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An elliptic surface of Mordell-Weil rank 8 over the rational numbers

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Abstract. – Néron showed that an elliptic surface with rank 8, and with base $B = P_1\mathbb{Q}$, and geometric genus =0, may be obtained by blowing up 9 points in the plane. In this paper, we obtain parameterizations of the coefficients of the Weierstrass equations of such elliptic surfaces, in terms of the 9 points. Manin also describes bases of the Mordell–Weil groups of these elliptic surfaces, in terms of the 9 points; we observe that, relative to the Weierstrass form of the equation,

$$Y^2 = X^3 + AX^2 + BX + C$$

(with $deg(A) \leq 2$, $deg(B) \leq 4$, and $deg(C) \leq 6$) a basis $\{(X_1, Y_1), \dots, (X_8, Y_8)\}$ can be found with X_i and Y_i polynomial of degree ≤ 2, ≤ 3, respectively. One explicit example is computed, showing that for almost every elliptic surface given by a Weierstrass equation of the above form, a basis may be found with X_i and Y_i polynomial of degree ≤ 2, ≤ 3, respectively.

1. Introduction

In 1950, Néron [6] showed that an elliptic surface with base $B = P_1\mathbb{Q}$ can be obtained by blowing up 9 rational points in the plane. If the points are in sufficiently general position, the elliptic surface has Mordell–Weil rank 8. Manin [4], described a basis of the Mordell–Weil group of this elliptic surface in terms of these points.

(Elliptic surfaces with higher rank have been found: in 1954, Néron [7] described a method of finding elliptic surfacs with rank at least 11; recently, Shioda [10] simplified Néron's recipe and made the construction explicit.)

How can one tell, for any particular example, if the surface has rank 8? We will use Shioda's formula, for which we need to know the Kodaira types of the singular fibers of the elliptic surface; and for this, we need to write the equation of the elliptic surface in Weierstrass form.

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In §2, we write the transformations necessary to change a cubic equation of a certain type into Weierstrass form:

$$Y^2 = X^3 + AX^2 + BX + C$$

(with $deg(A) \leq 2$, $deg(B) \leq 4$, and $deg(C) \leq 6$); we also show what happens to sections of the original surface, when it is transformed into this form. In particular, we exhibit a basis, consisting of sections of the form $\{\sigma_i = (X_i, Y_i)\}$, with X_i and Y_i polynomials of degree $\leq 2, 3$, respectively. We also state our main theorem, Theorem 1: that if $E \longrightarrow B = P_1\mathbb{C}$ is almost any elliptic surface with geometric genus $p_g = 0$ (and hence given by an equation of the form above), then its Mordell–Weil group has a basis as above. (Shioda and Oguiso have published a generalization of this result: The Mordell–Weil group of *every* rational elliptic surface is generated by sections of this type; cf. [11], Corollary 2.3.)

In §3, we describe the computer program we use to check if a given elliptic surface has Mordell–Weil rank 8. We also write the equation of one elliptic surface with rank 8, together with a basis of the given type.

2. The construction of the elliptic surface

A corollary to Max Noether's Theorem (see Fulton [2]) states the following: if two cubic curves, C_1 and C_2, intersect in the 9 points $P_0, \ldots, P_8$, and if the cubic curve C contains $P_0, \ldots, P_7$, then C contains P_8 also. On the other hand, 9 points in general determine a unique cubic curve, and 8 points in general determine a family of cubic curves, parameterized by a line. Taken together, these two statements imply that, in general, 8 points in the plane determine a unique 9^{th} point, through which all members of the family of cubics determined by the 8 points must pass.

The blow–up of 9 points in the plane may be described as the surface defined by the family of cubics going through 9 such points; alternatively, if F_1 and F_2 are polynomials which determine C_1 and C_2, members of such a family, then an equation for the elliptic surface is

$$uF_1 + vF_2 = 0,$$

where u and v may be viewed as homogeneous coordinates for the base of the elliptic surface.

Let $P_0, \ldots, P_7$ be 8 points in the projective plane, in sufficiently general position. 'Sufficiently general' includes such things as 'no three on a line, no six on a conic'. Without loss of generality, we may assume that $P_0 =$

$(0,1,0)$, $P_1 = (1,0,0)$, and $P_2 = (0,0,1)$. Then every cubic through these 8 points has an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey = ux^2y + vxy^2, \tag{1}$$

where u and v are necessarily not both zero. (When we literally choose the 5 points $P_3 \ldots, P_7$, we will solve for the coefficients a, b, c, d, and e, as linear forms in u and v.) This equation defines the elliptic surface we want, with rank 8 for $P_3, \ldots, P_7$ 'sufficiently general'. For the rest of this section, we shall assume that the points $P_3, \ldots, P_7$, are in sufficiently general position.

Let P_8 be the 9^{th} point as discussed above. Each of the points P_i determines a section, σ_i, of the elliptic surface. Manin [4] shows that, with σ_0 chosen as the zero–section (i.e. the identity of the Mordell–Weil group), the sections $\sigma_1, \ldots, \sigma_8$ generate a subgroup of index 3 (see also [1]).

Let P_9 be the point obtained by finding the '3^{rd}' point of intersection of the line tangent to P_0 in each fiber. Manin shows that the section, σ_9, defined by this point, together with the sections $\sigma_1, \ldots, \sigma_7$ generates the full Mordell–Weil group.

(In the following, we will use lower case x and y to refer to solutions of (1), and capital X and Y to refer to solutions of (2), (see below). Equal subscripts, then, refer to different forms of the same section.)

We use standard techniques (see Mordell [5]) to put (1) into Weierstrass form. In particular, reparameterize the curve by the following steps:

The line through P_9 with slope m, intersects the curve in two other points, whose x–coordinates are found by solving a quadratic equation. These other two points are rational if and only if the discriminant, $disc$, of the quadratic polynomial in x, is a square, ($disc$ is a cubic polynomial in m). Then the curve is birationally equivalent to $Y^2 = disc$. We change variables slightly and get:

LEMMA 1. *The equation of the elliptic surface defined by (1) may be rewritten as*

$$Y^2 = X^3 + AX^2 + BX + C, \tag{2}$$

where

$$A = 4dv + 8ac + 4eu + b^2,$$

$$B = 16(a^2c^2 + acdv + aceu + deuv) + 8(bcdu + ab^2c + abev),$$

and

$$C = 16(cdu + abc + aev)^2.$$

Furthermore, if (x, y) is a solution of (1), then (X, Y) is a solution of (2), where

$$\frac{X}{4} = (yu - a)c + v(uxy - ax - by - d) + vy^2,$$

and

$$\frac{Y}{4} = (-2uv^2xy^2 - 2v^3y^3) + (-4cuv + 3bv^2)y^2 + (-2cu^2 + buv + 2av^2)xy$$
$$+ (bcu + 2acv - b^2v + 2dv^2 - 2euv)y + (2acu - abv)x + (cdu - bdv + aev).$$

LEMMA 2. *The section σ_9 is given by*

$$x_9 = c/v,$$

and

$$y_9 = \frac{(ac^2 + cdv)}{(uc^2 - bcv - ev^2)}.$$

One can see that the section does not have X_9 and Y_9 polynomial, since its intersection with the zero section is non–empty. However, using the bilinear form defined by Cox and Zucker, we determined that the section $\sigma_{10} = \sigma_9 - (\sigma_1 + \sigma_2)$ must have X_{10} and Y_{10} polynomial.

LEMMA 3. *The section σ_{10} as above has coordinates for (1) given by*

$$x_{10} = -\frac{d}{a}$$

and

$$y_{10} = 0$$

and it has coordinates for (2) given by

$$X_{10} = -4ac,$$

and

$$Y_{10} = 4(eav - cdu).$$

It is clear that $\sigma_1, \ldots, \sigma_7$, and σ_{10} form a basis, if indeed the elliptic surface has rank 8.

Every elliptic surface, $E \longrightarrow B = P_1\mathbb{C}$, with $p_g = 0$ can be given by a Weierstrass equation

$$Y^2 = 4X^3 - G_2X - G_3,$$

with $deg(G_i) \leq 2i$ (see Kas [3]); Shioda's formula ([9]) shows that almost all such elliptic surfaces have Mordell–Weil rank 8. Since rational sections can be found such that $(X,\ Y) =$ (quadratic, cubic) by satisfying algebraic conditions on the coefficients $\alpha_2, \alpha_1, \alpha_0$, of $x = \alpha_2 u^2 + \alpha_1 u + \alpha_0$ (see Schwartz [8]), we may conclude that, if, for one example a basis may be found for which $(X,\ Y) =$ (quadratic, cubic), then for almost all elliptic surfaces a basis can be found among the sections with $(X,\ Y) =$ (quadratic, cubic). Our main theorem is proved by Proposition 1, in which we find such an example:

THEOREM 1. *Almost all elliptic surfaces, $E \longrightarrow B = P_1\mathbb{C}$, with $p_g = 0$, have bases consisting of sections which are polynomial, with $deg(X) \leq 2$, and $deg(Y) \leq 3$.*

3. A numerical example

We wrote a computer program to run on the Rider College DEC 2060 computer, which takes as input the coordinates of five points in the plane. The program then determines whether the elliptic surface determined by these five points, together with the points $(0, 1, 0)$, $(1, 0, 0)$, and $(0, 0, 1)$, has Mordell–Weil rank 8; if the Mordell–Weil rank is 8, then the sections found in §2 indeed give a basis of the Mordell–Weil Group.

In particular, the program solves for the coefficients $(a, b, c, d, e,$ and $k)$ in the equation

$$ax^2 + bxy + cy^2 + dx + ey = k(ux^2y + vxy^2), \tag{3}$$

(k is the common denominator needed, so that the program can use integer arithmetic). The program then transforms the equation into Weierstrass form

$$Y^2 = X^3 - G_2X - G_3,$$

and computes the discriminant, $\Delta = 4G_2^3 - 27G_3^2$ and its derivative, Δ'.

Since the rank of the Mordell–Weil group is given by Shioda's formula [9] :

$$r \leq 4g - 4 + t_1 + 2t_2 - 2p_g,$$

(where $g =$ genus of the base $= 0$,

$t_1 =$ number of singular fibers of type I_b (with $b > 0$),

$t_2 =$ number of singular fibers of other types, and

$p_g =$ geometric genus of the Elliptic surface),

with equality in the case of $p_g = 0$, a sufficient condition for the rank to be 8 is for Δ to have 12 simple roots. We would like to check this condition by checking that the greatest common divisor of Δ and Δ' (found *via* the Euclidean algorithm) is a constant, but, in fact, the program finds the greatest common divisor of Δ and Δ' modulo various small primes; if the greatest common divisor is a constant modulo any of these primes, then in fact, the greatest common divisor is a constant, Δ has no multiple roots, and the rank of the Mordell–Weil group is 8.

This program was written in BASIC, with arrays to handle the multiple precision integer arithmetic. The values which follow were computed using double precision real variables in FORTRAN. (When this paper was written, we were unaware of programs such as Maple and Macsyma which would do symbolic computations; we have since used Maple to check the computations in this paper.)

PROPOSITION 1. *The elliptic surface determined by the 8 points (0,1,0), (1,0,0), (0,0,1), (1,1,1), (-1,2,1), (-2,-1,1), (2,-2,1), (3,4,1) (by the method described above) has rank 8.*

An equation for this elliptic surface is

$$ax^2 + bxy + cy^2 + dx + ey = k(ux^2y + xy^2),$$

where

$$\begin{aligned} a &= 334u + 291, \\ b &= -237u - 261, \\ c &= -298u - 405, \\ d &= 6u - 257, \\ e &= 76u + 513, \\ &\text{and} \\ k &= -119. \end{aligned}$$

When this equation is put into Weierstrass form, an equation for this elliptic surface is

$$Y^2 = X^3 + AX^2 + BX + C,$$

where

$$
\begin{aligned}
A &= -776263u^2 - 1899234u - 752387, \\
B &= 172908582912u^4 + 843345351168u^3 + 1397771166672u^2 \\
&\quad + 896911919448u + 201659710536, \\
C &= (23801856u^3 + 66743748u^2 + 50462886u + 12995478)^2.
\end{aligned}
$$

A basis for the Mordell–Weil group of this elliptic surface is given in the following table:

$$
\begin{aligned}
X_1 &= 0 \\
Y_1 &= 95207424u^3 + 266974992u^2 + 201851544u + 51981912 \\
X_2 &= 398128u^2 + 890808u + 349088 \\
Y_2 &= 851088u^3 - 48057912u^2 - 113383200u - 39130056 \\
X_3 &= 539976u^2 + 1186404u + 420012 \\
Y_3 &= 95747400u^3 + 255741948u^2 + 174984264u + 42321636 \\
X_4 &= 681824u^2 + 778472u + 188676 \\
Y_4 &= -228658976u^3 + 43792952u^2 + 279601924u + 92547252 \\
X_5 &= 256280u^2 + 606160u + 252936 \\
Y_5 &= -87520216u^3 - 293100808u^2 - 280965664u - 74233152 \\
X_6 &= 114432u^2 + 822264u + 1101168 \\
Y_6 &= 122556672u^3 + 379159704u^2 - 140977872u - 759578904 \\
X_7 &= 965520u^2 + 2367360u + 1173996 \\
Y_7 &= 555760464u^3 + 2063380032u^2 + 2397693396u + 863213148 \\
X_{10} &= 398128u^2 + 887952u + 471420 \\
Y_{10} &= -851088u^3 + 23215472u^2 - 42541548u - 71058708
\end{aligned}
$$

The smallest prime which does not divide the discriminant of Δ is 37.

Bibliographie

[1] D. A. COX and S. ZUCKER, *Intersection numbers of sections of elliptic surfaces*, Invent. Math., **53** (1979), 1–44.

[2] W. FULTON, *Algebraic Curves, an introduction to algebraic geometry*, Benjamin, W. A. (1969), (Mathematics lecture note series).

[3] A. KAS, *On the deformation types of regular elliptic surfaces*, Preprint (1976).

[4] Ju. I. MANIN, *The Tate height of points on an Abelian variety; its variants and applications*, AMS Translations (series 2) **59** (1966), 82–110.

[5] L. J. MORDELL, *Diophantine Equations*, Academic Press, London (1969).

[6] A. NÉRON, *Les propriétés du rang des courbes algébriques dans les corps de degré de transcendance fini*, Centre National de la Recherche Scientifique, (1950), 65–69.

[7] A. NÉRON, *Propriétés arithmétiques de certaines familles de courbes algébriques*, Proc. Int. Congress, Amsterdam, **III** (1954), 481–488.

[8] C. F. SCHWARTZ, *A Mordell–Weil group of rank 8, and a subgroup of finite index*, Nagoya Math. J. **93** (1984), 19–26.

[9] T. SHIODA, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.

[10] T. SHIODA, *An infinite family of elliptic curves over $\mathbb{Q}$ with large rank via Néron's method*, Invent. Math. **106** (1991), 109–119.

[11] K. OGUISO and T. SHIODA, *The Mordell–Weil lattice of a rational elliptic surface*, Comment. Math. Univ. St. Pauli **40** (1991), 83–99.

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