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Characterizing weakly symmetric spaces as Gelfand pairs

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Abstract. Let (M, G, μ) be a Riemannian weakly symmetric space. Fix a base point $x_0 \in M$ and denote by H to be the compact isotropy subgroup of G at x_0. It is proven that $L^1(H \backslash G/H)$ is commutative, i.e. (G, H) is a Gelfand pair. This extends É. Cartan's result for Riemannian symmetric spaces. Conversely, if (G, H) is a Riemannian weakly symmetric pair, then $M = G/H$ can be made to be Riemannian weakly symmetric. An application of this result is presented.

1. Introduction

In this paper, we characterize Riemannian weakly symmetric spaces as a special class of Gelfand pairs. The notion of a weakly symmetric space was introduced by A. Selberg in [8] as a generalization to that of a symmetric space. He also constructed examples of weakly symmetric (but not symmetric) spaces by endowing the total space of principal S^1-bundles over the Siegel half-space with a twisted action by the group $Sp(n, \mathbb{R}) \times S^1$ (cf. [6]). The topic has recently gained much interest with discoveries of many new examples, especially by J. Berndt and L. Vanhecke in [1], who found an alternative characterization of weak symmetry based on the reversal of geodesics, and by W. Ziller in [12], who replaced geodesics with tangent vectors. Recently, the author in [7] was able to extend Selberg's results to hermitian symmetric spaces based on Ziller's infinitesimal characterization.

Let M be a Riemannian manifold, G a locally compact transitive Lie group of isometries of M, and μ a fixed isometry of M (not necessarily in G) such that (M, G, μ) is a Riemannian weakly symmetric space. Fix a base point $x_0 \in M$ and let H be the compact isotropy subgroup of G at x_0. Selberg was able to prove in [8] that $D(G/H)$, the space of G-invariant differential operators on $M = G/H$, is commutative. If we further assume that G is connected, then a result of E.G.F. Thomas [9] says that commutativity of $D(G/H)$ is equivalent to commutativity of $L^1(H \backslash G/H)$, the space of integrable functions on G bi-invariant under H. Therefore, under this additional assumption, it follows that if $(M = G/H, G, \mu)$ is weakly symmetric, then (G, H) is a Gelfand pair. In this paper, we remove the restriction that G must be connected and give a simple direct proof of this

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result as stated in the following theorem, which extends É. Cartan's result for Riemannian symmetric spaces (cf. [10], Cor. 2.7).

Theorem 1.1. *If (M, G, μ) is a Riemannian weakly symmetric space and H is the compact isotropy subgroup of G at a base point of M, then $L^1(H \backslash G/H)$ is commutative, i.e. (G, H) is a Gelfand pair.*

Our argument proceeds as follows. We first define a Riemannian weakly symmetric space (M, G, μ) and demonstrate that it is better characterized as a homogeneous weakly symmetric space G/H. We then establish an equivalence between G/H with weakly symmetric pairs (G, H). Our theorem will follow by showing that when H is compact, these weakly symmetric pairs (G, H) are in fact Gelfand pairs. We also prove as a converse that a Riemannian weakly symmetric pair (G, H) produces a Riemannian weakly symmetric space $M = G/H$ and go on to apply this result in an example.

2. Proof of Theorem

Let us begin with some preliminaries. We shall always assume in this paper that G is a locally compact Lie group, H a closed subgroup of G and e the identity element in G. We also denote by $\mathrm{Inn}_H(G)$ to be the subgroup of $\mathrm{Aut}(G)$ consisting of inner automorphisms of the form $I_h : g \mapsto hgh^{-1}$, where $h \in H$ and $g \in G$.

Definition 2.1. (Selberg [8]) The triple (M, G, μ) is called a *Riemannian weakly symmetric space* if the following properties are satisfied:

(i) M is a Riemannian manifold, G is a locally compact transitive Lie group of isometries of M, and μ is a fixed isometry of M (not necessarily in G) satisfying $\mu G \mu^{-1} = G$ and $\mu^2 \in G$,
(ii) Given any two points x and y in M, there exists an element $g \in G$ such that

$$gx = \mu y \text{ and } gy = \mu x. \tag{1}$$

Unfortunately, this definition of Selberg does not make any reference to the isotropy subgroups of G with respect to its action on M. Since such a subgroup of G must be specified in order to make the connection with Gelfand pairs, we revise his definition to better fit this framework.

Definition 2.2. G/H is called a *homogeneous weakly symmetric space* if there exists an analytic diffeomorphism μ of the homogeneous manifold $M = G/H$ such that

(i) $\mu G \mu^{-1} = G$, $\mu(eH) = eH$ and $\mu^2 \in H$,
(ii) Given any two points x and y in M, there exists an element $g \in G$ such that

$$gx = \mu y \text{ and } gy = \mu x. \tag{2}$$

The following lemma shows that indeed Selberg's definition is compatible with ours.

Lemma 2.3. *Let (M, G, μ) be a Riemannian weakly symmetric space. Fix any base point $x_0 \in M$ and let H be the compact isotropy subgroup of G at x_0. Then there exists an isometry $\tilde{\mu}$ of $M = G/H$ which makes G/H a homogeneous weakly symmetric space.*

Proof. Set $y_0 = \mu x_0$. By transitivity of G, there exists an element $\tilde{g} \in G$ such that $\tilde{g} y_0 = x_0$. Define $\tilde{\mu} = \tilde{g}\mu$. Then $\tilde{\mu}$ fixes x_0 and satisfies $\tilde{\mu} G \tilde{\mu}^{-1} = G$. We next identify M with the homogeneous space G/H and the point x_0 with the coset eH so that $\tilde{\mu}(eH) = eH$ and $\tilde{\mu}^2 \in H$. By assumption, given any two points x and y of M, there exists an element $g \in G$ such that $gx = \mu y$ and $gy = \mu x$. It follows that $\tilde{g} g x = \tilde{\mu} y$ and $\tilde{g} g y = \tilde{\mu} x$. Hence, G/H is a homogeneous weakly symmetric space with respect to $\tilde{\mu}$. ■

We next establish an equivalence between homogeneous weakly symmetric spaces and weakly symmetric pairs. These pairs possess special symmetry conditions that will ensure that they are Gelfand pairs.

Definition 2.4. (G, H) is called a *weakly symmetric pair* if there exists an automorphism θ of G such that

(i) $\theta(H) \subseteq H$ and $\theta^2 \in \mathrm{Inn}_H(G)$,
(ii) $H\theta(g)H = Hg^{-1}H$ for all $g \in G$.

Furthermore, (G, H) is called a *Riemannian* weakly symmetric pair if $\mathrm{Ad}_G(H)$ is compact.

Lemma 2.5. *(G, H) is a weakly symmetric pair if and only if G/H is a homogeneous weakly symmetric space.*

Proof. Let (G, H) be a weakly symmetric pair and θ the corresponding automorphism of G. We define a map μ of the homogeneous manifold $M = G/H$ by

$$\mu(gH) = \theta(g)H, \qquad gH \in G/H.$$

Then μ is an analytic diffeomorphism since θ is an automorphism. It is also clear that $\mu(eH) = \theta(e)H = eH$ and $\mu^2 \in H$ since $\theta^2 \in \mathrm{Inn}_H(G)$. This proves property (i) of a homogeneous weakly symmetric space.

To prove property (ii), we will first show that it holds for any point $x_0 \in M$ and the origin eH. Since G is transitive on M, there exists an element $g_1 \in G$ such that $g_1 x_0 = g_1 g_0 H = eH$. Also, by assumption, we have $H\theta(g_0)H = Hg_0^{-1}H$. This allows us to write $h_0\theta(g_0)H = g_0^{-1}H$ for some element $h_0 \in H$. Setting $\tilde{g} = h_0^{-1} g_0^{-1}$, it follows that

$$\tilde{g}(eH) = h_0^{-1}(g_0^{-1}H) = h_0^{-1} h_0 \theta(g_0) H = \mu x_0, \tag{3}$$

and

$$\tilde{g} x_0 = h_0^{-1} g_0^{-1}(g_0 H) = eH = \mu(eH). \tag{4}$$

For the arbitrary case, fix x and y to be any two points of M and let g_0 be the element of G which maps y to eH. Denote by x_0 the image of x under

g_0. By our previous work, there exists an element $\tilde{g}$ which satisfies (3) and (4) for the two points x_0 and eH. Setting $g = \theta(g_0^{-1})\tilde{g}g_0$, it follows that

$$gx = \mu y \text{ and } gy = \mu x.$$

Hence, G/H is a homogeneous weakly symmetric space with respect to μ.

Conversely, let G/H be a homogeneous weakly symmetric space and μ the corresponding analytic diffeomorphism of G/H. We define a homomorphism θ of G by

$$\theta(g) = \mu g \mu^{-1}, \qquad g \in G.$$

Then θ is an automorphism of G since μ is an analytic diffeomorphism and $\mu G \mu^{-1} = G$. It is clear that $\theta(H) \subseteq H$ and $\theta^2 \in \mathrm{Inn}_H(G)$ since μ fixes the point $x_0 = eH$ and $\mu^2 \in H$. This proves property (i) of a weakly symmetric pair. Furthermore, we have

$$\mu(gH) = \mu g \mu^{-1}(eH) = \theta(g)(eH) = \theta(g)H, \qquad gH \in G/H.$$

To prove property (ii), fix any element $g_0 \in G$ and denote $x_0 = g_0 H$. By assumption, there exists an element $g \in G$ such that $g(eH) = \mu x_0 = \theta(g_0)H$ and $gx_0 = gg_0H = \theta(e)H = eH$. It follows that $HgH = H\theta(g_0)H$ and $Hgg_0H = HeH$. This allows us to write $h_1 g g_0 h_2 = e$, for some $h_1, h_2 \in H$. In other words, $h_1 g = h_2^{-1} g_0^{-1}$ or $HgH = Hg_0^{-1}H$. Hence, $Hg_0^{-1}H = H\theta(g_0)H$ and (G,H) is a weakly symmetric pair with respect to θ. ∎

We are now ready to make the connection between Riemannian weakly symmetric spaces with Gelfand pairs.

Definition 2.6. Let H be a compact subgroup of G. Then (G,H) is called a *Gelfand pair* if the algebra $L^1(H\backslash G/H)$ is commutative under the convolution product.

Lemma 2.7. *If (G,H) is a weakly symmetric pair and H is compact, then (G,H) is a Gelfand pair.*

Remark 2.8. We mention that this lemma is a minor extension of a result due to J. Faraut [2], Prop. I.2., which in turn is an extension of the following classical result due to I.M. Gelfand [4]:

Theorem 2.9. ([4]) *Let θ be an involutive automorphism of $G = KP$ where K is compact, $\theta(k) = k$ for all $k \in K$ and $\theta(p) = p^{-1}$ for all $p \in P$. Then (G,K) is a Gelfand pair.*

Proof. (of Lemma 2.7) We follow the argument used in [2], Prop. I.2. Let $f \in C_c(G)$, the space of continuous functions on G with compact support, and define

$$f^\theta(g) = f(\theta(g)) \text{ and } f^\sharp(g) = f(g^{-1}).$$

Let dg be the Haar measure on G. The maps

$$f \mapsto \int_G f(g)dg \text{ and } f \mapsto \int_G f^\theta(g)dg$$

are left-invariant positive Radon measures on G. But it is well known that such a Radon measure is unique up to a constant factor. It follows that there exists a positive constant c such that

$$\int_G f^\theta(g)dg = c\int_G f(g)dg. \tag{5}$$

Now, choose any $f \in C_c(H\backslash G/H)$, the space of functions in $C_c(G)$ bi-invariant under H, such that $\int_G f(g)dg \neq 0$. Since $\theta(H) \subseteq H$ and $\theta^2 \in \mathrm{Inn}_H(G)$ for a weakly symmetric pair, we must have $(f^\theta)^\theta = f$. This implies $c^2 = 1$ or $c = 1$ because of (5). It follows that $(f_1 * f_2)^\theta = f_1^\theta * f_2^\theta$ for any two functions $f_1, f_2 \in C_c(G)$.

Next, we notice that property (ii) of a weakly symmetric pair implies $f^\theta = f^\sharp$ for any $f \in C_c(H\backslash G/H)$. It follows that

$$\int_G f(g^{-1})dg = \int_G f(\theta(g))dg = \int_G f(g)dg$$

and shows G is unimodular (cf. [2], Prop. I.1). Since $\sharp$ is an anti-automorphism, it follows that $(f_1 * f_2)^\sharp = f_2^\sharp * f_1^\sharp$ for any two functions $f_1, f_2 \in C_c(H\backslash G/H)$. Hence, $f_1 * f_2 = f_2 * f_1$ and $C_c(H\backslash G/H)$ is commutative. According to [9], Prop. 2, this is equivalent to commutativity of $L^1(H\backslash G/H)$. ■

Proof. (of Theorem 1.1) Apply Lemmas 2.3, 2.5 and 2.7 in the order listed and use the fact that H is compact. ■

3. Criterion for Weak Symmetry

Notice that Lemma 2.5 also lets us state conditions as to when a pair (G, H) produces a Riemannian weakly symmetric space $M = G/H$.

Lemma 3.1. *If (G, H) is Riemannian weakly symmetric pair, then $M = G/H$ can be made to be Riemannian weakly symmetric.*

Proof. Because of Lemma 2.5, G/H is a homogeneous weakly symmetric space with respect to the analytic diffeomorphism μ induced from θ. The argument now copies that used in the classical situation where (G, H) is a Riemannian symmetric pair (cf. [5], Ch. IV, §3, Prop. 3.4). Since $\mathrm{Ad}_G(H)$ is compact, there exists a Riemannian structure Q on the homogeneous manifold $M = G/H$ such that it is invariant under the action of G by left-translation of cosets. Let N consist of those elements $n \in G$ such that n acts as the identity mapping on M. Then there exists a closed Lie subgroup $\widetilde{G} \subset I(M)$, the full isometry group of M, such that $\widetilde{G}$ is isomorphic to G/N. Furthermore, it is clear that μ is an isometry of M since θ is an automorphism of G. Hence, $(M, \widetilde{G}, \mu)$ is Riemannian weakly symmetric. ■

As an application, we shall use Lemma 3.1 to find examples of Riemannian weakly symmetric spaces. Following [3] and [7], we let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra of hermitian compact or non-compact type. Then $\mathfrak{g}$ is real semisimple. Decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ into its ± 1-eigenspaces. Then $\mathfrak{k}$ is a compact Lie algebra and $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{z}_{\mathfrak{k}}$, where $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ is the semisimple part and $\mathfrak{z}$ is the center of $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Let G be a connected real Lie group with Lie algebra $\mathfrak{g}$ and denote by K, K_s, Z_K and A be the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{k}_s$, $\mathfrak{z}_{\mathfrak{k}}$ and $\mathfrak{a}$. Here, K_s is compact and K and Z_K are compact if and only if G has finite center.

According to [3], Prop. 2.1, there exists an involutive automorphism γ of $\mathfrak{g}$, which we shall assume lifts to G, such that $\gamma(K_s) = K_s$, $\gamma(a) = a^{-1}$ for all $a \in A$ and $\gamma(k) = k^{-1}$ for all $k \in Z_K$. This gives the following result, which is implied in [3], Theorem 3.1, and independently proven by J.A. Wolf and A. Korányi in an unpublished work [11] using a similar argument. We mention that both of these papers also go on to prove that (G, K_s) is a Gelfand pair.

Lemma 3.2. *If the hermitian symmetric space G/K contains no irreducible factors of tube type, then (G, K_s) is a Riemannian weakly symmetric pair with respect to $\theta = \gamma$.*

Proof. It is clear that (G, K_s) satisfies property (i) of a weakly symmetric pair. As for property (ii), Theorem 3.3 of [3] says that if G/K contains no irreducible factors of tube type, then it is equivalent to the following property:

$$K_s k g k^{-1} K_s = K_s g K_s \quad \text{for all } k \in Z_K \text{ and } g \in G. \tag{6}$$

Now, G has the well-known decomposition $G = KAK$. Also, $K = K_s Z_K$. It is now easy to check using these two decompositions and (6) that

$$K_s \theta(g) K_s = K_s g^{-1} K_s \quad \text{for all } g \in G. \tag{7}$$

This proves property (ii). ∎

As a result, it follows that there exists an isometry group $\widetilde{G} \cong G/N$ such that $(M = G/K_s, \widetilde{G}, \mu)$ is Riemannian weakly symmetric because of Lemma 3.1, where μ is the isometry of M induced from θ.

Remark 3.3. The author has already proven in [7] that $M = G/K_s$ is weakly symmetric, but with respect to an isometry group larger than G, namely $G^1 := G \times Z_K$. More precisely, it was proven that $(M, \widetilde{G}^1, \mu)$ is Riemannian weakly symmetric and proven without the restriction that G/K not contain any irreducible factors of tube type. It is in this sense that our result here that M is weakly symmetric with respect to $\widetilde{G}$ is new. Furthermore, it was also proven in [7] that M is not Riemannian symmetric with respect to a special one-parameter family of Riemannian metrics in the case where G/K is a classical irreducible bounded symmetric domain.

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