Journal of Lie Theory
Volume **8** (1998) 393–397

On the Number of Control Sets on Flag Manifolds of the Real Simple Lie Groups

Carlos José Braga Barros

Communicated by J. D. Lawson

Abstract. In this paper we determine upper bounds for the number of control sets on flag manifolds of a real non-compact simple Lie group whose Lie algebra is a real form of a complex simple Lie algebra. The estimates for the number of control sets are based on the results of San Martin and Tonelli [6]. They are determined by computing the orbits of a subgroup of the Weyl Group.

1. Introduction

One of the first questions in the study of nonlinear control systems is the controllability of a system. Nonlinear systems may posses several regions in the state space in which a local controllability property holds, without being globally controllable. An important conceptual tool is to study the control sets. The notion of control sets can be abstracted to arbitrary semigroup actions, and in particular, to actions of subsemigroups of Lie groups on their homogeneous spaces.

An important problem in the geometric theory of semigroups and in the geometric control theory is the determination of the number of control sets for semigroup actions. Control sets for control systems have been studied by Colonius and Kliemann [4]. In particular an upper bound for the number of control sets for control semigroups acting on projective spaces has been obtained. An improved version of this result was given by Barros and San Martin [2], where smaller upper bounds have been given depending on the group that is acting on the projective space. The theory of control sets for control systems has been developed by Albertini, Sontag, Colonius, Kliemann and Wirth [1, 4, 3, 10]. A similar theory for semigroup actions on homogeneous spaces has been developed by San Martin and Tonelli [6, 7].

In this paper we determine upper bounds for the number of effective control sets on the flag manifolds of the real simple non-compact Lie groups. We use some results of San Martin and Tonelli [6]. These upper bounds are determined by the orbits of a subgroup of the Weyl Group. The diagram for the restricted roots

ISSN 0949–5932 / $2.50

allows us to determine the upper bounds for the number of control sets. The central idea is that we can decompose the diagram corresponding to a subset of the set of simple roots in other diagrams of a known type.

2. Preliminaries

In this section we present the main result that will be used in the paper. We refer to [6, 7] for the theory of control sets for semigroup actions and [9] for the theory of semisimple Lie groups and their parabolic subgroups.

Thus let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition, where $\mathfrak{k}$ is a maximal compact subalgebra and $\mathfrak{s}$ its complement with respect to the Cartan-Killing form. Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra, and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber. Denote by Σ^+ the corresponding system of positive roots. The simple system of roots generating Σ^+ is denoted by Π. The set of all roots is $\Sigma = \Sigma^+ \cup (-\Sigma^+)$. For a root $\lambda \in \Sigma$, we let $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \lambda(H)X\}$ be its root space, and put

$$\mathfrak{n}^+ = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda,$$

which gives rise to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}^+$. The global Iwasawa decomposition reads $G = KAN^+$, with $K = \exp \mathfrak{k}$, $A = \exp \mathfrak{a}$ and $N^+ = \exp \mathfrak{n}^+$. Let M be the centralizer of $\mathfrak{a}$ (or A) in K, and M^* the normalizer. The finite group $W = M^*/M$ is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$.

Given a subset $\Theta \subset \Pi$ we denote by P_Θ the parabolic subgroup associated to Θ and by $B_\Theta = G/P_\Theta$ the corresponding flag manifold.

Let S be a semigroup with interior points in G, i.e., $\mathrm{int}_G(S) \neq \emptyset$. The number of control sets for S on the flag manifolds of G are given in terms of the Weyl group W. San Martin and Tonelli in [6] proved that for each $w \in W$ there is a control set D_w for S on B. Furthermore, each control set for S on B is D_w for some $w \in W$. The unique invariant control set on B is D_1, $1 \in W$. It is also shown in [6] that the subset defined by

$$W(S) = \{w \in W : D_w = D_1\}$$

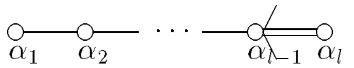
is a subgroup of W. The subgroup $W(S) \subset W$ gives information about the structure of S and in particular about its effective control sets on the flag manifolds. Based on the information provided by $W(S)$ we have the following result of [6, Corollary 5.2].

Proposition 1. *Let S be a semigroup with interior points in G. The number of control sets in a boundary $B_\Theta = G/P_\Theta$ is the order of the set of double cosets $W(S)\backslash W/W_\Theta$ where W_Θ is the subgroup of W generated by the reflections defined by Θ. Therefore an upper bound for the number of control sets on a flag manifold B_Θ is the order of W/W_Θ.*

3. The upper bounds

In this section we determine upper bounds for the number of control sets on flag manifolds of a real non-compact simple Lie group whose Lie algebra is a real form of a complex simple Lie algebra. We analyze the diagram of the simple system of roots. They can be obtained from [5, Table 9]. The Dynkin diagram associated to the system of roots enables us to determine the order of W_Θ, the subgroup of the Weyl group generated by the reflections defined by Θ. The point is that the Dynkin diagram corresponding to Θ is composed of diagrams of a known type. Thus the order of W_Θ is the product of the orders of the Weyl groups corresponding to the diagrams. We refer to [5] for the orders of the Weyl groups. In order to illustrate our method we present the explicit calculation for C_l:

1. (The simple system $\Pi = C_l$) The set of simple roots is $\Pi = \{\lambda_1 - \lambda_2, ..., \lambda_{l-1} - \lambda_l, 2\lambda_l\}$. We use the notation $\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, ..., l-1$ and $\alpha_l = 2\lambda_l$. The Dynkin diagram is

   $C_l, l \geq 3$ 

   The Weyl group W has $2^l l!$ elements. We consider $\Pi(i,j) = \{\lambda_r - \lambda_{r+1} : i \leq r \leq j \text{ and } j < l\}$. Any $\Theta \subset \Pi$ can be written as one of the disjoint union $\Theta = \Pi(i_1, j_1) \cup ... \cup \Pi(i_k, j_k) \cup \{2\lambda_l\}$ or $\Theta = \Pi(i_1, j_1) \cup ... \cup \Pi(i_k, j_k)$ with $j_n + 1 < i_{n+1}$ for every $n = 1, ..., k-1$ and $k \leq l-1$. If we consider $\Theta = \Pi(i_1, j_1) \cup ... \cup \Pi(i_k, j_k)$ the Dynkin diagram decomposes in k diagrams of the type $A_{j_n - i_n + 1}$ and therefore $\mid W_\Theta \mid = (j_1 - i_1 + 2)!...(j_k - i_k + 2)!$. It follows that an upper bound for the number of effective control sets for S on B_Θ is $2^l l!/(j_1 - i_1 + 2)!...(j_k - i_k + 2)!$. Let us assume that $\Theta = \Pi(i_1, j_1) \cup ... \cup \Pi(i_k, j_k) \cup \{2\lambda_l\}$. We have two possibilities: $j_k = l-1$ or $j_k < l-1$. If $j_k = l-1$ the Dynkin diagram decomposes on $k-1$ diagrams of the type $A_{j_n - i_n + 1}$ and a diagram of the type $C_{l - i_k + 1}$ corresponding to $\Pi(i_k, j_k) \cup \{2\lambda_l\}$. Thus $\mid W_\Theta \mid = (j_1 - i_1 + 2)!...(j_{k-1} - i_{k-1} + 2)!(l - i_k + 1)! 2^{l - i_k + 1}$ and the number of effective control sets in B_Θ is at most $(2^{i_k - 1} l!)/(j_1 - i_1 + 2)!...(j_{k-1} - i_{k-1} + 2)!(l - i_k + 1)!$. If $j_k < l-1$ the Dynkin diagram decomposes in k diagrams of the type $A_{j_n - i_n + 1}$ and the isolated root $2\lambda_l$. Therefore the order of W_Θ is $(j_1 - i_1 + 2)!...(j_k - i_k + 2)!.2$ and an upper bound for the number of control sets on B_Θ is $2^{l-1} l!/(j_1 - i_1 + 2)!...(j_k - i_k + 2)!$.

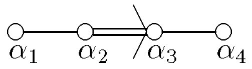
Now, we present a table with upper bounds for the number of control sets on flag manifolds of a Lie group whose Lie algebras have the simple system of roots of type A_l, B_l, C_l and D_l. We use the notation $n_p^k = p!/(j_1 - i_1 + 2)!...(j_k - i_k + 2)!$, $\Pi_k = \Pi(i_1, j_1) \cup ... \cup \Pi(i_k, j_k)$ and $\Pi(i,j)$ as above.

Π	Θ	j_k	$\lvert W/W_\Theta \rvert$
A_l	Π_k		n_{l+1}^k
B_l	$\Pi_k \cup \{\lambda_l\}$	$l-1$	$(2^{i_k-1}/(l-i_k+1)!)n_l^{k-1}$
		$< l-1$	$2^{l-1}n_l^k$
	Π_k		$2^l n_l^k$
C_l	$\Pi_k \cup \{2\lambda_l\}$	$l-1$	$(2^{i_k-1}/(l-i_k+1)!)n_l^{k-1}$
		$< l-1$	$2^{l-1}n_l^k$
	Π_k		$2^l n_l^k$
D_l	$\Pi_k \cup \{\lambda_{l-1}+\lambda_l\}$	$l-1$	$(2^{i_k-1}/(l-i_k+1)!)n_l^{k-1}$
		$< l-1$	$2^{l-2}n_l^k$
	Π_k		$2^{l-1}n_l^k$

In case of a flag manifold of a Lie Group with exceptional Lie algebra the calculation of the number of control sets on these manifolds follows the same method as above. As an illustration we make the calculation for the simple system $\Pi = F_4$.

1. (The simple system $\Pi = F_4$). The Dynkin Diagram is

F_4



With the simple system of roots $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The order of the Weyl group is 1152. If $\Theta = \Pi - \{\alpha_4\}$ we get the Dynkin diagram of B_3. An upper bound for the number of control sets for S on B_Θ is $1152/(2^3 3!) = 24$. For $\Theta = \Pi - \{\alpha_1\}$ we get the Dynkin diagram of C_3 and there exist at most 24 control sets for S on B_Θ. Now, take $\Theta = \Pi - \{\alpha_3\}$ or $\Theta = \Pi - \{\alpha_2\}$. In these cases the Dynkin diagram corresponding to Θ consists of an isolated root and a diagram of type A_2. Therefore an upper bound for the number of control sets on B_Θ will be $1152/((2)(3!)) = 96$. For $\Theta = \Pi - \{\alpha_1, \alpha_2\}$ or $\Theta = \Pi - \{\alpha_3, \alpha_4\}$ the Dynkin diagram corresponding to Θ is of type A_2 and the number of control sets for S on B_Θ is at most $1152/3! = 192$. For $\Theta = \Pi - \{\alpha_1, \alpha_3\}$, $\Theta = \Pi - \{\alpha_2, \alpha_3\}$ or $\Theta = \Pi - \{\alpha_2, \alpha_4\}$ we have two isolated roots and the order of W_Θ will be 4. Therefore an upper bound for the number of control sets on B_Θ will be $1152/4 = 288$. If we take $\Theta = \Pi - \{\alpha_1, \alpha_4\}$ then the Dynkin diagram for Θ is of type B_2 and there are at most $1152/(2^2 2!) = 144$ control sets on B_Θ. It remains to look at the case where Θ consist of only one root. In this case the order of W_Θ is 2 and there are at most $1152/2 = 576$ control sets on B_Θ.

References

[1] Albertini, F. and E. D. Sontag, *Discrete time transitivity and accessibility: Analytic systems*, SIAM J. Control Optimiz. **31** (1993), 1599–1622.

[2] Barros, C. J. B. and L. A. B. San Martin, *On the number of control sets on projective spaces.* Systems & Control Letters (Elsevier) **29** (1996) 21–26.

[3] Colonius, F. and W. Kliemann, *Some aspects of control systems as dynamical systems*, Journal of Dynamics and Differential Equations **5** (1993), 469-494.

[4] —, *Linear control semigroups acting on projective spaces*, Journal of Dynamics and Differential Equations **5** (1993), 495–528.

[5] Onishchik, A. L., and E. B. Vinberg, "Lie Groups and Algebraic Groups," Springer-Verlag, Berlin etc., 1990.

[6] San Martin,L. A. B, and P. A. Tonelli, *Semigroup actions on homogeneous spaces*, Semigroup Forum, **50** (1995) 59–88.

[7] San Martin, L. A. B., *Control sets and semigroups in semisimple Lie groups*, In: K. H. Hofmann, J. D. Lawson, and E. B. Vinberg, Eds., "Semigroups in Algebra, Geometry and Analysis," De Gruyter Verlag, Berlin 1995, 275–291.

[8] Varadarajan, V. S., "Lie Groups, Lie Algebras and their Representations," Prentice-Hall, Englewood Cliffs, 1974.

[9] Warner, G., "Harmonic Analysis on Semisimple Lie Groups," Springer-Verlag, Berlin etc., 1972.

[10] Wirth, F. R., "Robust Stability of Discrete-Time Systems under Time-Varying Perturbations," Dissertation, University of Bremen, 1995.

Departamento de Matemática
Universidade Estadual de Maringá
Cx.Postal 331
87.020-900 Maringá-Pr
Brasil

Received July 24, 1997
and in final form January 28, 1998