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Limit theorems for the Matsumoto Zeta-function

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Résumé. On démontre deux théorèmes limites fonctionnels pondérés pour la fonction introduite par K. Matsumoto.

Abstract. In this paper two weighted functional limit theorems for the function introduced by K. Matsumoto are proved.

Let $\mathbb{N}$ denote the set of all natural numbers. For any $m \in \mathbb{N}$, we define a positive integer $g(m)$. Let $a_m^{(j)}$ be complex numbers and $f(j,m)$, $1 \le j \le g(m)$, $m \in \mathbb{N}$, be natural numbers. Now we can define the polynomials

$$A_m(X) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)}\, X^{f(j,m)}\right)$$

of degree $f(1,m) + \cdots + f\big(g(m), m\big)$. Let $s = \sigma + it$ be a complex variable, and let p_n denote the nth prime number. In [5] K. Matsumoto introduced the following zeta–function

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}\left(p_m^{-s}\right)$$

and under some hypotheses on $g(m)$, $a_m^{(j)}$ and $\varphi(s)$ he proved the limit theorems for $\log \varphi(s)$ in the complex plane $\mathbb{C}$.

Let B denote a number (not always the same) bounded by a constant. Suppose that

$$(1) \qquad g(m) = B p_m^{\alpha}\,, \qquad \left|a_m^{(j)}\right| \le p_m^{\beta}$$

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with some non–negative constats α and β. Let meas $\{A\}$ denote the Lebesgue measure of the set A, and, for $T > 0$, let

$$\nu_T^t(\dots) = \frac{1}{T}\,\text{meas}\,\big\{t \in [0,T]\,,\dots\big\}$$

where in place of dots we indicate a condition satisfied by t. Under the condition (1) $\varphi(s)$ is a holomorphic function in the half–plane $\sigma > \alpha + \beta + 1$ with no zeros. Let, for $\sigma > \beta$,

$$\log\varphi(s) = -\sum_{m=1}^{\infty}\sum_{j=1}^{g(m)} \text{Log}\big(1 - a_m^{(j)}\,p_m^{-f(j,m)s}\big)\,,$$

and let R denote a closed rectangle in the complex plane with the edges parallel to the axes.

Theorem A. (Matsumoto [5]). *Suppose that $\sigma_o > \alpha + \beta + 1$. Then there exists the limit*

$$\lim_{T\to\infty} \nu_T^t\big(\log\varphi(\sigma_o + it) \in R\big)\,.$$

If we assume that $\varphi(s)$ can be meromorphically continued to the region $\sigma \geq \rho_o$, $\rho_o < \alpha + \beta + 1$, then the theorem analogue to theorem A in this region was also proved in [5]. More precisely, let $\varphi(s)$ by meromorphic in the half–plane $\sigma \geq \rho_o \geq \alpha + \beta + \frac{1}{2}$, all poles in this region be included in a compact set, and, for $\sigma \geq \rho_o$,

$$\varphi(\sigma + it) = B\,|\,t\,|^{\delta} \tag{2}$$

with some positive δ. Moreover, let

$$\int_0^T \big|\varphi(\rho_o + it)\big|^2\,dt = BT\,.$$

We put

$$G = \big\{s \in \mathbb{C}, \sigma \geq \rho_o\big\} \setminus \bigcup_{s'=\sigma'+it'} \big\{s = \sigma + it', \rho_o \leq \sigma \leq \sigma'\big\}$$

where $s' = \sigma' + it'$ runs all possible zeros and poles of $\varphi(s)$ in the strip $\rho_o \leq \sigma \leq \alpha + \beta + 1$. Define $\varphi(\sigma_o + it_o)$ for $\sigma_o + it_o \in G$ by the analytic continuation along the path $s = \sigma + it_o$, $\sigma \geq \sigma_o$.

THEOREM B. (Matsumoto, [5]). *Suppose that $\sigma_o \geq \rho_o$ then there exists the limit*

$$\lim_{T\to\infty} \nu_T^t\big(\sigma_o + it \in G,\ \log\varphi(\sigma_o + it) \in R\big)\,.$$

In fact Theorems A and B are the limit theorems in sense of weak convergence of probability mesures on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ where by $\mathcal{B}(S)$ we denote the class of Borel sets of the space S. Our aim is to give a generalization of Theorems A and B in sense of [1] and [4], i.e. to prove the weighted functional limit theorems for the function $\varphi(s)$.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $d(s_1, s_2)$ be a metric on $\mathbb{C}_\infty$ given by the formulae

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1+|s_1|^2}\,\sqrt{1+|s_2|^2}}\,, \quad d(s, \infty) = \frac{2}{\sqrt{1+|s|^2}}\,, \quad d(\infty, \infty) = 0\,.$$

Here s, s_1, $s_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$. Let $D = \{s \in \mathbb{C}, \sigma > \alpha + \beta + 1\}$. Denote by $H(D)$ the space of analytic on D functions $f : D \to (\mathbb{C}_\infty, d)$ equiped with the topology of uniform convergence on compacta. In this topology a sequence $\{f_n, f_n \in H(D)\}$ converges to the function $f \in H(D)$ if

$$d\big(f_n(s), f(s)\big) \to 0$$

as $n \to \infty$ uniformly in s on compact subsets of D. On the other hand let T_o be a fixed positive number, and let $w(\tau)$ be a positive function of bounded variation on $[T_o, \infty)$. Let us put

$$U = U(T, w) = \int_{T_o}^{T} w(\tau)\, d\tau\,.$$

Suppose that $\lim\limits_{T\to\infty} U(T, w) = \infty$ and consider the probability measure

$$P_{T,w}(A) = \frac{1}{U}\int_{T_o}^{T} w(\tau)\, I_{\{\tau:\,\varphi(s+i\tau)\in A\}}\, d\tau\,, \qquad A \in \mathcal{B}\big(H(D)\big)\,.$$

Here I_A denote the indicator function of the set A.

Theorem 1. *There exists a probability measure* P_w *on* $\big(H(D), \mathcal{B}(H(D))\big)$ *such that the measure* $P_{T,w}$ *converges weakly to* P_w *as* $T \to \infty$.

Now let $D_1 = \{s \in \mathbb{C}, \sigma > \rho_o\}$ where $\alpha + \beta + \frac{1}{2} \le \rho_o < \alpha + \beta + 1$. Denote by $M(D_1)$ the space of meromorphic on D_1 functions $f : D_1 \to (\mathbb{C}_\infty, d)$ equiped with the topology of uniform convergence on compacta. Suppose that for the functions $w(\tau)$ and $\varphi(s)$ the estimate

$$\int_{T_o}^{T} w(\tau) \left|\varphi(\sigma + i\tau + it)\right|^2 d\tau = BU(1 + |\,t\,|) \tag{3}$$

is satisfied for all $\sigma > \rho_o$ and all $t \in \mathbb{R}$. Consider the probability measure

$$Q_{T,w}(A) = \frac{1}{U} \int_{T_o}^{T} w(\tau)\, I_{\{\varphi(s+i\tau) \in A\}}\, d\tau\,, \qquad A \in \mathcal{B}\big(M(D_1)\big)\,.$$

Theorem 2. *Let the function* $\varphi(s)$ *be meromorphic in the half–plane* $\sigma > \rho_o$. *Suppose that all poles in this region are included in a compact set, and that the estimates (2) and (3) hold. Then there exists a probability measure* Q_w *on* $\Big(M(D_1), \mathcal{B}\big(M(D_1)\big)\Big)$ *such that the measure* $Q_{T,w}$ *converges weakly to* Q_w *as* $T \to \infty$.

For the proof of Theorems 1 and 2 we will use the following auxiliary results.

Let S_1 and S_2 be two metric spaces. Let $h : S_1 \to S_2$ be a measurable function. Then every probability measure P on $\big(S_1, \mathcal{B}(S_1)\big)$ induces on $\big(S_2, \mathcal{B}(S_2)\big)$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_2)$. Let P, P_n be the probability measures on $\big(S_1, \mathcal{B}(S_1)\big)$.

Lemma 1. *Let* $h : S_1 \to S_2$ *be a continuous function. Then the weak convergence of* P_n *to* P *implies that of* $P_n h^{-1}$ *to* Ph^{-1} *as* $n \to \infty$.

Proof. This lemma is a particular case of Theorem 5.1 from [2].

Now let S be a separable metric space with a metric ρ, and let Y_n, X_{1n}, X_{2n}, ... be the S–valued random elements defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following assertion is true (Theorem 4.2 from [2]).

Lemma 2. *Suppose that* $X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$ *for each* k *and also* $X_k \xrightarrow[k\to\infty]{\mathcal{D}} X$. *If for every* $\varepsilon > 0$

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}\big(\rho(X_{kn}, Y_n) \geq \varepsilon\big) = 0\,,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Let γ denote the unite circle on complex plane that is $\gamma = \{s \in \mathbb{C}, \mid s \mid = 1\}$, and let P be a probability measure on $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$, $r \in \mathbb{N}$. The Fourier transform $g(k_1, \dots, k_r)$ of the measure P is defined by equality

$$g(k_1, \dots, k_r) = \int\limits_{\gamma^r} x_1^{k_1} \dots x_r^{k_r}\, dP$$

where $k_j \in \mathbb{Z}$, $x_j \in \gamma$, $j = 1, \dots, r$.

Lemma 3. *Let* $\{P_n\}$ *be a sequence of probability measures on* $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$ *and let* $\{g_n(k_1, \dots, k_r)\}$ *be a sequence of corresponding Fourier transforms. Suppose that for every set* $(k_1, \dots, k_r)$ *of integers the limit* $g(k_1, \dots, k_r) = \lim\limits_{n\to\infty} g_n(k_1, \dots, k_r)$ *exists. Then there exists a probability measure* P *on* $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$ *such that* P_n *converges weakly to* P *as* $n \to \infty$. *Moreover,* $g(k_1, \dots, k_r)$ *is the Fourier transform of* P .

Proof. Lemma is a special case of the continuity theorem for probability measures on compact Abelian groups, see, for example, [3].

The family $\{P\}$ of probability measures on $\big(S, \mathcal{B}(S)\big)$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence.

The family $\{P\}$ is tight if for arbitrary $\epsilon > 0$ there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all P from $\{P\}$.

Lemma 4. (The Prokhorov theorem). *If the family of probability measures* $\{P\}$ *is tight, then it is relatively compact.*

Proof of the lemma is contained in [2].

Let G be a region in $\mathbb{C}$. The family of functions regular on G is said to be compact on G if every sequence of this family contains a subsequence which converges uniformly on every compact subset $K \subset G$.

Lemma 5. *If the family of functions regular on* G *is uniformly bounded on every compact subset* $K \subset G$, *then it is compact on* G.

Proof can be found, for example, in [6].

Proof of Theorem 1. As it was noted in [5], the function $\varphi(s)$ in the half–plane $\sigma > \alpha + \beta + 1$ is presented by absolutely convergent Dirichlet series

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}.$$

First we will prove a limit theorem in the space $H(D)$ for the Dirichlet polynomial

$$\varphi_n(s) := \sum_{m=1}^{n} \frac{b_m}{m^s}.$$

Let

$$P_{T,\varphi_n,w}(A) = \frac{1}{U} \int\limits_{T_o}^{T} w(\tau)\, I_{\{\tau:\, \varphi_n(s+it)\in A\}}\, d\tau\,, \qquad A \in \mathcal{B}\big(H(D)\big).$$

We will prove that there exists a probability measure $P_{\varphi_n,w}$ on $\big(H(D), \mathcal{B}(H(D))\big)$ such that $P_{T,\varphi_n,w}$ converges weakly to $P_{\varphi_n,w}$ as $T \to \infty$.

Let $p_1, \dots, p_r$ be the distinct primes which divide the product

$$\prod_{m=1}^{n} m\,,$$

and let

$$\Omega_r = \prod_{j=1}^{r} \gamma_{p_j}$$

where γ_{p_j} is the unite circle on $\mathbb{C}$ for all $j = 1, \dots, r$. Let us define the function $h_n : \Omega_r \to H(D)$ by the formula

$$h_n(x_1, \dots, x_r) = \sum_{k=1}^{n} \frac{b_k}{k^s \prod\limits_{p_j^{\alpha_j} \| k} x_j^{\alpha_j}}$$

for $(x_1, \dots, x_r) \in \Omega_r$. Here $p^\alpha \| k$ means that $p^\alpha | k$ but $p^{\alpha+1} \nmid k$. Clearly, the function h_n is continuous on Ω_r and

$$\varphi_n(s + i\tau) = h_n\big(p_1^{i\tau}, \dots, p_r^{i\tau}\big)\,. \tag{4}$$

Now we define on $\big(\Omega_r, \mathcal{B}(\Omega_r)\big)$ the probability measure

$$\mu_{T,w}(A) = \frac{1}{U} \int_{T_o}^{T} w(\tau)\, I_{\{\tau : (p_1^{i\tau}, \dots, p_r^{i\tau}) \in A\}}\, d\tau\,.$$

Then the Fourier transform $g_T(k_1, \dots, k_r)$, $k_j \in \mathbb{Z}$, $j = 1, \dots, r$, of $\mu_{T,w}$ is given by the formula

$$g_T(k_1, \dots, k_r) = \int_{\Omega_r} x_1^{k_1} \dots x_r^{k_r}\, d\mu_{T,w} = \frac{1}{U} \int_{T_o}^{T} w(\tau) \prod_{j=1}^{r} p_j^{ik_j\tau}\, d\tau\,.$$

Hence, integrating by parts and using the properties of the function $w(\tau)$, we fing that

$$g_T(k_1, \dots, k_r) = \begin{cases} 1 & \text{if} \quad (k_1, \dots, k_r) = (0, \dots, 0), \\ \dfrac{B}{U\Big|\sum\limits_{j=1}^{r} k_j \log p_j\Big|} & \text{if} \quad (k_1, \dots, k_r) \neq (0, \dots, 0). \end{cases}$$

Since the logarithms of prime numbers are linearly independent over the field of rational numbers, whence we deduce that

$$g_T(k_1, \dots, k_r) \to \begin{cases} 1 & \text{if} \quad (k_1, \dots, k_r) = (0, \dots, 0)\,, \\ 0 & \text{if} \quad (k_1, \dots, k_r) \neq (0, \dots, 0) \end{cases}$$

as $T \to \infty$. Therefore, by Lemma 3 the measure $\mu_{T,w}$ converges weakly to the Haar measure m_r on $\big(\Omega_r, \mathcal{B}(\Omega_r)\big)$ as $T \to \infty$. Taking into account the continuity of the function h_n and the formula (4), we obtain in view of Lemma 1 that the measure $P_{T,\varphi_n,w}$ converges weakly to the measure $m_r h_n^{-1}$ as $T \to \infty$.

Let K be a compact subset of the half–plane D. Since the Dirichlet series for $\varphi(s)$ is absolutely and hence uniformly convergent for $\sigma > \alpha + \beta + 1$, we have that

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_o}^{T} w(\tau) \sup_{s \in K} \big|\varphi(s + i\tau) - \varphi_n(s + i\tau)\big|\, d\tau = 0. \tag{5}$$

Let θ be a random variable defined on the probability space $\big(\mathcal{A}, \mathcal{B}(\mathcal{A}), \mathbb{P}\big)$ with values on $[T_o, T\,]$ and having the distribution

$$\mathbb{P}(\theta \in A) = \frac{1}{U} \int_{T_o}^{T} w(\tau)\, d\tau\,, \qquad A \in \mathcal{B}\big([T_o, T\,]\big)\,.$$

We take

$$X_{T,n,w}(s) = \varphi_n(s + i\theta).$$

Then from weak convergence of the measure $P_{T,\varphi_n,w}$ we have that

$$X_{T,n,w}(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,w} \tag{6}$$

where X_n denote an $H(D)$–valued random element with the distribution

$$m_r h_n^{-1} := P_{\varphi_n,w} \, .$$

It is well known that there is a sequence $\{K_n\}$ of compact subsets of D such that

$$D = \bigcup_{n=1}^{\infty} K_n \, ,$$

and the sets K_n can be choosen to satisfy the following conditions:

a) $K_n \subset K_{n+1}$,

b) if K is a compact and $K \subset D$, then $K \subset K_n$ for some n.

For $f, g \in H(D)$ let

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

where

$$\rho_n(f,g) = \sup_{s\in K_n} \big| f(s) - g(s) \big| \, .$$

Then from the remark made above we have that ρ is a metric on $H(D)$ which induces its topology.

Let $l \in \mathbb{N}$. Then by Chebyshev's inequality

$$\mathbb{P}\Big(\sup_{s\in K_l} \big| X_{T,n,w}(s) \big| > M_l \Big) \le \frac{1}{M_l U} \int_{T_o}^{T} w(\tau) \sup_{s\in K_l} \big| \varphi_n(s+i\tau) \big| \, d\tau \, .$$

Therefore

$$\begin{aligned} &\overline{\lim_{T\to\infty}} \mathbb{P}\Big(\sup_{s\in K_l} \big| X_{T,n,w}(s) \big| > M_l \Big) \\ &\qquad \le \frac{1}{M_l} \sup_{n\ge 1} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_o}^{T} w(\tau) \sup_{s\in K_l} \big| \varphi_n(s+i\tau) \big| \, d\tau \, . \end{aligned} \tag{7}$$

Since the series for $\varphi(s)$ converges absolutely on D, we have that

$$\sup_{n\geq 1} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_o}^{T} w(\tau) \sup_{s\in K_l} \left|\varphi_n(s+i\tau)\right| d\tau \leq R_l < \infty \,. \tag{8}$$

Let ε be an arbitrary positive number, and let

$$M_l = \frac{R_l 2^l}{\varepsilon} \,.$$

Then (7) and (8) give

$$\overline{\lim_{T\to\infty}} \mathbb{P}\Big(\sup_{s\in K_l} \left|X_{T,n,w}(s)\right| > M_l \Big) \leq \frac{\varepsilon}{2^l} \tag{9}$$

for all $l \in \mathbb{N}$. Let the function $h : H(D) \to \mathbb{R}$ be defined by the formula

$$h(f) = \sup_{s\in K_l} |f(s)| \,, \qquad f \in H(D) \,.$$

Then, clearly, h is continuous, and

$$\sup_{s\in K_l} \left|X_{T,n,w}(s)\right| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K_l} \left|X_{n,w}(s)\right|$$

according to (6), continuity of h and Lemma 1. Hence, using the inequality (9), we find that

$$\mathbb{P}\Big(\sup_{s\in K_l} \left|X_{n,w}(s)\right| > M_l \Big) \leq \frac{\varepsilon}{2^l} \tag{10}$$

for all $l \in \mathbb{N}$. Let us define

$$H_\varepsilon = \Big\{ f \in H(D),\ \sup_{s\in K_l} |f(s)| \leq M_l,\ l \geq 1 \Big\} \,.$$

Then the family of functions H_ε is uniformly bounded on every compact $K \subset D$. But by Lemma 5 then it is compact on D. In view of (10)

$$\mathbb{P}\big(X_{n,w}(s) \in H_\varepsilon \big) \geq 1 - \varepsilon \,,$$

or, since $P_{n,w}$ is the distribution of the random element $X_{n,w}$,

$$P_{n,w}(H_\varepsilon) \geq 1 - \varepsilon$$

for all $n \geq 1$. This gives the tightness of the family of probability measures $\{P_{n,w}\}$. Hence by Lemma 4 it is relatively compact. Let $\{P_{n',w}\}$ be a subsequence of $\{P_{n,w}\}$ such that $P_{n',w}$ converges weakly to P_w as $n \to \infty$. Then, clearly,

$$X_{n',w} \xrightarrow[n' \to \infty]{\mathcal{D}} P_w \,. \tag{11}$$

The convergence of the series for $\varphi(s)$ is uniform on the half–plane $\sigma \geq \alpha + \beta + 1 + \delta$ for every $\delta > 0$. Hence we find that

$$\begin{aligned} &\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_o}^{T} w(\tau)\, I_{\left\{\tau:\, \rho\left(\varphi(s+i\tau),\, \varphi_n(s+i\tau)\right) \geq \varepsilon\right\}}\, d\tau \leq \\ &\leq \lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{\varepsilon U} \int_{T_o}^{T} w(\tau)\, \rho\big(\varphi(s+i\tau)\,, \varphi_n(s+i\tau)\big)\, d\tau = 0\,. \end{aligned} \tag{12}$$

Let us put

$$X_{T,w}(s) = \varphi(s+i\theta)\,.$$

Then from (12) we get

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \mathbb{P}\Big(\rho\big(X_{T,n,w}(s),\, X_{T,w}(s)\big) \geq \varepsilon\Big) = 0 \tag{13}$$

for every $\varepsilon > 0$. Thus, by Lemma 2 and (6), (11), (13) it follows that

$$X_{T,w}(s) \xrightarrow[T \to \infty]{\mathcal{D}} P_w \,,$$

what proves the theorem.

Proof of Theorem 2. Part 1. First we suppose that $\varphi(s)$ is analytic in D_1. Then we will prove that the probability measure

$$\widehat{Q}_{T,w}(A) = \frac{1}{U} \int_{T_o}^{T} w(\tau)\, I_{\{\tau:\, \varphi(s+i\tau) \in A\}}\, d\tau\,, \qquad A \in \mathcal{B}\big(H(D_1)\big)\,,$$

converges weakly to some measure $\widehat{Q}_w$ on $\Big(H(D_1),\, \mathcal{B}\big(H(D_1)\big)\Big)$ as $T \to \infty$.

Let $\sigma_1 > \frac{1}{2}$ and $n \in \mathbb{N}$. We define the function

$$l_n(s) = \frac{s}{\sigma_1}\,\Gamma\Big(\frac{s}{\sigma_1}\Big)n^s$$

in the strip $-\sigma_1 \le \sigma \le \sigma_1$. Here, as usual, $\Gamma(z)$ denote the Euler gamma–function. Moreover, let, for $\sigma > \rho_o$

$$\varphi_{1\,n}(s) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \varphi(s+z)\,l_n(z)\,\frac{dz}{z}\,.$$

Since

$$\Gamma(s) = B|\,t\,|^{\sigma - \frac{1}{2}}\,\mathrm{e}^{-\frac{\pi|t|}{2}}$$

uniformly in σ, $\sigma' \le \sigma \le \sigma''$, as $|\,t\,| \to \infty$, it follows from (2) that the integral for $\varphi_{1\,n}(s)$ exists. Let

$$Q_{T,n,w}(A) = \frac{1}{U}\int\limits_{T_o}^{T} w(\tau)\,I_{\{\tau:\,\varphi_{1n}(s+i\tau)\in A\}}\,d\tau, \qquad A \in \mathcal{B}\big(H(D_1)\big)\,.$$

Consider the weak convergence of the measure $Q_{T,n,w}$. Since $\sigma > \rho_o \ge \alpha + \beta + \frac{1}{2}$ and $\sigma_1 > \frac{1}{2}$, then $\sigma + \sigma_1 > \alpha + \beta + 1$, and the function $\varphi(s+z)$, for $\mathrm{Re} z = \sigma_1$, is presented by absolutely convergent series

$$\varphi(s+z) = \sum_{m=1}^{\infty} \frac{b_m}{m^{s+z}}\,.$$

Let us consider the series

$$\sum_{m=1}^{\infty} \frac{a_n(m)\,b_m}{m^s} \tag{14}$$

where

$$a_n(m) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_n(s)}{sm^s}\,ds\,.$$

Since

$$a_n(m) = Bm^{-\sigma_1}\int\limits_{-\infty}^{\infty} \big|l_n(\sigma + it)\big|\,dt = Bm^{-\sigma_1}\,,$$

the series (14) converges absolutely for $\sigma > \alpha + \beta + \frac{1}{2}$. Thus, interchanging sum and integral in the definition of $\varphi_{1\,n}(s)$, we find

$$\varphi_{1\,n}(s) = \sum_{m=1}^{\infty} \frac{a_n(m)\, b_m}{m^s}\,. \tag{15}$$

It is well known that, for positive numbers a and b,

$$\frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s}\, ds = \mathrm{e}^{-a}\,.$$

Therefore it is easy to calculate that

$$a_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1} \right\}.$$

Consequently, the equality (15) may be written as

$$\varphi_{1\,n}(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s} \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1} \right\},$$

the series being absolutely convergent for $\sigma > \rho_o$. Thus, repeating the proof of Theorem 1, we deduce that there exists a probability measure $Q_{n,w}$ on $\Big(H(D_1), \mathcal{B}\big(H(D_1)\big)\Big)$ such that $Q_{T,n,w}$ converges weakly to $Q_{n,w}$ as $T \to \infty$.

Let K be a compact subset of D_1. We will prove that

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{U} \int\limits_{T_o}^{T} w(\tau) \sup_{s\in K} |\varphi(s+i\tau) - \varphi_{1\,n}(s+i\tau)|\, d\tau = 0\,. \tag{16}$$

We begin by changing the contour in the integral for $\varphi_{1\,n}(s)$. The integrand has a simple pole at $z = 0$. Let $\sigma \geq \rho_o + \varepsilon$, $\varepsilon > 0$, when $s \in K$, and we put $\sigma_2 = \rho_o + \frac{\varepsilon}{2}$. Then by the residue theorem

$$\varphi_{1\,n}(s) = \frac{1}{2\pi i} \int\limits_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} \varphi(s+z)\, l_n(z)\, \frac{dz}{z} + \varphi(s)\,. \tag{17}$$

Denote by L a simple closed contour lying in D_1 and enclosing the set K, and let δ denote the distance of L from the set K. Then by the Cauchy formula we have

$$\sup_{s\in K}|\varphi(s+i\tau)-\varphi_{1\,n}(s+i\tau)| \le \frac{1}{2\pi\delta}\int\limits_L |\varphi(z+i\tau)-\varphi_{1\,n}(z+i\tau)|\,dz\,.$$

Therefore

$$\frac{1}{U}\int\limits_{T_o}^{T} w(\tau)\sup_{s\in K}|\varphi(s+i\tau)-\varphi_{1\,n}(s+i\tau)|\,d\tau =$$

$$= \frac{B}{U\delta}\int\limits_L |dz|\int\limits_{T_o}^{T} w(\tau)\,|\varphi(z+i\tau)-\varphi_{1\,n}(z+i\tau)|\,d\tau = \tag{18}$$

$$= \frac{B|L|}{U\delta}\sup_{u+iv\in L}\int\limits_{T_o}^{T} w(\tau)\,|\varphi(u+iv+i\tau)-\varphi_{1\,n}(u+iv+i\tau)|\,d\tau\,.$$

Here by $|\,L\,|$ we denote the lenght of L. By the formula (17)

$$\varphi(u+iv+i\tau)-\varphi_{1\,n}(u+iv+i\tau) = B\int\limits_{-\infty}^{\infty}|\varphi(\sigma_2+iv+i\tau+iy)|\,|\,l_n(\sigma_2-u+iy)|\,dy\,.$$

Thus, in view of (3),

$$\frac{1}{U}\int\limits_{T_o}^{T} w(\tau)\left|\varphi(u+iv+i\tau)-\varphi_{1\,n}(u+iv+i\tau)\right|d\tau =$$

$$= B\int\limits_{-\infty}^{\infty}\left|l_n(\sigma_2-u+iy)\right|\frac{1}{U}\int\limits_{T_o}^{T} w(\tau)\left|\varphi(\sigma_2+iv+i\tau+iy)\right|d\tau\,dy =$$

$$= B\int\limits_{-\infty}^{\infty}\left|l_n(\sigma_2-u+iy)\right|\left(\frac{1}{U}\int\limits_{T_o}^{T} w(\tau)\left|\varphi(\sigma_2+iv+i\tau+iy)\right|^2 d\tau\right)^{\frac{1}{2}}dy =$$

$$= B\int\limits_{-\infty}^{\infty}\left|l_n(\sigma_2-u+iy)\right|\left(1+|\,v\,|+|\,y\,|\right)^{\frac{1}{2}}dy\,.$$

Consequently, since v is bounded by a constant,

$$\sup_{u+iv\in L}\int_{T_o}^{T} w(\tau)\left|\varphi(u+iv+i\tau)-\varphi_{1\,n}(u+iv+i\tau)\right|d\tau = \tag{19}$$
$$= B\sup_{\sigma\le-\frac{\varepsilon}{2}}\int_{-\infty}^{\infty}\left|l_n(\sigma+i\tau)\right|\bigl(1+|\,\tau\,|\bigr)\,d\tau = o(1)$$

as $n\to\infty$. The contour L can be choosen so that, for $s\in L$, the inequalities $\sigma\ge\rho_o+\frac{3\varepsilon}{4}$ and $\delta\ge\frac{\varepsilon}{4}$ hold. Thus, the relation (16) is a consequence of (18) and (19).

Now we already can prove a limit theorem for the measure $Q_{T,n,w}$. Let

$$X_{T,n,w}(s)=\varphi_{1\,n}(s+i\theta)$$

where the random variable θ is defined in the proof of Theorem 1. Since $Q_{T,n,w}$ converges weakly to $Q_{n,w}$ as $T\to\infty$, hence we have that

$$X_{T,n,w}\ \xrightarrow[T\to\infty]{\mathcal{D}}\ X_{n,w} \tag{20}$$

where $X_{n,w}$ is an $H(D_1)$– valued random element with the distribution $Q_{n,w}$. Since the series for $\varphi_{1\,n}(s)$ is absolutely convergent, we obtain similarly as in the proof of Theorem 1 that the family of measures $\{Q_{n,w}\}$ is tight, and therefore by Lemma 4 it is relatively compact. Applying the Chebyshev inequality, we deduce from (16) that, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\overline{\lim_{T\to\infty}}\frac{1}{U}\int_{T_o}^{T} w(\tau)\,I_{\left\{\tau:\,\rho\left(\varphi(s+i\tau),\,\varphi_{1\,n}(s+i\tau)\right)\ge\varepsilon\right\}}\,d\tau\le \tag{21}$$
$$\le\lim_{n\to\infty}\overline{\lim_{T\to\infty}}\frac{1}{\varepsilon U}\int_{T_o}^{T} w(\tau)\,\rho\bigl(\varphi(s+i\tau),\,\varphi_{1\,n}(s+i\tau)\bigr)\,d\tau = 0\,.$$

Here ρ is a metric on $H(D_1)$ defined by the same manner as in the proof of Theorem 1. Let us take

$$Y_{T,w}(s)=\varphi(s+i\theta)\,.$$

Then the relation (21) can be written in the form

$$\lim_{n\to\infty}\overline{\lim_{T\to\infty}}\,\mathbb{P}\Bigl(\rho\bigl(X_{T,n,w}(s),\,Y_{T,w}(s)\bigr)\ge\varepsilon\Bigr)=0\,. \tag{22}$$

Let $\{Q_{n',w}\}$ be a subsequence of $\{Q_{n,w}\}$ which converges weakly to the measure Q_w, say, as $n' \to \infty$. Then we have that

$$X_{n',w}(s) \xrightarrow[n' \to \infty]{\mathcal{D}} Q_w .$$

From this, using Lemma 2 again, we obtain in view of (20) and (22) that

$$Y_{T,w}(s) \xrightarrow[T \to \infty]{\mathcal{D}} Q_w .$$

The latter relation is an equivalent of the weak convergence of $Q_{T,w}$ to Q_w as $T \to \infty$.

Part 2. Let ψ and φ be two functions satisfying the hypotheses of Part 1. On $\Big(H^2(D_1), \mathcal{B}\big(H^2(D_1)\big)\Big)$ we define the probability measure

$$Q^1_{T,w}(A) + \frac{1}{U}\int_{T_o}^{T} w(\tau)\, I_{\left\{\tau:\, \left(\psi(s+i\tau),\, \varphi(s+i\tau)\right)\in A\right\}}\, d\tau .$$

Then, reasoning similarly as in Part 1, we can prove that there exists a probability measure Q^1_w on $\Big(H^2(D_1), \mathcal{B}\big(H^2(D_1)\big)\Big)$ such that $Q^1_{T,w}$ converges weakly to Q^1_w as $T \to \infty$.

Part 3. Now let $\varphi(s)$ be as in Theorem 2. Since all poles of $\varphi(s)$ are included in a compact set, the number r of these poles in finite. Denote they by $s_1, \dots, s_r$, and let

$$\psi(s) = \prod_{j=1}^{r} \left(1 - 2^{s_j - s}\right) \varphi(s) .$$

Then the function $\psi(s)$ is analytic in D_1, and, for $\sigma > \alpha + \beta + 1$, it is given by an absolutely convergent Dirichlet series. The estimate (3) for $\psi(s)$ is satisfied, too. Consequently, by Part 2 the measure $Q^1_{T,w}$ converges weakly to some measure Q^1_w as $T \to \infty$.

Let the function $h : H^2(D_1) \to M(D_1)$ be given by the formula

$$h(f_1, f_2) = \frac{f_1}{f_2}, \qquad f_1,\, f_2 \in H(D_1) .$$

Since the metric d satisfies the equality

$$d(f_1, f_2) = d\left(\frac{1}{f_1}, \frac{1}{f_2}\right),$$

the function h is continuous. Therefore, from Lemma 1 we find that the measure $Q_{T,w}$ converges weakly to $Q_w^1 h^{-1}$ as $T \to \infty$. Theorem 2 is proved.

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