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# Commuting Operators and Class Functions

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**Abstract.** Given a unimodular locally compact group $G$, we associate two algebras of operators, $R(G)$ and $S(G)$. Operators in $R(G)$ are multiplication operators and those in $S(G)$ are defined by group translations on $G \times G$. The construction of these algebras follow closely the so-called 'group-measure' construction of von Neumann algebras due to Murray and von Neumann. In this paper, we show that the 'normalizer' (a study began by Judith Packer) of $S(G)$ in $R(G)$ is identified with the space of class functions on $G$.

## 1. Introduction

The so-called 'group-measure' construction due to Murray and von Neumann associates to a group action a von Neumann algebra. Many times properties of the group action are reflected in the inner structure of the corresponding von Neumann algebra. In [2], Packer analyzed von Neumann algebras associated to ergodic free actions of discrete abelian groups. An appropriately defined 'normalizer' of a subalgebra of group translation operators in the corresponding von Neumann algebra is characterized by Packer in terms of maximal quotient actions having pure point spectrum (see the definition of the 'normalizer' at the end of this section). In this paper we deviate from free actions of discrete groups and consider instead the action of unimodular groups by inner-conjugation. In this case we show that the 'normalizer' in a subalgebra of multiplication operators can be identified with the space of class functions on $G$.

Specifically let $G$ be a unimodular locally compact group with Haar measure $\mu$. Suppose $G$ acts by inner-conjugation on itself from the right, $s \cdot g = g^{-1}sg$ for all $s, g \in G$. We investigate several subalgebras of the corresponding von Neumann algebra. The construction of these subalgebras follows closely, but not precisely, the construction given in [2], which involves the classic 'group-measure' construction of von Neumann algebras due to Murray and von Neumann. Let $L^2(G^2, \mu^2)$ be the Hilbert space of complex-valued square integrable functions on the cartesian product $G^2 = G \times G$, where $\mu^2$ is the product measure $\mu \times \mu$, and let

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$L^\infty(G,\mu)$ be the Banach space of complex-valued bounded measurable functions on $G$ normed by the essential norm. Define a unitary group representation of $G$ on $L^2(G^2,\mu^2)$ by setting $U_g f(s,x) = f(s\cdot g, g^{-1}x)$ and construct a representation of $L^\infty(G,\mu)$ on $L^2(G^2,\mu^2)$ by defining $T_\gamma f(s,x) = \gamma(s)f(s,x)$ for $f \in L^2(G^2,\mu^2)$, $\gamma \in L^\infty(G,\mu)$, $x \in G$. The von Neumann algebra $F(G)$, which corresponds to the action of $G$ by inner-conjugation, is generated by the group translation operators $\{U_g : g \in G\}$ and the multiplication operators $\{T_\gamma : \gamma \in L^\infty(G,\mu)\}$. One of the subalgebras of $F(G)$ we investigate is $S(G)$, which consists of linear operators of the form $\sum_{i=1}^{n} a_i U_{g_i}$ where $n$ is some positive integer, $a_i \in \mathbb{C}$, and $g_i \in G$. A second one is $R(G) = \{T_\gamma | \gamma \in L^\infty(G,\mu)\}$, which is an abelian von Neumann algebra of multiplication operators.

In this paper we are interested in describing the von Neumann subalgebra $N(G)$ of $R(G)$ generated by all unitary operators $T_\gamma$ in $R(G)$ which satisfy

$$T_\gamma S(G) T_\gamma^{-1} = S(G). \tag{1}$$

The action of $G$ by inner-conjugation is reflected in the structure of $N(G)$. This is the main result of this paper and is stated in the following theorem.

**Theorem 1.** $N(G) = \{T_\gamma \mid \gamma \in L^\infty(G), \quad \gamma \text{ is a class function}\}$

When $G$ is a finite group, the theorem has been established by the author [4]. The present case is slightly more complicated than the finite case because of the presence of measure-theoretic considerations. We shall find useful a common technique utilized in the study of Borel group actions and groupoids to modify functions on sets of measure zero.

We remark that in Packer's article [2], $S(G)$ is the von Neumann algebra generated by $\{U_g : g \in G\}$ while in this paper $S(G)$ is the algebra generated by $\{U_g : g \in G\}$. The 'normalizer' characterized by Packer is the von Neumann subalgebra of $F(G)$ generated by the unitary operators $m \in F(G)$ satisfying $mS(G)m^{-1} = S(G)$. In this paper, we provide in Theorem 1 a description of the 'normalizer' of $S(G)$ in $R(G)$.

## 2. Proof of Theorem 1.

First, we shall describe the unitary operators that generate $N(G)$.

**Lemma 2.** *Let $T_\gamma$ be a unitary operator satisfying (1). Then $T_\gamma U_g T_\gamma^* = U_g$ for all $g$. Moreover, $\gamma(s\cdot g) = \gamma(s)$ a.e. $s$, for each $g$.*

**Proof.** Let $g \in G$. It follows from (1) that there exists a finite subset $I_g \subseteq G$ such that

$$T_\gamma U_g T_\gamma^* = \sum_{h\in I_g} a_h U_h \tag{2}$$

where $a_h \neq 0$ for all $h \in I_g$. If $f \in L^2(G^2,\mu^2)$, then

$$\begin{aligned}(T_\gamma U_g T_\gamma^* f)(s,x) &= \gamma(s)\overline{\gamma(s\cdot g)} f(s\cdot g, g^{-1}x) \\ &= \sum_{h\in I_g} a_h f(s\cdot h, h^{-1}x) \quad a.e. \ (s,x).\end{aligned} \tag{3}$$

First, we show $I_g = \{g\}$. If it were not and $g \notin I_g$, then there exists an open neighborhood $V$ of the identity $e$ whose closure $\overline{V}$ is compact and $h^{-1}V \cap g^{-1}V = \emptyset$, the empty set, for all $h \in I_g$. By continuity we can choose an open neighborhood $O$ of $e$ such that $\overline{O} \subseteq V$ and $s \cdot g \in V$ whenever $s \in O$. By Urysohn's lemma, there exists a continuous function $\delta$ on $G$ such that on $g^{-1}\overline{O}$ its value is identically 1 and its support is contained in $g^{-1}V$. Define a function $f \in L^2(G^2, \mu^2)$ by letting

$$f(s,x) = \begin{cases} \delta(x) & \text{if } s \in V \\ 0 & \text{otherwise.} \end{cases}$$

Since $T_\gamma$ is a unitary operator, then $|\gamma(s)| = 1 = |\gamma(s \cdot g)|$ a.e. $s$. For all $x \in O$ and for a.e. $s$ in $O$, $(T_\gamma U_g T_\gamma^* f)(s,x) = \gamma(s)\overline{\gamma(s \cdot g)}\delta(g^{-1}x) = \gamma(s)\overline{\gamma(s \cdot g)} \neq 0$ while $\sum_{h \in I_g} a_h f(s \cdot h, h^{-1}x) = \sum_{h \in I_g} a_h \cdot 0 = 0$ since $h^{-1}x \notin g^{-1}V$. This contradicts (3) since $\mu(O) > 0$; hence, $g \in I_g$.

We may rewrite (2) as

$$T_\gamma U_g T_\gamma^* = a_g U_g + \sum_{h \in I_g - \{g\}} a_h U_h \tag{4}$$

where the second term in the right-hand side is zero if $I_g - \{g\} = \emptyset$. Suppose there exists an $h \in I_g - \{g\}$. Similarly, there exists an open neighborhood $U$ of $e$ whose closure is compact such that $g^{-1}U \cap h^{-1}U = \emptyset$ and $y^{-1}U \cap h^{-1}U = \emptyset$ for any $y \in I_g - \{g,h\}$. Let $W$ be an open neighborhood of $e$ such that $\overline{W} \subseteq U$ and $s \cdot h \in U$ whenever $s \in W$. Choose a continuous function $\delta_h$ on $G$ whose value on $h^{-1}\overline{W}$ is identically 1 and whose support is contained in $h^{-1}U$. Define a function $f_h \in L^2(G^2, \mu^2)$ by setting

$$f_h(s,x) = \begin{cases} \delta_h(x) & \text{if } s \in U \\ 0 & \text{otherwise.} \end{cases}$$

For all $s, x \in W$, $(T_\gamma U_g T_\gamma^* f_h)(s,x) = \gamma(s)\overline{\gamma(s \cdot g)} f_h(s \cdot g, g^{-1}x) = \gamma(s)\overline{\gamma(s \cdot g)} \cdot 0 = 0$ and $a_g f_h(s \cdot g, g^{-1}x) + a_h f_h(s \cdot h, h^{-1}x) + \sum_{y \in I_g - \{g,h\}} a_y f_h(s \cdot y, y^{-1}x) = a_g 0 + a_h \delta_h(h^{-1}x) + \sum_{y \in I_g - \{g,h\}} a_y 0 = 0 + a_h + 0 = a_h$. This contradicts (4) since $a_h \neq 0$; hence, $I_g = \{g\}$ and $T_\gamma U_g T_\gamma^* = a_g U_g$.

Note, $\gamma(s)\overline{\gamma(s \cdot g)} = a_g$ a.e. $s$, for each $g$ and the mapping given by $\chi(g) = a_g$ defines a character of $G$ since $a_{g_1 g_2} U_{g_1 g_2} = a_{g_1} a_{g_2} U_{g_1 g_2}$. To complete the proof of the lemma, we must show that $\chi$ is the trivial character.

Since $G$ acts on the right of $G$ i.e. $s \cdot g = g^{-1}sg$, $G \times G$ is a groupoid with a partially defined multiplication given by $(s, g_1)(s \cdot g_1, g_2) = (s, g_1 g_2)$. Our use of the theory of groupoids shall be very brief. Mainly, we shall use a technical tool to show the existence of the function given in (5). Let $E = \{(s,g) \mid \gamma(s \cdot g) = \chi(g)^{-1}\gamma(s)\}$. If $(s, g_1)$ and $(s \cdot g_1, g_2) \in E$, then $\gamma(s \cdot g_1 g_2) = \chi(g_2^{-1})\gamma(s \cdot g_1) = \chi(g_2^{-1})\chi(g_1^{-1})\gamma(s) = \chi((g_1 g_2)^{-1})\gamma(s)$. Thus, $(s, g_1 g_2) \in E$ and $E$ is a conull, multiplicatively closed Borel subset of the groupoid $G \times G$. By [1, Lemma A.4] and [3, Lemma 5.2], there exists a conull Borel subset $S_o \subseteq G$ such that

1. if $s, s\cdot g \in S_o$ then $(s,g)\in E$,

2. $S_o\cdot G$ is a Borel subset and there exists a Borel mapping $\theta : S_o\cdot G \to G$ satisfying $\theta(s)=e$ if $s\in S_o$ and $s\cdot\theta(s)\in S_o$ for $s\in S_o\cdot G$.

Let $F : G\to\mathbb{C}$ be the Borel function given by

$$F(s)=\begin{cases}\chi(\theta(s))\gamma(s\cdot\theta(s)) & \text{if } s\in S_o\cdot G\\ 1 & \text{otherwise.}\end{cases}\tag{5}$$

If $s\in S_o\cdot G$, then $(s\cdot\theta(s),\theta(s)^{-1}g\theta(s\cdot g))\in E$ and $\gamma(s\cdot g\theta(s\cdot g))=\chi(\theta(s)^{-1}g\theta(s\cdot g))^{-1}\gamma(s\cdot\theta(s))$. Thus, whenever $s\in S_o\cdot G$, $F(s\cdot g)=\chi(\theta(s\cdot g))\gamma(s\cdot g\theta(s\cdot g))=\chi(\theta(s\cdot g))\chi(\theta(s)^{-1}g\theta(s\cdot g))^{-1}\gamma(s\cdot\theta(s))=\chi(g)^{-1}F(s)$. Choose a conull Borel subset $Z_o\subseteq G$ for which $|\gamma(s)|=1$ for all $s\in Z_o$ . If $s\in Z_o\cap S_o$ and by choosing $g=s$, then $s\cdot g=s$ and $\gamma(s)=F(s)=F(s\cdot g)=\chi(g)^{-1}F(s)=\chi(g)^{-1}\gamma(s)$. Then $\chi(s)=1$ for all $s\in Z_o\cap S_o$. Since the closed subgroup generated by $Z_o\cap S_o$ is $G$, $\chi$ is the trivial character. ∎

We now return to the proof of the theorem. Note, $L^\infty(G)^G=\{\gamma\in L^\infty(G,\mu):\gamma(s\cdot g)=\gamma(s)\text{ a.e. } s, \text{ for each } g\}$ is a closed $*$-subalgebra of $L^\infty(G,\mu)$. Let $A=\{\gamma\in L^\infty(G)^G : |\gamma(s)|=1 \text{ a.e. } s\}$ and let $\mathcal{A}$ be the complex vector space which consists of operators in $R(G)$ of the form $\sum_{i=1}^{n} a_iT_{\gamma_i}$ where $n$ is some positive integer, $a_i\in\mathbb{C}$, and $\gamma_i\in A$.

Let $\gamma\in L^\infty(G)^G$ be a real-valued function such that $\|\gamma\|_\infty\neq 0$. Consider the function $\alpha$ defined by

$$\alpha(s)=\begin{cases}\dfrac{\gamma(s)+i\sqrt{\|\gamma\|_\infty^2-\gamma(s)^2}}{\|\gamma\|_\infty} & \text{if } |\gamma(s)|\leq\|\gamma\|_\infty\\ 1 & \text{otherwise.}\end{cases}\tag{6}$$

Clearly, $\alpha\in A$ and $\gamma(s)=\dfrac{\|\gamma\|_\infty\left(\alpha(s)+\overline{\alpha(s)}\right)}{2}$ a.e. $s$, Thus, $T_\gamma\in\mathcal{A}$ for all real-valued functions $\gamma\in L^\infty(G)^G$. Since we can express each function in $L^\infty(G)^G$ as a linear combination of two real-valued functions belonging $L^\infty(G)^G$, then $\mathcal{A}=\{T_\gamma:\gamma\in L^\infty(G)^G\}$.

In particular, $\mathcal{A}$ is a von Neumann algebra. By Lemma 2, $N(G)\subseteq\mathcal{A}$. Since one can directly show that each unitary operator in $\mathcal{A}$ commutes with every operator in $S(G)$, then $\mathcal{A}\subseteq N(G)$. Hence, $N(G)=\{T_\gamma:\gamma\in L^\infty(G)^G\}$. ∎

**Remarks.** It would be interesting to know whether the theorem still holds for the case $S(G)$ is defined to be the von Neumann algebra generated by $\{U_g : g\in G\}$.

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