r > 0$ is as above. From this follows

$$\int_T^{2T} E^2(x)\,dx \ge c\frac{r^2}{4}T^2,$$

which contradicts (10). Hence it suffices to prove Theorem 4.

Proof of Theorem 4. Observe that (9) follows from (8) by replacing f by $-f$. It suffices therefore to prove (8).

Let us put

$$g(x) = f(x) - ax^{\theta} \qquad (x \ge 1)$$

and

$$G(s) = \int_1^{\infty} g(x)x^{-s-1}\,dx \qquad (\sigma > \theta).$$

Then

$$G(s) = F(s) - \frac{a}{s-\theta}. \tag{11}$$

Further, let us denote

$$g_+(x) = \max(g(x), 0) \quad , \quad g_-(x) = \max(-g(x), 0) \qquad (x \geq 1),$$

$$G_+(s) = \int_1^\infty g_+(x) x^{-s-1}\, dx \quad , \quad G_-(s) = \int_1^\infty g_-(x) x^{-s-1}\, dx \qquad (\sigma > \theta).$$

Then

$$G = G_+ - G_-.$$

Now, suppose that (8) is not true. With

$$A = \{x \geq 1 : g(x) > 0\},$$

$$M(T) = \mu(A \cap [1, T])$$

it means that

$$M(T) = o(T/\eta(T)) \tag{12}$$

as $T \to \infty$.

For any real t, $\sigma = \theta + \delta$, $\delta > 0$ we have

$$\begin{aligned}
(\sigma - \theta)|G_+(\sigma + it)| &\leq \delta \int_1^\infty g_+(x) x^{-\sigma-1}\, dx \\
&= \delta \int_A g(x) x^{-\theta-\delta-1}\, dx \\
&= \delta \int_A \left(g(x) \frac{x^{-1/2-\theta-\delta/2}}{\sqrt{\eta(x)}} \right) \left(\sqrt{\eta(x)} x^{-1/2-\delta/2} \right) dx \\
&\leq \delta \left(\int_A g^2(x) \frac{x^{-1-2\theta-\delta}}{\eta(x)}\, dx \right)^{1/2} \left(\int_A \eta(x) x^{-1-\delta}\, dx \right)^{1/2} \\
&= \delta \alpha^{1/2} \beta^{1/2}, \qquad (13)
\end{aligned}$$

say. From (5) we have

$$\begin{aligned}
\alpha &\leq \int_1^\infty g^2(x) \frac{x^{-1-2\theta-\delta}}{\eta(x)}\, dx \\
&= \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} g^2(x) \frac{x^{-1-2\theta-\delta}}{\eta(x)}\, dx \\
&\leq \sum_{k=0}^\infty \frac{2^{-k(1+2\theta+\delta)}}{\eta(2^k)} \int_{2^k}^{2^{k+1}} g^2(x)\, dx \\
&\leq \sum_{k=0}^\infty 2^{-k\delta} \ll 1/\delta. \qquad (14)
\end{aligned}$$

Integrating by parts it follows that

$$\begin{aligned}\beta &= \int_1^\infty \eta(x)x^{-1-\delta}\,dM(x)\\ &= \eta(x)M(x)x^{-1-\delta}\Big|_1^\infty - \int_1^\infty M(x)\,d(\eta(x)x^{-1-\delta})\\ &= -\int_1^\infty M(x)\,d(\eta(x)x^{-1-\delta}).\end{aligned}$$

But observe that for any $1 \le x_1 \le x_2$ we have

$$\begin{aligned}-M(x_1)(\eta(x_2)x_2^{-1-\delta} - \eta(x_1)x_1^{-1-\delta}) &\le M(x_1)\eta(x_1)(1+\delta)\int_{x_1}^{x_2} x^{-2-\delta}\,dx\\ &\le (1+\delta)\int_{x_1}^{x_2} \eta(x)M(x)x^{-2-\delta}\,dx,\end{aligned}$$

because the functions η and M are nondecreasing. Hence, by well-known properties of the Riemann-Stieltjes integral, it follows that

$$\beta \le (1+\delta)\int_1^\infty \eta(x)M(x)x^{-2-\delta}\,dx. \tag{15}$$

Next, let us fix a positive ε. From (12) it follows that there exists $x_0 \ge 1$ such that

$$M(x) \le \varepsilon^2 \frac{x}{\eta(x)}$$

for $x \ge x_0$. From (15) we have therefore

$$\begin{aligned}\beta &\ll \int_1^{x_0} \eta(x)M(x)x^{-2}\,dx + \varepsilon^2\int_{x_0}^\infty x^{-1-\delta}\,dx\\ &\le B(\varepsilon) + \varepsilon^2/\delta, \qquad (16)\end{aligned}$$

say. Hence, from (13), (14) and (16) we have for any fixed $\varepsilon > 0$

$$(\sigma-\theta)|G_+(\sigma+it)| \ll \sqrt{\delta B(\varepsilon)} + \varepsilon.$$

Consequently, (12) implies that

$$\lim_{\sigma\to\theta+}(\sigma-\theta)G_+(\sigma+it) = 0 \tag{17}$$

for all real t. Observe also that for $\sigma > \theta$ we have

$$|G_-(\sigma+it)| \le \int_1^\infty g_-(x)x^{-\sigma-1}\,dx = G_-(\sigma). \tag{18}$$

From (7), (11), (17) and (18) we deduce further that for a certain sequence of σ tending to θ from above we have

$$\begin{aligned} r &\le (\sigma-\theta)|F(\sigma+it_0)|+o(1)=(\sigma-\theta)|G(\sigma+it_0)+\frac{a}{\sigma+it_0-\theta}|+o(1) \\ &\le (\sigma-\theta)|G(\sigma+it_0)|+o(1) \\ &\le (\sigma-\theta)|G_+(\sigma+it_0)|+(\sigma-\theta)|G_-(\sigma+it_0)|+o(1) \\ &\le (\sigma-\theta)G_-(\sigma)+o(1)=(\sigma-\theta)(G_+(\sigma)-G(\sigma))+o(1) \\ &= -(\sigma-\theta)G(\sigma)+o(1)=a-(\sigma-\theta)F(\sigma)+o(1) \\ &\le a-\liminf_{\sigma\to\theta+}(\sigma-\theta)F(\sigma)+o(1). \end{aligned}$$

This contradicts our assumption that $a < r + \liminf_{\sigma\to 0^+}(\sigma-\theta)F(\sigma)$. Therefore (8) follows.

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Jerzy Kaczorowski, Bogdan Szydło
Faculty of Mathematics and Computer Science
A. Mickiewicz University
ul. Matejki 48/49
60-769 Poznań, Poland
e-mail: kjerzy@math.amu.edu.pl