Journal of Lie Theory
Volume **8** (1998) 95–110


# Kazhdan Constants Associated with Laplacian on Connected Lie Groups.

**M. E. B. Bekka, P-A. Cherix[1], P. Jolissaint**

Communicated by A. Valette

**Abstract.** Let $G$ be a finite dimensional connected Lie group. Fix a basis $\{X_i\}_{i=1,\cdots,n}$ of the Lie algebra $\mathfrak{g}$ and form the associated Laplace operator $\Delta = -\sum_{1\leq i\leq n} X_i^2$ in the enveloping algebra $U(\mathfrak{g})$. Let $\pi$ be a strongly continuous unitary representation of $G$; let $\overline{d\pi(\Delta)}$ be the closure of the essentially self-adjoint operator $d\pi(\Delta)$. We show that $\pi$ almost has invariant vectors if and only if 0 belongs to the spectrum of $\overline{d\pi(\Delta)}$. From this, we deduce that $G$ has Kazhdan's property $(T)$ if and only if there exists $\epsilon > 0$ such that, for any unitary representation without non zero fixed vectors, one has $\epsilon < \min\{\mathrm{Sp}(\overline{d\pi(\Delta)})\}$. This answers positively a question of Y. Colin de Verdière. It also allows us to define new Kazhdan constants, that we compare to the classical ones.

## 1. Introduction

In 1967, Kazhdan introduced property $(T)$, a fixed point property of unitary representations for locally compact groups. More precisely,

**Definition 1.1.** Let $G$ be a locally compact group.

1. Let $\pi : G \to U(H_\pi)$ be a strongly continuous unitary representation, $\epsilon > 0$ and let $K \subset G$ be a compact subset of $G$; a vector $\xi \in H_\pi^1$, the set of vectors of length 1 in $H_\pi$, is $(\epsilon, K)$*-invariant* if
$$\sup\{\| \pi(g)\xi - \xi \| \mid g \in K\} < \epsilon.$$
2. $\pi$ *has almost invariant vectors* if for every $\epsilon$ and $K$ as above, there exists an $(\epsilon, K)$-invariant vector.
3. $G$ *has property (T)*, if for all representations with almost invariant vectors, there exists a nonzero fixed vector.

[1]Supported by Grant 20-43215.95 of the Swiss National Fund for Scientific Research

ISSN 0949–5932 / \$2.50 

This property has strong consequences both from practical and theoretical points of view (see [10]). In graph theory, the property $(T)$ permits the construction of family of expanders via the knowledge of the Kazhdan constant (see [11]). Let $(\pi, H_\pi)$ be a representation of $G$ and let $K$ be a compact generating set of $G$. We define $\kappa(G,K,\pi)$ as follows :

$$\kappa(G,K,\pi) = \inf_{\xi \in H_\pi^1} \max_{s \in K} \| \pi(s)\xi - \xi \| .$$

The Kazhdan constant of the group $G$ relatively to $K$ is defined by:

$$\kappa(G,K) = \inf\{\kappa(G,K,\pi) \mid \pi \in \widetilde{G}^*\}$$

where $\widetilde{G}^*$ is the set of equivalence classes of unitary representations of $G$ on separable Hilbert spaces, without nonzero fixed vectors.

**Proposition 1.2.** *For $G$ a locally compact group and $K$ a compact generating set of $G$, the following assertions are equivalent :*

1. *$G$ has property $(T)$*
2. *$\kappa(G,K) > 0$*

A proof of this result is given in [4].

In general, it is difficult to see whether a group $G$ has property $(T)$. In this paper we give an equivalent definition of property $(T)$ for connected Lie groups. This approach was suggested by Colin de Verdière [1].

## 2. Definitions and first properties

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and enveloping algebra $U(\mathfrak{g})$. We use on $G$ a right invariant Haar measure $dg$.

Any unitary representation $(\pi, H_\pi)$ of $G$ induces a representation $d\pi$ of $\mathfrak{g}$ on the subspace $\mathcal{C}^\infty(\mathcal{H}_\pi)$ of $\mathcal{C}^\infty$-vectors of $\pi$, i.e. the subspace of vectors $\xi \in H_\pi$ for which the function $x \mapsto \pi(x)\xi$ is a $\mathcal{C}^\infty$ function. $d\pi$ extends to a representation of $U(\mathfrak{g})$ on the same subspace $\mathcal{C}^\infty(\mathcal{H}_\pi)$.

Let $\xi$ and $\eta$ be two vectors in $H_\pi$, we denote by $\varphi_{\xi,\eta}$ the function on $G$ defined by $\varphi_{\xi,\eta}(g) = \langle \pi(g)\xi \mid \eta\rangle$. We call $\varphi_{\xi,\eta}$ the coefficient of $\pi$ associated to $\xi$ and $\eta$.

**Lemma 2.1.** *Using the same notations as before, if $\xi$ is in $\mathcal{C}^\infty(\mathcal{H}_\pi)$, $\eta$ is in $H_\pi$ and $X$ is in $\mathfrak{g}$, then $\varphi_{\xi,\eta}$ satisfies :*

1. $X\varphi_{\xi,\eta} = \varphi_{d\pi(X)\xi,\eta}$ ,
2. $\Delta\varphi_{\xi,\eta} = \varphi_{d\pi(\Delta)\xi,\eta}$ ,
3. $\Delta^n\varphi_{\xi,\eta} = \varphi_{d\pi(\Delta)^n\xi,\eta}$ *for every* $n \geq 1$.

**Proof.** By definition, we get for every $g$ in $G$ :

$$\begin{aligned}(X\varphi_{\xi,\eta})(g) &= \lim_{t\to 0} \frac{\varphi_{\xi,\eta}(g\exp(tX)) - \varphi_{\xi,\eta}(g)}{t} \\ &= \lim_{t\to 0} \frac{1}{t}\left[\left\langle \pi(\exp(tX))\xi \,|\, \pi(g^{-1})\eta\right\rangle - \left\langle \xi|\pi(g^{-1})\eta\right\rangle\right] \\ &= \left\langle d\pi(X)\xi \,|\, \pi(g^{-1})\eta\right\rangle = \varphi_{d\pi(X)\xi,\eta}(g)\end{aligned}$$

This proves (1). The assertions (2) and (3) follow from the fact that $d\pi(X)\xi$ is a $\mathcal{C}^\infty$ vector if $\xi$ is. ∎

**Lemma 2.2.** *Let $X$ be an element of $\mathfrak{g}$ and $\psi$ be a $\mathcal{C}^\infty$ function on $G$ such that $\psi$ and $X\psi$ lie in $L^1(G)$; then $\int_G (X\psi)(g)dg = 0$.*

**Proof.** Suppose first that $\psi \in \mathcal{C}_0^\infty(G)$, the space of $\mathcal{C}^\infty$ with compact support on $G$ :

$$\begin{aligned}\int_G (X\psi)(g)dg &= \int_G \left(\lim_{t\to 0} \frac{\psi(g\exp(tX)) - \psi(g)}{t}\right) dg \\ &= \lim_{t\to 0} \int_G \frac{\psi(g\exp(tX)) - \psi(g)}{t} dg = 0.\end{aligned}$$

The two last equalities hold because $Supp(\psi)$ is compact and because the measure $dg$ is invariant by right multiplication. Now, if $\psi \in \mathcal{C}^\infty(G)$ is such that $\psi$ and $X\psi$ lie in $L^1(G)$, there exists a compact subset $K$ of $G$ such that $\int_{G-K} |\psi(g)|dg < \epsilon/3$ and $\int_{G-K} |(X\psi)(g)|dg < \epsilon/3$. So we have : $|\int_G (X\psi)(g)dg| < |\int_K (X\psi)(g)dg| + \epsilon/3$.

For $\chi \in \mathcal{C}_0^\infty(G)$, $\chi \equiv 1$ on $K$, $0 \le \chi \le 1$, $|(X\chi)(g)| \le 1$, we have

$$\begin{aligned}\int_K (X\psi)(g)dg &= \int_K (X\psi)(g)\chi(g)dg \\ &= \int_K (X(\psi\chi))(g)dg \ \ (\text{because } X\chi = 0 \text{ on } K) \\ &= -\int_{G-K} (X(\psi\chi))(g)dg \ \ (\text{because } \int_G X(\psi\chi)(g)dg = 0) \\ &= -\int_{G-K} (X\psi)(g)\chi(g)dg - \int_{G-K} (X\chi)(g)\psi(g)dg\end{aligned}$$

But $|\int_{G-K} (X\psi)(g)\chi(g)dg| \le \int_{G-K} |(X\psi)(g)|dg \le \epsilon/3$ and $|\int_{G-K} (X\chi)(g)\psi(g)dg| \le \int_{G-K} |\psi(g)|dg \le \epsilon/3$. ∎

**Corollary 2.3.** *Let $X$ be an element of $\mathfrak{g}$ and $\psi$ be a $\mathcal{C}^\infty$ function on $G$ such that $\psi$ and $X\psi$ lie in $L^1(G)$. If $\xi$ belongs to $\mathcal{C}^\infty(H_\pi)$ and $\eta$ belongs to $H_\pi$, then :*

$$\int_G \left\langle \xi|\pi(g^{-1})\eta\right\rangle (X\psi)(g)dg = -\int_G \left\langle d\pi(X)\xi|\pi(g^{-1})\eta\right\rangle \psi(g)dg.$$

**Proof.** The corollary follows from 2.2, 2.1 (1) and from the equality :

$$\int_G \varphi_{\xi,\eta}(g)(X\psi)(g)dg + \int_G (X\varphi_{\xi,\eta})(g)\psi(g)dg = \int_G (X(\varphi_{\xi,\eta}\psi))(g)dg = 0. \quad ∎$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$; set $\Delta = -\sum_{i=1}^n X_i^2 \in U(\mathfrak{g})$ The operator $d\pi(\Delta)$ is essentially selfadjoint [8], and we denote by $\overline{d\pi(\Delta)}$ its closure and by $D(\overline{d\pi(\Delta)})$ the domain of $\overline{d\pi(\Delta)}$.

**Corollary 2.4.** *If $\psi$ is a $\mathcal{C}^\infty$ function on $G$ such that $\psi$, $X_i\psi$ and $X_i^2\psi$ belong to $L^1(G)$ for every $i = 1, \ldots, n$, then for every $\xi$ in $D(\overline{d\pi(\Delta)})$ and for every $\eta$ in $H_\pi$ :*

$$\int_G \varphi_{\xi,\eta}(g)(\Delta\psi)(g)dg = \int_G \varphi_{\overline{d\pi(\Delta)}\xi,\eta}(g)\psi(g)dg.$$

**Proof.** By lemmas 2.1 and 2.3, the equality is true for $\xi \in \mathcal{C}^\infty(H_\pi)$. If $\xi \in D(\overline{d\pi(\Delta)})$, there exists a sequence $(\xi_k)_{k\geq 1}$ in $\mathcal{C}^\infty(H_\pi)$ such that

$$\| \xi_k - \xi \| + \| \overline{d\pi(\Delta)}\xi_k - \overline{d\pi(\Delta)}\xi \| \underset{k\to\infty}{\longrightarrow} 0.$$

$$\begin{aligned}
\text{Then :} \qquad & | \int \varphi_{\xi,\eta}(g)(\Delta\psi)(g)dg - \int \left\langle \overline{d\pi(\Delta)}\xi | \pi(g^{-1})\eta \right\rangle \psi(g)dg | \\
\leq \quad & | \int \left\langle \xi - \xi_k | \pi(g^{-1})\eta \right\rangle (\Delta\psi)(g)dg | \\
& + | \int \left\langle \xi_k | \pi(g^{-1})\eta \right\rangle (\Delta\psi)(g)dg \\
& - \int \left\langle \overline{d\pi(\Delta)}\xi_k | \pi(g^{-1})\eta \right\rangle \psi(g)dg | \\
& + | \int \left\langle \overline{d\pi(\Delta)}\xi_k - \overline{d\pi(\Delta)}\xi | \pi(g^{-1})\eta \right\rangle \psi(g)dg | \\
\leq \quad & (\| \xi - \xi_k \| \| \Delta\psi \|_1 + \| \overline{d\pi(\Delta)}(\xi_k - \xi) \| \| \psi \|_1) \| \eta \| \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This finishes the proof. ∎

**Proposition 2.5.** *The following conditions are equivalent :*

1. *$\pi$ has a non zero fixed vector.*
2. *0 is an eigenvalue for $d\pi(\Delta)$.*
3. *0 is an eigenvalue for $\overline{d\pi(\Delta)}$.*

**Proof.** The implications 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) are obvious.

We prove now 3) $\Rightarrow$ 1) : Let $\xi$ be a non zero vector in $Ker(\overline{d\pi(\Delta)})$. Let $\eta \in H_\pi$ and $\psi \in \mathcal{C}_0^\infty(G)$. Then, using Corollary 2.4, we have in the sense of weak derivatives :

$$(\Delta\varphi_{\xi,\eta}, \psi) = (\varphi_{\xi,\eta}, \Delta\psi) = (\varphi_{\overline{d\pi(\Delta)}\xi,\eta}, \psi) = 0.$$

Therefore, $\Delta\varphi_{\xi,\eta} = 0$ as a distribution, and since $\Delta$ is hypo-elliptic (see [8]), $\varphi_{\xi,\eta}$ is a $\mathcal{C}^\infty$ function on $G$.

Since $\eta$ is arbitrary, this implies that $g \mapsto \pi(g)\xi$ is weakly $\mathcal{C}^\infty$ and so, by a lemma due to Poulsen (see [15]), $g \mapsto \pi(g)\xi$ is strongly $\mathcal{C}^\infty$. Therefore, $\overline{d\pi(\Delta)}\xi = d\pi(\Delta)\xi = 0$. Hence,

$$\sum_{i=1}^n \| d\pi(X_i)\xi \|^2 = \langle d\pi(\Delta)\xi | \xi \rangle = 0.$$

Since $\{X_i\}_{i=1,\ldots,n}$ is a basis of $\mathfrak{g}$, this implies that $d\pi(X)\xi = 0$ for every element of $\mathfrak{g}$. Hence, $\pi(g)\xi = \xi$ for every $g$ in $V$, a suitable neighbourhood of $e$ in $G$. As $G$ is connected, $V$ generates $G$ and so $\xi$ is fixed under the action of $G$. ∎

## 3. Spectral characterisation of almost invariant vectors

The goal of this section is to prove :

**Theorem 3.1.** *For a unitary representation $(\pi, H_\pi)$ of $G$, the following conditions are equivalent :*

1. *$\pi$ almost has invariant vectors.*
2. *$0$ is an approximate eigenvalue of $d\pi(\Delta)$.*
3. *$0$ is a spectral value of $\overline{d\pi(\Delta)}$.*

**Proof.** The equivalence between 2) $\Leftrightarrow$ 3) is clear, since $\overline{d\pi(\Delta)}$ is the closure of $d\pi(\Delta)$.

2) $\Rightarrow$ 1)

If 0 is an approximate eigenvalue of $d\pi(\Delta)$, there exists a sequence $\{\xi_m\}_{m\geq 0}$ of unit vectors in $\mathcal{C}^\infty(H_\pi)$ such that $\lim_{m\to+\infty} \| d\pi(\Delta)\xi_m \| = 0$. So,

$$\lim_{m\to+\infty} \| d\pi(X_i)\xi_m \| = 0 \text{ for every } i = 1, \ldots, n.$$

Let $V$ defined by

$$V = \{ \prod_{1\leq i\leq n} \exp(t_i X_i) \mid -1 \leq t_i \leq 1\}.$$

$V$ is a compact neighbourhood of $e$, see [5].

Moreover, for every $X$ in $\mathfrak{g}$ and every $t \geq 0$, we have :

$$\pi(\exp(tX))\xi_m - \xi_m = \int_0^t \pi(\exp(sX)) d\pi(X)\xi_m ds\,.$$

Let $\epsilon > 0$. Then, for $0 \leq t \leq 1$, we have :

$$\| \pi(\exp(tX_i))\xi_m - \xi_m \| \leq t \| d\pi(X_i)\xi_m \| \leq \epsilon/n$$

for $i = 1, \cdots, n$ and $m$ sufficiently large.

This implies that $\| \pi(g)\xi_m - \xi_m \| \leq \epsilon$ for every $g$ in $V$ as soon as $m$ is large enough.

So we proved that, for every $\epsilon > 0$, there exist $(\epsilon, V)$-invariant vectors. As $G$ is connected, $V$ generates $G$ and one easily deduces that $\pi$ has $(\epsilon, K)$-invariant vectors for any compact subset $K$ of $G$.

The remainder of this section is devoted to the proof of the implication 1) $\Rightarrow$ 3) which is much more involved. For this, we need to recall some facts about the heat kernel associated with $\Delta$.

Let $h$ denote the closure of $\Delta$ acting on $L^2(G)$. As $-h$ is a selfadjoint and negative definite operator, by the Hille-Yosida theorem (see [16]), $-h$ is the infinitesimal generator of a strongly continous semi-group of contractions $T(t)$.

It is known (see [13]) that, for every $t > 0$, $T(t)$ is given by the convolution from the right with a function $p_t$, the heat kernel. The function $p_t$ is a smooth function on $G$ with positive values and integral 1. We briefly recall its construction.

Let $\delta$ denote the modular function associated with our right Haar measure, thus :

$$\int_G \delta(x)f(xy)dy = \int_G f(y)dy,$$

for every integrable function $f$ over $G$. The right regular representation over $L^2(G)$ is given by :

$$(\rho(x)\xi)(y) = \xi(yx).$$

The left regular representation is then given by :

$$(\lambda(x)\xi)(y) = \delta(x^{-1})^{1/2}\xi(x^{-1}y).$$

If $\Delta = -\sum_{i=1}^{n} X_i^2$ and if $h$ is the closure of $\Delta$ on $L^2(G)$, then $h$ and $\lambda(x)$ commute for every $x$ in $G$.

For every $t \geq 0$, let $T(t) = \exp(-th)$.

We claim that for every $t > 0$, $T(t)$ is a regularising operator, that is, for every $\xi \in L^2(G)$, for every integer $m \geq 1$, $h^m T(t)\xi \in D(h)$. Indeed, let $h = \int_0^\infty \lambda dE(\lambda)$ be the spectral decomposition of $h$. As the function $\lambda \mapsto \lambda^{2m}\exp(-2t\lambda)$ is bounded on $\mathbb{R}_+$, we have :

$$\int_0^\infty \lambda^{2m}\exp(-2t\lambda)dE_\xi(\lambda) = \| h^m T(t)\xi \|^2 < \infty, \forall \xi \in L^2(G).$$

By Sobolev's lemma, $T(t)\xi \in \mathcal{C}^\infty(G)$, $\forall \xi \in L^2(G)$.

By Schwartz Kernel theorem, there exists a $\mathcal{C}^\infty$ function $p'_t : G \times G \to \mathbb{R}_+$ such that

$$(T(t)\xi)(x) = \int_G p'_t(x,y)\xi(y)dy, \forall \xi \in L^2(G).$$

Hence ,

$$\begin{aligned}(\lambda(x)T(t)\xi)(x') &= \delta(x^{-1})^{1/2}(T(t)\xi)(x^{-1}x') \\ &= \delta(x^{-1})^{1/2}\int_G p'_t(x^{-1}x',y)\xi(y)dy\end{aligned}$$

Moreover

$$\begin{aligned}(T(t)\lambda(x)\xi)(x') &= \int_G p'_t(x',y)(\lambda(x)\xi)(y)dy \\ &= \delta(x^{-1})^{1/2}\int_G p'_t(x',y)\xi(x^{-1}y)dy \\ &= \delta(x^{-1})^{1/2}\int_G \delta(x)p'_t(x',xy)\xi(y)dy\end{aligned}$$

As $T(t)$ and $\lambda(x)$ commute, we deduce that

$$(*)\ p'_t(x^{-1}x',y) = \delta(x)p'_t(x',xy)\, \forall x, x', y \in G.$$

Set $p_t(x) = p'_t(e,x)$.

The above relation $(*)$ shows that, for all $y \in G$,

$$p_t(x^{-1}y)\delta(x^{-1}) = p'_t(e,x^{-1}y)\delta(x^{-1}) = p'_t(x,y).$$

Hence

$$\begin{aligned}(T(t)\xi)(x) &= \int p'_t(x,y)\xi(y)dy = \int p_t(x^{-1}y)\delta(x^{-1})\xi(y)dy \\ &= \int p_t(y)\xi(xy)dy \text{ for } \xi \in L^2(G).\end{aligned}$$

Recall that if $\pi$ is a representation of $G$, then $\pi(f)$ is the operator defined for $f \in L^1(G)$ by

$$\pi(f)\eta = \int f(y)\pi(y)\eta dy \text{ for every } \eta \in H_\pi.$$

Taking $\pi = \rho$ and $f = p_t$, we see that :

$$(\rho(p_t)\xi)(x) = \int p_t(y)(\rho(y)\xi)(x)dy = \int p_t(y)\xi(xy)dy = (T(t)\xi)(x).$$

As $T(t) = T(t)^*$, we have $p_t = p_t^*$, and $p_t$ is a solution of the heat equation :

$$\frac{\partial p_t}{\partial t} = -\Delta p_t.$$

**Lemma 3.2.** *$\Delta p_t$ belongs to $L^1(G)$ and*

$$\lim_{s\to 0^+} \| \frac{p_{t+s} - p_t}{s} + \Delta p_t \|_1 = 0, \text{ for every } t > 0.$$

**Proof.** We denote by $\rho_1$ the right regular representation of $G$ on $L^1(G)$. Following ([13], theorem 4, p.599), we set for $t > 0$ and for $f \in L^1(G)$ :

$$P^t f = \int_G p_t(y)\rho_1(y)f dy.$$

By lemma 7.1 and theorem 4 of [13], $P^t f$ is an analytic vector in the following sense : there exists $s > 0$ such that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{1\le i_1,\ldots,i_m\le n} \| d\rho_1(X_{i_1})\ldots d\rho_1(X_{i_m})P^t f\|_1 \, s^m < \infty.$$

In particular, $p_t = P^{t/2}p_{t/2}$ is analytic in the preceding sense, and $hp_t \in L^1(G)$. The last assertion is a consequence of mean value theorem and of Lebesgue's dominated convergence theorem. ∎

Let now $\pi$ be a unitary representation of $G$. Set

$$S(t) = \pi(p_t), \text{ if } t > 0 \text{ and } S(0) = \mathbb{I}_{H_\pi}.$$

$(S(t))_{t\ge 0}$ is a strongly continuous semi-group on $H_\pi$ with infinitesimal generator $A$. Moreover $S(t)^* = S(t)$ for every $t$ because $p_t = p_t^*$.

By Corollary 10.6, p.41 of [14], $A$ is selfadjoint. More precisely the following is true.

**Lemma 3.3.** *With the same notations, one has $A = -\overline{d\pi(\Delta)}$, where $\overline{d\pi(\Delta)}$ denotes the closure of $d\pi(\Delta)$.*

**Proof.** Let $\xi$ be in $D(\overline{d\pi(\Delta)})$. We claim that, for every $t>0$, $S(t)\xi$ belongs to $D(A)$ and

$$AS(t)\xi = -S(t)\overline{d\pi(\Delta)}\xi.$$

Indeed, let $\eta$ in $H_\pi$. Then, for $0 < s \leq 1$ :

$$\begin{aligned}
\left\langle \frac{S(s)S(t)\xi - S(t)\xi}{s} | \eta \right\rangle &= \frac{1}{s}\langle \pi(p_{t+s})\xi - \pi(p_t)\xi|\eta\rangle \\
&= \frac{1}{s}\int (p_{t+s}(g) - p_t(g)) \left\langle \xi|\pi(g^{-1})\eta\right\rangle dg \\
&= \int \frac{p_{t+s}(g) - p_t(g)}{s}\varphi_{\xi,\eta}(g)dg.
\end{aligned}$$

By lemma 3.2 and corollary 2.4,

$$\begin{aligned}
\lim_{s\to 0^+} \left\langle \frac{S(s)S(t)\xi - S(t)\xi}{s}|\eta \right\rangle &= \int (-\Delta p_t)(g)\varphi_{\xi,\eta}(g)dg \\
&= -\left\langle S(t)\overline{d\pi(\Delta)}\xi|\eta\right\rangle.
\end{aligned}$$

By theorem 1.3, p.43 of [14], $S(t)\xi$ belongs to $D(A)$ and $AS(t)\xi = -S(t)\overline{d\pi(\Delta)}\xi$. This proves the claim. But $\| S(t)\xi - \xi \| \underset{t\to 0}{\longrightarrow} 0$ and $\| AS(t)\xi - (-\overline{d\pi(\Delta)}\xi) \| = \| -S(t)\overline{d\pi(\Delta)}\xi + \overline{d\pi(\Delta)}\xi \| \underset{t\to 0}{\longrightarrow} 0.$

Since $A$ is a closed operator, $\xi$ belongs to $D(A)$ and $A\xi = -\overline{d\pi(\Delta)}\xi$.

So $-\overline{d\pi(\Delta)} \subset A$, and as they are selfadjoint, we have $-\overline{d\pi(\Delta)} = A$. ∎

**Lemma 3.4.** *If $(\pi, H_\pi)$ is a unitary representation of a locally compact group $G$ with almost invariant vectors and if $\mu$ is a probability measure over $G$, the spectrum of the operator $\pi(\mu) = \int \pi(g)d\mu(g)$ contains 1.*

**Proof.** Let $(\xi_k)_{k\geq 1}$ a sequence of unit vectors in $H_\pi$ such that $\varphi_{\xi_k,\xi_k}$ tends to 1 uniformly over every compact set of $G$. We show that

$$\lim_{n\to\infty} \| \pi(\mu)\xi_k - \xi_k \| = 0.$$

Let $\epsilon > 0$; there exists a compact set $K$ in $G$ such that $\mu(G-K) \leq \epsilon$. So we have

$$\| \pi(\mu)\xi_k - \xi_k \| \leq \int_K \| \pi(g)\xi_k - \xi_k \| \, d\mu(g) + 2\epsilon < 3\epsilon$$

for $k$ sufficiently large. ∎

As we now show, the converse is true under some restrictions on $\mu$. Although we shall not use this result, we give the proof as we think it is of interest.

**Lemma 3.5.** *Let $G$ be a locally compact group and let $\mu$ be a absolutely continuous probability measure with respect to the Haar measure and such that the support generates topologically $G$. Let $\pi$ be a unitary representation on a Hilbert space $H_\pi$. Then $\pi$ has almost invariant vectors, if the spectrum of the operator $\pi(\mu) = \int \pi(g)d\mu(g)$ contains 1.*

**Proof.** By replacing $\mu$ by $\frac{1}{2}(\mu+\check{\mu})$, we can assume that $\mu$ is symmetric. Then $\pi(\mu)$ is selfadjoint and there exist unit vectors $\xi_n$ such that $\|\pi(\mu)\xi_n - \xi_n\| \to 0$.

So we have :

$$\lim_{n\to\infty} \int_G (1 - \mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle)d\mu(x) = 0.$$

As $\mu \geq 0$ and $1 - \mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle \geq 0$, there exist a subsequence of $\{\xi_n\}_{n\geq 0}$ (that we will also denote by $\{\xi_n\}_{n\geq 0}$), such that

$$\lim_{n\to\infty} 1 - \mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle \geq 0$$

for $\mu$-almost every $x \in G$.

On an other hand, by compactness of the unit ball of $L^\infty$ gifted with the weak$^*$ topology, there exists a subsequence of $\{\xi_n\}_{n\geq 0}$ (still denoted by $\{\xi_n\}_{n\geq 0}$) and a positive type function $\varphi$ on $G$ such that

$$\lim_{n\to\infty} \varphi(x) - \mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle \geq 0$$

almost everywhere with respect to the Haar measure (this is true first for the weak$^*$ topology $\sigma(L^\infty, L^1)$ and the claim follows by the same arguments as before).

As $\mu$ is absolutely continuous, we have $\varphi = 1$ $\mu$-almost everywhere. As $\varphi$ is a measurable, positive definite function, $\varphi$ is continuous. Hence $\varphi = 1$ on the support of $\mu$.

Therefore, $\varphi = 1$ on the closed subgroup generated by $Supp(\mu)$ which is $G$, by assumption.

So $\lim\limits_{n\to\infty}$ $\mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle = 1$ almost everywhere on $G$ and by Lebesgue's dominated convergence, this is also true in the $\sigma(L^\infty, L^1)$ topology. By [2], Theorem 13.5.2, it follows that $\lim\limits_{n\to\infty}$ $\mathrm{Re}\,\langle\pi(x)\xi_n|\xi_n\rangle = 1$ uniformly on every compact subset of $G$ and, hence, $\lim\limits_{n\to\infty}$ $\|\pi(x)\xi_n - \xi_n\| = 0$ uniformly on compact subsets of $G$. This shows that $\pi$ almost has invariant vectors. ∎

Now we are able to finish the proof of Theorem 3.1.

If $\pi$ almost has invariant vectors, by lemma 3.4, the spectrum of $\pi(p_t)$ contains $1$, for every $t > 0$.

Now $-\overline{d\pi(\Delta)}$ is the infinitesimal generator of the semi-group $(\pi(p_t))_{t\geq 0}$ so that $0$ is in the spectrum of $\overline{d\pi(\Delta)}$ by functional calculus and lemma 3.3. ∎

One may wonder whether it is necessary to use, as we did, arguments involving the heat kernel to prove 1) $\Rightarrow$ 3). For instance, one might think that if, for a compact neighbourhood $V$ of $e$ in $G$, $\{\xi_n\}_{n\geq 0}$ is a family of $(1/n, V)$-invariant $\mathcal{C}^\infty$ vectors, then $\|d\pi(\Delta)\xi_n\|$ should tend to $0$.
The following example shows that this is not always true.

**Example 3.6.** Let $G = \mathbb{R}$ be the real line. We define first, the following family of unitary representations of degree $2$ :

$$s \mapsto \pi_n(s) = \begin{pmatrix} \exp(is/n) & 0 \\ 0 & \exp(isn) \end{pmatrix}.$$

We consider the unitary representation $\pi$ defined by $s \mapsto \bigoplus_{n\geq 0} \pi_n(s)$.

We also define $\xi_n = (0,\ldots,0,\sqrt{\frac{n-1}{n}},\sqrt{\frac{1}{n}},0,\ldots)$ where the nonzero components of $\xi_n$ are in $2n^{th}$ and $(2n+1)^{th}$ places.

By construction the $\xi_n$ have norm 1. Let $I$ be a compact subset of $\mathbb{R}$. Then, for every $s$ in $I$, we have :

$$\begin{aligned} \| \pi(s)\xi_n - \xi_n \|^2 &= |(\exp(is/n)-1)\sqrt{\frac{n-1}{n}}\,|^2 + |(\exp(isn)-1)\sqrt{\frac{1}{n}}\,|^2 \\ &= |(\exp(is/n)-1)|^2\,\frac{n-1}{n} + |\exp(isn)-1|^2\,\frac{1}{n}. \end{aligned}$$

As $|\exp(is/n)-1|^2 \underset{n\to\infty}{\longrightarrow} 0$ uniformly for $s$ in $I$ and as $|\exp(isn)-1|^2 \leq 4$, the $\xi_n$ are $(\epsilon_n, I)$-invariant vectors for a sequence $\{\epsilon_n\}_{n\geq 0}$ with $\epsilon_n > 0$ and $\lim \epsilon_n = 0$.

However this family $\xi_n$ does not satisfy $\| d\pi(\Delta)\xi_n \| \underset{n\to\infty}{\longrightarrow} 0$.

In fact,

$$d\pi(\Delta)\xi_n = (0,\ldots,0,\frac{1}{n^2}\sqrt{\frac{n-1}{n}}, n^2\sqrt{\frac{1}{n}}, 0,\ldots).$$

Thus, $\| d\pi(\Delta)\xi_n \| \approx n^{3/2}$; so it does not tend to 0.

**Lemma 3.7.** *Let $h$ be a selfadjoint operator on a Hilbert space $H$, with domain $D(h)$, and such that its spectrum is bounded from below. Then*

$$\min(\mathrm{Sp}\, h) = \inf_{\xi\in D(H)^1} \langle h\xi|\xi\rangle \ , \textit{ where } D(H)^1 = \{\xi \in D(H) |\ \|\xi\| = 1\}.$$

**Proof.** As $h$ is selfadjoint, its residual spectrum is empty. So every spectral value is an approximate eigenvalue. For every spectral value $\lambda$, there exists a sequence $\{\xi_n\}_{n\geq 0}$ in $D(h)^1$ such that $\langle h\xi_n|\xi_n\rangle \to \lambda$.

Hence,

$$\inf_{\xi\in D(h)^1} \langle h\xi|\xi\rangle \ \leq\ \lambda\,.$$

Let $\lambda_0 = \min(\mathrm{Sp}(h))$ (the minimum exists since the spectrum of $h$ is real, closed and bounded below). If we apply the last inequality to $\lambda_0$, we get :

$$\inf_{\xi\in D(h)^1} \langle h\xi|\xi\rangle \ \leq\ \lambda_0\,.$$

As for the other inequality, let $h = \int \lambda dE(\lambda)$ be the spectral decomposition of $h$; we have $\langle h\xi|\xi\rangle = \int \lambda\,\langle dE(\lambda)\xi|\xi\rangle$ and $\langle \xi|\xi\rangle = \int \langle dE(\lambda)\xi|\xi\rangle$ for every fixed $\xi$ in $D(h)$. Then, for $\xi$ in $D(h)^1$,

$$\langle h\xi|\xi\rangle = \int_{\mathrm{spec}(h)} \lambda\,\langle dE(\lambda)\xi|\xi\rangle \ \geq \lambda_0 \int_{\mathrm{spec}(h)} \langle dE(\lambda)\xi|\xi\rangle \ = \lambda_0\,.$$

This finishes the proof. ∎

**Definition 3.8.** For a unitary representation $\pi$ of $G$, we define the constant $k(\overline{d\pi(\Delta)}, G)$ by

$$k(\overline{d\pi(\Delta)}, G) = \inf_{\xi \in D(\overline{d\pi(\Delta)})^1} \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle .$$

**Corollary 3.9.** *The following holds :*

$$k(\overline{d\pi(\Delta)}, G) = \min \mathrm{Sp}(\overline{d\pi(\Delta)}) = \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \sum_{i=1}^n \| d\pi(X_i)\xi \|^2$$

**Proof.** The first equality comes from the preceding lemma. On the other hand, it is clear that

$$k(\overline{d\pi(\Delta)}, G) \leq \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle = \inf_{\xi \in [\mathcal{C}^\infty(H_\pi)]^1} \sum_{i=1}^n \| d\pi(X_i)\xi \|^2 .$$

To obtain the reverse inequality, it suffices to show that, for $\epsilon > 0$, there exists a $\mathcal{C}^\infty$ vector $\eta$ of norm 1 such that

$$|k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\eta | \eta \rangle \,| < \epsilon .$$

By definition, there exists $\xi$ in $D(\overline{d\pi(\Delta)})$ of norm 1 such that

$$0 \leq \left\langle \overline{d\pi(\Delta)}\xi | \xi \right\rangle - k(\overline{d\pi(\Delta)}, G) < \epsilon/3 .$$

As $\overline{d\pi(\Delta)}$ is the closure of $d\pi(\Delta)$, there exists a $\mathcal{C}^\infty$ vector $\eta$ of norm 1 which is arbitrarily close to $\xi$ with respect to the graph norm. As,

$$\begin{aligned} |k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\eta|\eta\rangle \,| \;\leq\; & |k(\overline{d\pi(\Delta)}, G) - \langle d\pi(\Delta)\xi|\xi\rangle \,| + \\ & |\left\langle \overline{d\pi(\Delta)}\xi|\xi \right\rangle - \left\langle \overline{d\pi(\Delta)}\xi|\eta \right\rangle | + \\ & |\left\langle \overline{d\pi(\Delta)}\xi|\eta \right\rangle - \langle d\pi(\Delta)\eta|\eta\rangle \,| \end{aligned}$$

this proves the claim. ∎

**Theorem 3.10.** *$G$ has property $(T)$ if and only if there exists an $\epsilon > 0$ such that $k(\overline{d\pi(\Delta)}, G) \geq \epsilon$ for every unitary representation $\pi$ of $G$ without non zero fixed vectors.*

**Proof.** $\Leftarrow$) If $\pi$ has almost invariant vectors, by thm 3.1, $k(\overline{d\pi(\Delta)}, G) = \min \mathrm{Sp}(\overline{d\pi(\Delta)}) = 0$. The assumption then implies that $\pi$ has non zero fixed vectors, i.e. $G$ has property $(T)$.
$\Rightarrow$) Assume by contradiction that there exists a sequence of unitary representations $\{\pi_n\}_{n\geq 0}$ without non zero fixed vector such that $k(\overline{d\pi_n(\Delta)}, G) \to 0$. We claim that the representation $\sigma = \bigoplus_{n\geq 0} \pi_n$ satisfies $k(\overline{d\sigma(\Delta)}, G) = 0$. By the assumption, there exists, for every $n$, a vector $\xi_n \in \mathcal{C}^\infty(H_{\pi_n})^1$ such that $\langle d\pi_n(h)\xi_n \,|\, \xi_n\rangle < 1/n + k(\overline{d\pi_n(\Delta)}, G)$.

The vector $\eta_n$ defined by $\eta_n = (0, \cdots, 0, \xi_n, 0 \cdots)$, with $\xi_n$ at the n$^{th}$ place, is a $\mathcal{C}^\infty$ vector in $H^1_\sigma$ and

$$\left\langle \overline{d\sigma(\Delta)}\eta_n \,|\, \eta_n \right\rangle \;=\; \left\langle \overline{d\pi_n(\Delta)}\xi_n \,|\, \xi_n \right\rangle \;<\; 1/n + k(\overline{d\pi_n(\Delta)}, G).$$

Hence, $k(\overline{d\sigma(\Delta)}, G) = 0$ and $0$ is in the spectrum of $\overline{d\sigma(\Delta)}$. By Theorem 3.1, this implies that $\sigma$ has almost invariant vectors. Since $G$ has property $T$, $\sigma$ has a nonzero fixed vector $\{\beta_n\}_{n\geq 0}$. Choose $n$ so that $\beta_n \neq 0$. Then $\beta_n$ is a non zero fixed vector for $\pi_n$, contradicting the assumption. ∎

We can define $K(\Delta, G) = \inf_{\pi\in\widetilde{G}^*} k(\overline{d\pi(\Delta)}, G)$, the Laplacian Kazhdan constant.

**Corollary 3.11.** *$G$ has property $(T)$ if and only if $K(\Delta, G) > 0$.*

This corollary is a direct consequence of the preceding proposition.

**Remark 3.12.** The above results (Theorem 3.1, Theorem 3.10 and Corollary 3.11) remain true if the vectors $X_1, \ldots, X_n$ are only assumed to generate $\mathfrak{g}$ as a Lie algebra. The relevant facts about the heat kernel associated with the sub-Laplacian $\Delta = -\sum_{i=1}^n X_i^2$ hold in this more general situation.

## 4. Comparison with the classical constants

**Proposition 4.1.** *Let $G$ be a connected Lie group and let $\{X_i\}_{i=1,\ldots,n}$ be a basis of its Lie algebra. Fix $\epsilon > 0$ and set $S = \{\exp(tX_i) \mid t \in [0, \epsilon], i \in \{1, \ldots, n\}\}$. Then, for any unitary representation $\pi$ of $G$, one has :*

$$\kappa(G, S, \pi) \leq \epsilon\sqrt{k(\overline{d\pi(\Delta)}, G)}.$$

*In particular $\kappa(G, S) \leq \epsilon\sqrt{K(\Delta, G)}$.*

**Proof.** Let $\pi$ be a unitary representation of $G$ and let $\xi \in \mathcal{C}^\infty(H_\pi)$.

$$\begin{aligned} \text{Then } \sup_{s\in S} \|\pi(s)\xi - \xi\| &= \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \|\pi(e^{tX_i})\xi - \xi\| \\ &= \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \left\|\int_0^t \pi(e^{sX_i})d\pi(X_i)\xi ds\right\| \\ &\leq \sup_{t\in[0,\epsilon]} \max_{i\in\{1\ldots n\}} \int_0^t \|\pi(e^{sX_i})d\pi(X_i)\xi\| \, ds \\ &\leq \epsilon \max_{i\in\{1\ldots n\}} \|d\pi(X_i)\xi\| \\ &\leq \epsilon\sqrt{\sum_{i=1}^n \|d\pi(X_i)\xi\|^2}\,. \end{aligned}$$

So

$$\begin{aligned} \kappa(G, S, \pi) &= \inf_{\xi\in H_\pi^1} \sup_{s\in S} \|\pi(s)\xi - \xi\| \\ &\leq \inf_{\xi\in(\mathcal{C}^\infty(H_\pi))^1} \sup_{s\in S} \|\pi(s)\xi - \xi\| \\ &\leq \epsilon \inf_{\xi\in(\mathcal{C}^\infty(H_\pi))^1} \sqrt{\sum_{i=1}^n \|d\pi(X_i)\xi\|^2} \\ &= \epsilon\sqrt{k(\overline{d\pi(\Delta)}, G)} \end{aligned}$$

This ends the proof. ∎

**Remark 4.2.** (i) 4.1 gives an alternative proof of implication ($\Rightarrow$) in 3.10.

(ii) Proposition 4.1 states that :

$$\kappa(G,S,\pi) \leq \epsilon\sqrt{k(\overline{d\pi(\Delta)},G)},$$

for every unitary representation $\pi$ of $G$. However the converse fails very strongly : there exists no continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, such that $f(0) = 0$ and

$$k(\overline{d\pi(\Delta)},G) \leq f(\kappa(G,S,\pi)),$$

for every connected Lie group $G$ and for every unitary representation $\pi$ of $G$. Indeed, $\kappa(G,S,\pi) \leq 2$ for any representation $\pi$, and we shall exhibit a sequence $(\pi_k)_{k\geq 1}$ of unitary representations of $\mathbb{R}$ such that $k(\overline{d\pi_k(\Delta)},G) = k^2$ for every $k \geq 1$.

Let $H_k = L^2([k,+\infty))$ et $T_k$ be the selfadjoint operator on $H_k$ defined by :

$$D(T_k) = \{\xi \in H_k \mid \int_k^\infty \lambda^2|\xi(\lambda)|^2 d\lambda < \infty\} \text{ and } (T_k\xi)(\lambda) = \lambda\,\xi(\lambda).$$

If $\pi_k(t) = \exp(itT_k)$, we have :

$$k(\overline{d\pi_k(\Delta)},\mathbb{R}) = \inf_{\xi\in D(T_k)^1} \left\langle T_k^2\xi \mid \xi\right\rangle = k^2.$$

**Example 4.3.** 1. Let $G = SL(2,\mathbb{R})$. Take, as basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, the matrices

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\Delta = -(H^2+V^2+W^2)$ is equal to $\frac{1}{4}(\omega + 2W^2)$ with $\omega$ the Casimir operator in $U(\mathfrak{sl}_2(\mathbb{R}))$.

Let us use the coordinates $(x,y,\theta)$ on $G$, where a group element is expressed as

$$g = \begin{pmatrix} \sqrt{y} & x \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \ (x,\theta\in\mathbb{R}, y>0).$$

Then, in these coordinates, we find, viewing $H,V,W$ as left invariant vector fields on $G$ :

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial\theta} + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}$$

(see S. Lang, [9], Chap. X, §1 and §2).

Let $\mathbb{H} \cong G/K$, $K = SO(2)$, be the Poincaré upper half space

$$\mathbb{H} = \{z = x+iy \mid x,y\in\mathbb{R}, y>0\},$$

with invariant measure $y^{-2}dxdy$.

Let $\pi$ be the left regular representation of $G$ on $L^2(\mathbb{H})$. As the functions on $\mathbb{H}$ are independent of $\theta$ on the right, we have

$$d\pi(\Delta) = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

which is the Laplace-Beltrami operator on $\mathbb{H}$ .

It is well known that $\mathrm{Sp}(d\pi(\Delta)) \subseteq [1/4; +\infty[$. Here is a very elementary argument for this due to Mac Kean [12]: Let $f$ be a real smooth function on $\mathbb{H}$ with compact support. Then

$$\begin{aligned} \frac{1}{4}\left(\int_0^\infty f(x,y)^2 y^{-2} dy\right)^2 &= \left(\int_0^\infty f(x,y)\frac{\partial f}{\partial y}(x,y) y^{-1} dy\right)^2 \\ &\leq \int_0^\infty f(x,y)^2 y^{-2} dy \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy \end{aligned}$$

So

$$\frac{1}{4}\int_0^\infty f(x,y)^2 y^{-2} dy \leq \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy.$$

Hence

$$\begin{aligned} \frac{1}{4} \| f \|^2_{L^2(\mathbb{H})} &\leq \int_{-\infty}^{+\infty} dx \int_0^\infty \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy \\ &\leq \int_{-\infty}^{+\infty} dx \int_0^\infty \left[\left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2\right] dy \\ &= -\int_{-\infty}^{+\infty} dx \int_0^\infty f(x,y)\left(\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right) dy \\ &= \langle d\pi(\Delta) f | f \rangle . \end{aligned}$$

This implies that $\lambda \geq 1/4$ for any $\lambda \in \mathrm{Sp}(\pi(\Delta))$.

It is actually known that $\inf \mathrm{Sp}(\overline{d\pi(\Delta)}) = 1/4$, that is, $k(\overline{d\pi(\Delta)}, G) = 1/4$ (see [9]).

2. Let $G$ be the (three-dimensional) Heisenberg group. Thus $G = \mathbb{R}^3$ with group law

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t' + \frac{1}{2}(pq' - qp')).$$

The left invariant vectors fields on $G$ corresponding to the coordinates $(p,q,t)$ are :

$$P = \frac{\partial}{\partial p} - \frac{1}{2}q\frac{\partial}{\partial t},\ Q = \frac{\partial}{\partial q} + \frac{1}{2}p\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

As is well known, $G$ has for each $h \in \mathbb{R}$, $h \neq 0$, an infinite dimensional unitary representation $\rho_h$ on $L^2(\mathbb{R})$ so that

$$d\rho_h(P) = hD, d\rho_h(Q) = M, d\rho_h(T) = \frac{h}{2\pi i}\mathbb{I},$$

where $D = \frac{1}{2\pi i}\frac{\partial}{\partial x}$ and $M$ is the multiplication operator by $x$ on $L^2(\mathbb{R})$.

A version of the Heisenberg Uncertainty Principle states that :

$$\| Mu \|_2^2 + \| Du \|_2^2 \geq \frac{1}{2\pi} \| u \|_2^2, \ \forall u \in L^2(\mathbb{R})$$

with equality if and only if $u$ is a multiple of the Gaussian $x \mapsto \exp(-\pi x^2)$ (see [3], Corollary (1.37)). Replacing in this inequality $u$ by the function $x \mapsto |h|^{1/4} u(h^{1/2}x)$ yields

$$\| Mu \|_2^2 + \| hDu \|_2^2 \geq \frac{|h|}{2\pi} \| u \|_2^2$$

with equality if $u(x) = |h|^{1/4} \exp(-\pi |h| x^2)$.

Thus, with $\Delta = -(P^2 + Q^2 + T^2)$, we see that

$$\langle d\rho_h(\Delta) u | u \rangle = \| Mu \|_2^2 + \| hDu \|_2^2 + \frac{|h|^2}{4\pi^2} \| u \|_2^2 \geq \left( \frac{|h|}{2\pi} + \frac{|h|^2}{4\pi^2} \right) \| u \|_2^2$$

with equality for the above Gaussian function.

Hence, we obtain the exact value for the Kazhdan constant

$$\begin{aligned} k(\overline{d\rho_h(\Delta)}, G) &= \inf\{\lambda \mid \lambda \in \mathrm{Sp}(d\rho_h(\Delta))\} \\ &= \left( \frac{|h|}{2\pi} + \frac{|h|^2}{4\pi^2} \right) \end{aligned}$$

3. The following well-known example was pointed out to us by A. Valette. Let $G$ be a connected compact semi-simple Lie group. Let $\{X_i\}_{i=1,\cdots,n}$ be a basis of its Lie algebra $\mathfrak{g}$, which is orthogonal relatively to the Killing form. Then $\Delta = -\sum_{i=1}^{n} X_i^2$ is the Casimir operator of $G$. As $\Delta \in Z(U(\mathfrak{g}))$, the Schur lemma insures that, for $\pi \in \hat{G}$, $d\pi(\Delta) = c\mathbb{I}$. This constant $c$ can be determined in the following way : if $\lambda$ is the highest weight of $\pi$ and $\rho$ is the half sum of the positive roots, as we can see in [7], p. 247, we have

$$k(d\pi(\Delta), G) = \langle \lambda + 2\rho \mid \lambda \rangle = \| \lambda + \rho \|^2 - \| \rho \|^2 .$$

**Remark 4.4.** Let $G$ be a simply connected, connected nilpotent Lie group, and let $\pi$ be a unitary irreducible representation of $G$. In [6], a bound is given for $k(\overline{d\pi(\Delta)}, G)$ in terms of the distance from 0 of the Kirillov orbit associated to $\pi$.

## References


[1] Colin de Verdière, Y., *Le trou spectral des graphes et leur propriété d'expansion*, Séminaire de théorie spectrale et géométrie, Grenoble, (1993–1994), 51–63.

[2] Dixmier, J., "Les $C^*$-algèbres et leurs représentations," Cahiers Scientifiques **29**, Gauthier-Villars, 1964.

[3] Folland, G., "Harmonic Analysis on Phase Space," Princeton University Press, 1989.

[4] de la Harpe, P., and A. Valette, "La propriété (T) pour les groupes localement compacts," Astérisque **175**, Société Mathématique de France, 1989.

[5] Hochschild G., "La Structure des Groupes de Lie," Monographies Universitaires de Mathématiques **27**, Dunod, 1968.

[6] Hulanicki, A., J. Jenkins, and J. Ludwig, *Minimum eigenvalues for positive Rockland operators*, Proc. Amer. Math. Soc. **94** (1985), 718–720.

[7] Jacobson, N., "Lie Algebras," Interscience, 1962.

[8] Jorgensen, P., "Operators and Representation Theory," Mathematics Studies **147**, North-Holland, 1988.

[9] Lang, S., "$SL_2(\mathbb{R})$," Addison Wesley, 1975.

[10] Lubotzky, A., "Discrete Groups, Expanding Graphs and Invariant Measures," Progress in Mathematics **125**, Birkhäuser, 1994.

[11] Margulis, G., *Explicit constructions of concentrators*, Problems Inform. Transmission **9-4** (1973), 325–332.

[12] Mc Kean, h., *Selberg trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972), 225–246.

[13] Nelson, E., *Analytic vectors*, Annals of Mathematics **70** (1959), 572–615.

[14] Pazy, A., "Semigroups of Linear Operators and Applications to Partial Differential Equations," Applied Mathematical Sciences **44**, Springer, 1983.

[15] Poulsen, N., *On $\mathcal{C}^\infty$-vectors and intertwining bilinear forms for representations of Lie groups*, J. of Functional Analysis **9** (1972), 87–120.

[16] Yosida K., "Functional Analysis," Die Grundlehren der mathematischen Wissenschaft **123**, Springer, 1965.

URA CNRS 399
Département de mathématiques
Université de Metz
Ile de Saulcy
F-57045 Metz, France
bekka@poncelet.univ-metz.fr

School of mathematics,
University of New South Wales
Sydney, 2052, Australia
pacherix@maths.unsw.edu.au

Institut de mathématiques,
Université de Neuchâtel
Rue Emile-Argand 11
CH-2007 Neuchâtel
Switzerland
Jolissaint@maths.unine.ch

Received April 16, 1997
and in final form June 11, 1997