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Finite groups of rotations
A supplement to the preceding article

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Abstract. This paper completes the work started in the preceding paper where the following question was asked. Given a finite set S of isometries of some affine Euclidean space. When is the group Γ generated by S discrete? In that paper we described an algorithm which reduced this question to the special case discussed here.

Suppose we are given a finite set S of rotations of Euclidean d–space. We ask when the group Γ generated by S is finite. We show here that there is an algorithm to decide this question and this algorithm has a number of steps bounded by a constant depending only on d and $\#S$.

The main ingredient of the proof is the existence of a Zassenhaus neighbourhood Ω of the identity in a Lie group G. Recall that by definition, a neighbourhood Ω of e in G is called a Zassenhaus neighbourhood if for every discrete subgroup Γ of G the intersection $\Omega \cap \Gamma$ is contained in a connected nilpotent subgroup of G, hence in an abelian compact connected subgroup if G is compact. So our approach is still another variation on a long standing and celebrated theme, namely the theorem of Jordan, which states that every finite subgroup of O_d contains an abelian subgroup of an index bounded by a number depending only on d. For recent work on explicit bounds and explicit Zassenhaus neighbourhoods of e in O_d see [2,3] and the references therein.

1. The algorithm

Theorem . *There is a function $f(d,m)$ with the following property. For every subset S of cardinality m of the group $O(d)$ there is an algorithm with at most $f(d,m)$ steps to decide if the subgroup of $O(d)$ generated by S is finite.*

Recall that in a Lie group G a neighbourhood Ω of e is called a *Zassenhaus neighbourhood* of e if for every discrete subgroup Γ of G the set $\Gamma \cap \Omega$ is contained

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in a nilpotent connected Lie subgroup of G. Thus, if G is compact, the set $\Gamma \cap \Omega$ is contained in a torus for every finite subgroup Γ of G. We make use of the result that every Lie group contains a Zassenhaus neighbourhood of e, cf. [4, 8.16].

We work in a fixed dimension d. Consequently, we suppress the dependence on d from the notation. Put $K = O(d)$. Let S be a finite subset of K. We may assume that S is symmetric, i.e. $S = S^{-1}$, and contains the identity element e. We define inductively $\Gamma_1 = S$, $\Gamma_{n+1} = S \cdot \Gamma_n = \{\beta\gamma\,;\, \beta \in S, \gamma \in \Gamma_n\}$. Put $\dot{\Gamma}_n = \Gamma_n \setminus \Gamma_{n-1}$.

Lemma 1. *There is a constant f_1 with the following property $\Gamma_{n+1} \subset \Gamma_n \cdot \Omega$ for some $n \leq f_1$.*

The strange notation f_1 for a constant should remind us that f_1 can be considered as a function of d.

Proof. Let U be an open symmetric neighbourhood of e in K such that $U^2 \subset \Omega$. If $\gamma U \cap \Gamma_n U = \emptyset$ for at least one $\gamma \in \dot{\Gamma}_{n+1}$ then $\mathrm{vol}\,(\Gamma_{n+1}U) \geq \mathrm{vol}\,(\Gamma_n U) + \mathrm{vol}\,(U)$, where vol is a Haar measure on K. Thus $f_1 = \mathrm{vol}\,(K)/\mathrm{vol}\,(U)$ will do. ∎

Now let n be as in Lemma 1. For every $\gamma \in \dot{\Gamma}_{n+1}$ choose an element $\beta_\gamma \in \Gamma_n$ such that

$$\alpha(\gamma) := \beta_\gamma^{-1} \cdot \gamma \in \Omega.$$

Let A be the subgroup of K generated by $\{\alpha(\gamma)\ ;\ \gamma \in \dot{\Gamma}_{n+1}\}$.

Lemma 2. $\Gamma = \Gamma_n \cdot A$.

Proof. We have $S\Gamma_n = \Gamma_{n+1} = \Gamma_n \cup \dot{\Gamma}_{n+1} \subset \Gamma_n \cup \Gamma_n \cdot A = \Gamma_n A$, hence $S\Gamma_n A \subset \Gamma_n A$ and thus $\Gamma\Gamma_n A \subset \Gamma_n A$ which implies $\Gamma \subset \Gamma_n A$. The converse inclusion is trivial. ∎

Corollary . Γ *is finite iff the following two conditions hold.*

a) *The $\alpha(\gamma), \gamma \in \dot{\Gamma}_{n+1}$, commute.*

b) *Every $\alpha(\gamma), \gamma \in \dot{\Gamma}_{n+1}$, is of finite order.*

Proof. If Γ is finite, then a) holds by definition of a Zassenhaus neighbourhood and b) is obvious. Conversely, if a) and b) hold then A is finite and hence Γ is finite by lemma 2. ∎

This implies the theorem: Compute inductively $\Gamma_n, n = 1, 2, \ldots$ check if for every $\gamma \in \dot{\Gamma}_{n+1}$ at least one $\beta^{-1}\gamma$, $\beta \in \Gamma_n$, is contained in Ω. We can assume that Ω is of the form $\{g \in K \mid \|1 - g\| < \varepsilon\}$ for some ε, where $\|\cdot\|$ is the operator norm corresponding to the Euclidean norm on $\mathbb{R}^d$. We know that for some $n \leq f_1$ we will find for every $\gamma \in \dot{\Gamma}_{n+1}$ an element $\beta \in \Gamma_n$ such that $\alpha(\gamma) := \beta^{-1}\gamma \in \Omega$. Now check conditions a) and b). ∎

References

[1] Abels, H. *Discrete groups of affine isolmetries*, Journal of Lie Theory **9** (1999), 1–29.

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