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Hardy spaces on two–sheeted covering semigroups

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Abstract. In this paper we study the minimal complex Lie semigroups associated with three classical series of groups by using a holomorphic continuation of a certain Cayley transform for the group. In particular we show, that for the symplectic group the odd part of the Hardy space on the double cover is isomorphic to the classical Hardy space on the Siegel upper half space corresponding to the symplectic group of twice the rank of the given group.

0. Introduction

In this paper we shall give a detailed description of the minimal complex Lie semigroups associated with three of the four classical series of groups with an Hermitian symmetric space. These were found by Ol'shanskiĭ ([**21**]) as were the associated Hardy spaces on these semigroups ([**22**]), and recently there has been much interest in analysis of this type of function space (see [**3**], [**4**], [**7**], [**8**], [**14**], [**15**], [**18**], [**19**], [**20**] and [**23**]). In particular one would like to calculate the Cauchy–Szegö kernel explicitly, and to compare these new Hardy spaces with those for classical bounded domains (see [**3**], [**18**], [**19**] and [**23**]). By using a natural Cayley transform, which might be thought of as a holomorphic continuation of a causal compactification of the Lie group, we show that for the symplectic group the odd part of the Hardy space on the double cover is indeed isomorphic to the classical Hardy space on the Siegel upper half space corresponding to the symplectic group of twice the rank of the given group. One of the main technical points in this work is the actual construction of the double cover semigroup, isomorphic to Howe's oscillator semigroup (see [**12**] and [**16**]), via a choice of square root of a certain Jacobian; this is close in spirit to the construction of the Riemann surface for $\sqrt{z}$. Thus for the metaplectic group we obtain an explicit formula for the Cauchy–Szegö kernel as well as the branching

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law for the classical Hardy space with respect to the product of two metaplectic groups. For the two remaining series of groups the Cayley transform goes into the tube domains for the Jordan algebras of complex resp. quaternion Hermitian matrices; in the first case the Ol'shanskiĭ Hardy space is never isomorphic to the classical one, and in the second case we show that even after a necessary passage to the double cover the classical Hardy space is strictly contained in the odd part of the Hardy space on the double cover. This is due to a decay condition at infinity in the complex semigroup which holds on the classical space but not on all of the semigroup Hardy space.

It should be interesting to pursue this line of investigation for other groups and symmetric spaces which admit holomorphic discrete series representations and Hardy spaces. First, one would like to find the analogue of the Cayley transform studied here; second one should use this to calculate Cauchy–Szegö kernels and to compare to classical Hardy spaces. For the first problem there has been recent interesting progress in a quite general setting by Wolfgang Bertram ([**1**]) and Frank Betten ([**2**]) (independently). For the second the Cayley type spaces were treated by Mohamed Chadli ([**3**]) and Ólafsson–Ørsted ([**20**]) (independently).

In the first section we recall the Ol'shanskiĭ Hardy spaces and their decomposition as highest weight modules and in section 2 we describe the contraction semigroup for the classical series we consider: $U(p,q)$, $Sp(r,\mathbb{R})$ or $SO^*(2l)$. Also here we construct the relevant Cayley transform. In section 3 we parametrize the scalar-valued holomorphic discrete series and its analytic continuation for the conformal group $G^\flat$, where $G^\flat$ is $SU(n,n)$, $Sp(2r,\mathbb{R})$ or $SO^*(4l)$ respectively in our three cases. Here $n = p+q$, and the large group $G^\flat$ acts by local order-preserving transformations on the smaller one G. One way to look at our study is that we consider the spectrum of G in certain unitary highest weight representations of $G^\flat$, in particular the Hardy space (the classical one) on the tube domain for $G^\flat$; this we begin in section 4, and in section 5 we give, first for the metaplectic group, the explicit construction of the double cover semigroup. This is only abstractly isomorphic to Howe's oscillator semigroup, and we need the present construction in terms of geometrically defined cocycles. This section in particular contains the formula for the Cauchy-Szegö kernel for the odd part of the Ol'shanskiĭ Hardy space on Howe's oscillator semigroup, namely:

$$K_{\mathrm{odd}}(\gamma_1,\gamma_2) = \mathrm{Det}(J - \gamma_2^* J\gamma_1)^{-(r+1/2)}$$

where the right-hand side is the usual matrix determinant. The square root is exactly well-defined because of our double covering. Finally sections 6 and 7 contain the last two series; and in none of these cases does the spectrum of G coincide for the classical and for the Ol'shanskiĭ Hardy space. In the $SO^*(2l)$ case a double cover similar to the metaplectic case is constructed, perhaps of independent interest. The results of this paper were announced in [**19**].

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1. The Ol'shanskiĭ Hardy spaces

Let $\mathfrak{g}$ be a simple Lie algebra over the reals $\mathbb{R}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. We shall suppose that $\mathfrak{k}$ has a non–zero center $\mathfrak{z}$; then $\mathfrak{z}$ is one dimensional and $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$.

Let $G_{\mathbb{C}}$ be the simply connected complex Lie group corresponding to $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g}$, and let G, K and T be the connected subgroups in $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$, respectively. By the Kostant–Paneitz–Vinberg Theorem ([**25**]), there are non–trivial regular cones C in $i\mathfrak{g}$ which are $\mathrm{Ad}(G)-$invariant, where regular means, convex, closed, pointed ($C \cap -C = \{0\}$) and generating ($C - C = i\mathfrak{g}$). Let $\mathrm{Cone}_G(i\mathfrak{g})$ be the set of all regular $\mathrm{Ad}(G)-$invariant cones in $i\mathfrak{g}$.

For such a cone C in $\mathrm{Cone}_G(i\mathfrak{g})$, Ol'shanskiĭ associates a semigroup $\Gamma(C) := G\exp(C)$ in $G_{\mathbb{C}}$, and for this semigroup he associates a "non-commutative" Hardy space $H^2(\Gamma(C))$ which is the set of holomorphic functions f on the complex manifold $\Gamma(C)^\circ = G\exp(C^\circ)$, the interior of $\Gamma(C)$, such that

$$\sup_{\gamma\in\Gamma(C)^\circ} \int_G |f(\gamma g)|^2\, dg < \infty.$$

Note that sometimes the order of g and γ is interchanged in this integral; we use this convention here - they are of course equivalent. For any $\gamma \in \Gamma(C)^\circ$ the linear functional $f \longmapsto f(\gamma)$ is continuous on $H^2(\Gamma(C))$, which is a Hilbert space, see for example [**15**]. Therefore by the Riesz representation theorem, there exists a vector $K_\gamma \in H^2(\Gamma(C))$ such that $(f, K_\gamma) = f(\gamma)$. The reproducing kernel K which is called the *Cauchy–Szegö kernel* is defined by

$$K(\gamma_1, \gamma_2) = K_{\gamma_2}(\gamma_1).$$

It is Hermitian, holomorphic in γ_1 and anti–holomorphic in γ_2.

Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the set of roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$. Let $\Delta^+ \subset \Delta$ be the set of positive roots relative to some order (namely the one where the center of $\mathfrak{k}$ comes first), $\Delta_{\mathfrak{k}}^+$ and $\Delta_{\mathfrak{p}}^+$ the set of positive compact and non-compact roots, respectively. Put $\mathfrak{t}_{\mathbb{R}} := i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$. We identify $\mathfrak{t}_{\mathbb{R}}$ with its own dual via the Cartan–Killing form. Then we can consider $\Delta \subset \mathfrak{t}_{\mathbb{R}}$. Let $\mathcal{P} \subset \mathfrak{t}_{\mathbb{R}}^* \simeq \mathfrak{t}_{\mathbb{R}}$ be the set of weights relative to T and let $\mathcal{R}$ be the set of all highest weights relative to $\Delta_{\mathfrak{k}}^+$,

$$\mathcal{R} = \{\lambda \in \mathcal{P} \quad | \quad (\forall \alpha \in \Delta_{\mathfrak{k}}^+)\ \langle \lambda, \alpha\rangle \geq 0\}. \tag{1.1}$$

Let ρ be the half sum of all positive roots. Then by Harish–Chandra ([**9**], [**10**], [**11**]) the holomorphic discrete series representations for the group G are those irreducible unitary representations of G that are square–integrable with a highest weight λ belonging to

$$\mathcal{R}' = \{\lambda \in \mathcal{R} \quad | \quad (\forall \beta \in \Delta_{\mathfrak{p}}^{+})\ \langle \lambda + \rho, \beta \rangle < 0\}. \tag{1.2}$$

We will say that $\lambda \in \mathcal{R}$ satisfies the *Harish–Chandra condition* if

$$\langle \lambda + \rho, \beta \rangle < 0, \quad \forall \beta \in \Delta_{\mathfrak{p}}^{+}. \tag{1.3}$$

By Vinberg ([**25**]), there exists in $\mathrm{Cone}_G(i\mathfrak{g})$ a unique (up to multiplication by -1) maximal cone C_{max}, such that

$$C_{max} \cap \mathfrak{t}_{\mathbb{R}} = c_{max} := \{X \in \mathfrak{t}_{\mathbb{R}} \quad | \quad (\forall \alpha \in \Delta_{\mathfrak{p}}^{+})\ \langle X, \alpha \rangle \geq 0\},$$

and a unique minimal cone $C_{min} = C_{max}^{*}$, such that $C_{min} \cap \mathfrak{t}_{\mathbb{R}} = c_{min}$ is the convex cone spanned by all α in $\Delta_{\mathfrak{p}}^{+}$.

A unitary representation π of G in a Hilbert space $\mathcal{H}$ is said to be $C-$*dissipative* if for all $X \in C$ and all $\xi \in \mathcal{H}^{\infty}$, the space of $\mathcal{C}^{\infty}$ vectors in $\mathcal{H}$,

$$(\pi(X)\xi|\xi) \leq 0.$$

We can now state the Theorem B of Olshanskiĭ ([**22**]) on the non–commutative Hardy spaces

Theorem 1.1. *The Hardy space $H^2(\Gamma(C))$ is a non–trivial Hilbert space for any $C \in \mathrm{Cone}_G(i\mathfrak{g})$.*
The representation of G in $H^2(\Gamma(C))$ can be decomposed into a direct sum of irreducible unitary representations of G. The components of this decomposition are precisely all the holomorphic discrete series representations of G which are $C-$dissipative. ■

The group $G \times G$ acts on $H^2(\Gamma(C))$ via left and right regular representations. Therefore

$$H^2(\Gamma(C)) = \bigoplus_{\lambda \in (C^* \cap \mathfrak{t}_{\mathbb{R}}) \cap \mathcal{R}'} \pi_\lambda \otimes \pi_\lambda^*, \tag{1.4}$$

where π_λ is the *contraction representation* of $\Gamma(C)$ corresponding to a unitary highest weight representation of G with highest weight λ. Recall here the correspondence as in [**21**] between holomorphic contraction representations of the semigroup and admissible unitary highest weight representations of the group. In particular, $\Gamma(C)$ acts by contractions on the Hardy space, and this action is holomorphic on the interior $\Gamma(C)^{\circ}$. Morever, the corresponding function of the Cauchy–Szegö kernel K of $H^2(\Gamma(C))$ can be written on $\Gamma(C)^{\circ}$ as follows

$$K(\gamma) := K(\gamma, e) = \sum_{\lambda \in (C^* \cap \mathfrak{t}_{\mathbb{R}}) \cap \mathcal{R}'} d_\lambda \mathrm{tr}(\pi_\lambda(\gamma)), \tag{1.5}$$

where d_λ denotes the *formal dimension* of the representation π_λ. The series for K converges uniformly on compact subsets in $\Gamma(C)^{\circ}$.

Remark 1.2. Whenever C is the minimal cone the decompositions (1.4) and (1.5) are over all the holomorphic discrete series, namely over $\lambda \in \mathcal{R}'$.

One of the most important problems in this area is to give an explicit formula for the function $K(\gamma)$.

2. The contraction semigroup in $G_{\mathbb{C}}$ and the Cayley transform

From now on we assume that G is one of the classical groups $U(p,q)$, $Sp(r,\mathbb{R})$ or $SO^*(2l)$. Let σ be an involution on $\mathfrak{g}_{\mathbb{C}}$ such that

$$\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} \quad | \quad \sigma(X) = -X\}.$$

Then

$$\sigma(X) = JX^*J,$$

where X^* is the adjoint matrix and

$$\begin{aligned} \text{for} \quad \mathfrak{g} = \mathfrak{u}(p,q) \quad J &= \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \\ \text{for} \quad \mathfrak{g} = \mathfrak{sp}(r,\mathbb{R}) \quad J &= \begin{pmatrix} -I_r & 0 \\ 0 & I_r \end{pmatrix} \\ \text{for} \quad \mathfrak{g} = \mathfrak{o}^*(2l) \quad J &= \begin{pmatrix} -I_l & 0 \\ 0 & I_l \end{pmatrix}. \end{aligned}$$

Remark 2.1. $U(p,q)$ is not a simple Hermitian Lie group. Since

$$U(p,q) \simeq \big(U(1) \times SU(p,q)\big)/\mathbb{Z}_{p+q},$$

the holomorphic discrete series representations of $U(p,q)$ are the holomorphic discrete series representations of the circle times the Hermitian group $SU(p,q)$ which are trivial on $(\zeta, \zeta^{-1}I_n)$ where $n = p+q$ and $\zeta^n = 1$. Therefore one can easily generalize the results of section 1 to the reductive group $U(p,q)$; details will be given in section 7.

Let C be the regular cone in $i\mathfrak{g}$ defined by

$$C := \{X \in i\mathfrak{g} \quad | \quad JX \leq 0\} \tag{2.1}$$

and let $\Gamma(C) := G\exp(C)$ be the corresponding Ol'shanskiĭ semigroup. An element γ of $G_{\mathbb{C}}$ is said to be a *$J-$contraction* (resp. a *strict $J-$contraction*) if $J - \gamma^* J\gamma \geq 0$ (resp. $J - \gamma^* J\gamma \gg 0$).

Proposition 2.2. *The semigroup $\Gamma(C)$ is the $J-$contraction semigroup,*

$$\Gamma(C) = \{\gamma \in G_{\mathbb{C}} \quad | \quad J - \gamma^* J\gamma \geq 0\},$$

and $\Gamma(C)^\circ$ is the semigroup of strict $J-$contractions,

$$\Gamma(C)^\circ = \{\gamma \in G_{\mathbb{C}} \quad | \quad J - \gamma^* J\gamma \gg 0\}.$$

Proof. The case $G = U(p,q)$ is done by Hilgert and Neeb in [**13**]. For $G = Sp(r,\mathbb{R})$, $\Gamma_{Sp(r,\mathbb{R})} = G_{\mathbb{C}} \cap \Gamma_{U(r,r)}$ and for $G = SO^*(2l)$, $\Gamma_{SO^*(2l)} = G_{\mathbb{C}} \cap \Gamma_{U(l,l)}$. ■

Let V be one of the *Jordan algebras* $\mathrm{Herm}(n,\mathbb{C})$, $\mathrm{Sym}(2r,\mathbb{R})$ or $\mathrm{Herm}(l,\mathbb{H})$ and let Ω be the corresponding *symmetric cone*. Then $\Omega = V^+$ is the set of positive definite matrices in V. The tube domain $T_\Omega := V + i\Omega$ is a Hermitian symmetric space isomorphic to $G^\flat/K^\flat$, where $G^\flat$ is $SU(n,n)$, $Sp(2r,\mathbb{R})$ or $SO^*(4l)$ respectively and $K^\flat$ the corresponding maximal compact subgroup, *i.e.* $S\big(U(n)\times U(n)\big)$, $U(2r)$ or $U(2l)$ respectively.

Let C be the *Cayley transform* defined by

$$\mathsf{C}(Z) := (Z - iJ)(Z + iJ)^{-1} \tag{2.2}$$

whenever the matrix $(Z + iJ)$ is invertible.

Proposition 2.3. *The Cayley transform C is a biholomorphic bijection from an open subset of the tube domain T_Ω onto the complex manifold $\Gamma(C)^\circ$. More precisely, if Σ denotes the hypersurface $\Sigma = \{Z \in T_\Omega \mid \det(Z + iJ) = 0\}$, then*

$$\mathsf{C}(T_\Omega \setminus \Sigma) = \Gamma(C)^\circ. \tag{2.3}$$

Proof. We use the similar arguments as in the proof of Lemma 1.1 in [**18**]. Here "det" denotes the *Koecher norm* in the Jordan algebra $V_\mathbb{C}$ (*cf.* [**5**]). ∎

A crucial point in this paper is to compare holomorphic functions on the tube domain with their pull-backs on the semigroup via the Cayley transform, and vice versa. In particular, it will be important to know the rate of growth of the functions near the singularity Σ above. Assuming $\gamma = \mathsf{C}(Z)$ we have that

$$Z + iJ = 2(I-\gamma)^{-1} iJ$$

so that to approach the singularity in the Z variable, means that $\det(I-\gamma)$ tends to infinity in the γ variable. Clearly this condition is invariant under conjugation with G, so we may reduce the question of the growth near the singularity to a question on the compact Cartan subspace. Suppose the holomorphic functions f and F are related by

$$f(Z) = \det(I-\gamma)^p F(\gamma)$$

so that F is holomorphic on $\Gamma(C)^\circ$ and f therefore holomorphic on $T_\Omega \setminus \Sigma$. Then for f to admit a holomorphic continuation to all of T_Ω it is necessary and sufficient that it stays bounded as the determinant factor tends to infinity, i.e. that F satisfies a decay condition related to p. This is what we shall make precise in the following.

Lemma 2.4. *Given the correspondence as above between f and F and suppose F is a matrix coefficient of a holomorphic representation of the semigroup relative to an orthonormal basis $v_0, v_1, .., $ of weight vectors for the compact Cartan subspace $\mathfrak{t}_{\mathbb{R}}$ such that $\gamma v_i = \gamma^{b_i} v_i$ and on $\exp(C^\circ \cap \mathfrak{t}_{\mathbb{R}})$ we have that $\gamma^{b_0} \geq \gamma^{b_i}$ for all i. Then the singularity of f can be removed provided $\det(I-\gamma)^p \gamma^{b_0}$ stays bounded when the determinant factor tends to infinity along $\exp(C^\circ \cap \mathfrak{t}_{\mathbb{R}})$.*

Proof. By Vinberg [**25**] each element of C° is conjugate to an element of $\mathfrak{t}_{\mathbb{R}}$, so to look at an arbitrary matrix coefficient along the compact Cartan subspace is sufficient to decide the removability of the singularity. Namely, we have

$$(\gamma \sum_i a_i v_i, \sum_i a_i v_i) \leq \gamma^{b_o} (\sum_i a_i v_i, \sum_i a_i v_i)$$

which shows the assertion, namely that the rate of growth in the direction of the singularity is controlled by b_0. ∎

3. The holomorphic discrete series for $G^\flat$

All definitions and results in this section are taken from [**5**] and [**24**].

Let N and R be the dimension and the rank of the Jordan algebra V. For a complex manifold $\mathcal{M}$ we denote by $\mathcal{O}(\mathcal{M})$ the space of holomorphic functions on $\mathcal{M}$.

The group $G^\flat$ acts on T_Ω via

$$g \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right),$$

and the scalar–valued holomorphic discrete series representations of $G^\flat$ are

$$\big(U_\lambda(g)f\big)(Z) = \det(CZ+D)^{-\lambda} f(g^{-1} \cdot Z), \quad g^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$$

for $\lambda \geq 2\frac{N}{R}$, which all are unitary and irreducible in the Hilbert spaces

$$\mathcal{H}_\lambda(T_\Omega) := \{f \in \mathcal{O}(T_\Omega) \quad | \quad \int_{T_\Omega} |f(X+iY)|^2 \det(Y)^{\lambda - 2\frac{N}{R}} dX dY < \infty\}.$$

Morever the reproducing kernel of $\mathcal{H}_\lambda(T_\Omega)$ is given by

$$K_\lambda^{T_\Omega}(Z,W) = \det\Big(\frac{Z-W^*}{2i}\Big)^{-\lambda}.$$

The *classical Hardy space* $H^2(T_\Omega)$ on T_Ω is defined as the space of holomorphic functions f on T_Ω such that

$$\sup_{Y \in \Omega} \int_V |f(X+iY)|^2 \, dX < \infty.$$

Proposition 3.1. *The Hardy space $H^2(T_\Omega)$ may be thought of as the space $\mathcal{H}_\lambda(T_\Omega)$ for $\lambda = \frac{N}{R}$, and the Cauchy–Szegö kernel of T_Ω is given by*

$$K(Z,W) = \det\Big(\frac{Z-W^*}{2i}\Big)^{-N/R}. \tag{3.1}$$

∎

We list here the groups G and the corresponding group $G^\flat$, Jordan algebra V, its rank R, its dimension N, and the Koecher norm det :

G	$G^\flat$	V	N	R	det
$Sp(r,\mathbb{R})$	$Sp(2r,\mathbb{R})$	$\mathrm{Sym}(2r,\mathbb{R})$	$r(2r+1)$	$2r$	Det
$SO^*(2l)$	$SO^*(4l)$	$\mathrm{Herm}(l,\mathbb{H})$	$l(2l-1)$	l	$\mathrm{Det}^{1/2}$
$U(p,q)$	$SU(n,n)$	$\mathrm{Herm}(n,\mathbb{C})$	n^2	n	Det

4. The conformal image of the holomorphic discrete series for $G^\flat$

As in [**18**], we wish to make a correspondence between holomorphic functions on T_Ω and holomorphic functions on $\Gamma(C)^\circ$. First, we can easily extend Proposition 1.5 in [**18**] to any Euclidean Jordan algebra.

Proposition 4.1. *Let $f \in \mathcal{O}(T_\Omega \setminus \Sigma)$ such that*

$$\int_{T_\Omega} |f(Z)|^2 \det(Y)^{\lambda - 2\frac{N}{R}} dX\, dY < \infty,$$

for $\lambda \geq 2\frac{N}{R}$. Then f is actually holomorphic on all of T_Ω. ■

Let $\lambda \geq 2\frac{N}{R}$ and introduce the Hilbert space

$$\mathcal{H}_\lambda(\Gamma(C)) = \big\{F \in \mathcal{O}(\Gamma(C)^\circ) \quad | \quad \int_{\Gamma(C)^\circ} |F(\gamma)|^2 d\nu_\lambda(\gamma) < \infty\big\},$$

where $d\nu_\lambda$ is the Lebesgue measure of $G_{\mathbb{C}}$ restricted to $\Gamma(C)^\circ$ times the density

$$\det(J - \gamma^* J\gamma)^{\lambda - 2\frac{N}{R}}.$$

The Cayley transform $\gamma = \mathsf{C}(Z)$ gives as in Lemma 2.4 a correspondence between $\mathcal{O}(T_\Omega \setminus \Sigma)$ and $\mathcal{O}(\Gamma(C)^\circ)$, namely,

$$f = \mathsf{C}_\lambda(F) \quad : \quad f(Z) = \det(Z + iJ)^{-\lambda} F(\gamma). \tag{4.1}$$

Theorem 4.2. *Let $\lambda \geq 2\frac{N}{R}$. If F belongs to $\mathcal{H}_\lambda(\Gamma(C))$, then the function $f = \mathsf{C}_\lambda(F)$ belongs to $\mathcal{H}_\lambda(T_\Omega)$ and the map C_λ is a unitary linear isomorphism from $\mathcal{H}_\lambda(\Gamma(C))$ onto $\mathcal{H}_\lambda(T_\Omega)$. Furthermore, $\mathcal{H}_\lambda(\Gamma(C))$ is a reproducing kernel Hilbert space with the kernel*

$$K_\lambda(\gamma_1, \gamma_2) = \det\big(J - \gamma_2^* J\gamma_1\big)^{-\lambda}.$$

Proof. This is just like Theorem 1.7 in [**19**], using the above Proposition. ■

5. The Hardy space of the metaplectic semigroup

In this section we assume that $G = Sp(r,\mathbb{R})$. Then the Hardy parameter is $\frac{N}{R} = r + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ and the Koecher norm "det" coincides with the usual matrix determinant "Det" . This suggests that the operator $\mathsf{C}_{\frac{N}{R}} = \mathsf{C}_{r+\frac{1}{2}}$,

$$f = \mathsf{C}_{r+\frac{1}{2}}(F) \ : \ f(Z) = \mathrm{Det}(Z+iJ)^{-(r+\frac{1}{2})}F(\gamma) \tag{5.1}$$

may be an intertwining operator between $H^2(T_\Omega)$ and the odd part of the Hardy space $H^2(\Gamma(C)_2)$ on the double covering $\Gamma(C)_2$ of $\Gamma(C)$.

The appearance of a square root in (5.1) means that we have to consider double coverings defined in terms of the determinant function on the right-hand side of (5.1). This is another main point of this paper. We shall use the following two general principles:

Principle 1. Suppose $G \times M \to M$ is a holomorphic action of a Lie group G on a simply connected complex manifold M, and that there is given a non-vanishing cocycle $\lambda : G \times M \to \mathbb{C}^*$ which is holomorphic in the second variable and satisfies

$$\lambda(g_1g_2, z) = \lambda(g_1, g_2\cdot z)\lambda(g_2, z)$$

for all $g_1, g_2 \in G, z \in M$. Then we can define

$$G_2 := \{(g, \omega(g,\cdot)) \ \mid \ g \in G, \omega(g,\cdot)^2 = \lambda(g,\cdot)\}$$

where $z \to \omega(g,z)$ is holomorphic on M. Since we are on a simply connected space, this holomorphic square root is well-defined up to a sign change. G_2 is a double covering of G, and it is a Lie group endowed with the product

$$(g_1, \omega(g_1,z))(g_2,\omega(g_2,z)) = (g_1g_2, \omega(g_1, g_2\cdot z)\omega(g_2,z))$$

with z on both sides viewed as a holomorphic variable. Note that this product is well-defined, since on the right-hand side we indeed have a holomorphic square root of $\lambda(g_1g_2,z)$. The same construction applies to Lie semigroups. This product is associative, since for either position of the parentheses the ω part of the element becomes

$$\omega(g_1, g_2g_3\cdot z)\omega(g_2, g_3\cdot z)\omega(g_3,z)$$

so $(h_1h_2)h_3 = h_1(h_2h_3)$ where $h_1 = (g_1, \omega(g_1,z))$ etc. This principle is also to be applied in (5.3). Note finally that ω defines a cocycle for G_2, namely $\omega(\widetilde{g},z) = \omega(g,z)$ where $\widetilde{g} = (g,\omega(g,z))$. By definition of the product in G_2 this satisfies the cocycle relation, similar to the one for λ above.

Principle 2. Let N be a complex manifold (not necessarily simply connected) and ϕ a nowhere vanishing holomorphic function on N. Then we can define

$$N_2 := \{(z,w) \ \mid \ z \in N, w \in \mathbb{C}, w^2 = \phi(z)\}$$

which will be a holomorphic double cover of N. By construction, $\phi(z)^{1/2}$ is a well-defined holomorphic function on N_2.

An abstract construction of double (and universal) covering semigroups can be found in much detail in [**13**]. For the open subset $\Gamma(C)_2{}^\circ$ we have a new and explicit construction; but first we recall (in a way suitable to our choices) how to obtain the double cover G_2 of G, based on a slight modification via the Cayley transform of Principle 1 above:

Let $J^\flat = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ and let $G^\flat := Sp(2r, \mathbb{R})$ be the group of all matrices in $Sp(2r, \mathbb{C})$ satisfying

$$g^* J^\flat g = J^\flat.$$

We imbed G in a natural way in $G^\flat$ as follows :

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & I_{2r} \end{pmatrix}.$$

We also view the Cayley transform C as the element of $G^\flat_{\mathbb{C}}$ given by the matrix

$$\mathsf{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2r} & -iJ \\ I_{2r} & iJ \end{pmatrix}.$$

Our precise definition of G_2 is to be the set of all pairs $(g, \omega(g^{\mathsf{C}}, \cdot))$ with $g \in Sp(r, \mathbb{R})$, $g^{\mathsf{C}} = \mathsf{C}^{-1}\, g\, \mathsf{C} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $\omega(g^{\mathsf{C}}, \cdot)^2 = \mathrm{Det}(C\ \cdot + D)^{-1}$ and $Z \longrightarrow \omega(g^{\mathsf{C}}, Z)$ is holomorphic on T_Ω. Note that this is analogous to the definition of the double cover of $SU(1,1)$, where we take all pairs $(g, \sqrt{cz+d})$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$ and $\sqrt{cz+d}$ a holomorphic choice of square root of the non-zero function $cz + d$ on the unit disc. Indeed, it sometimes is convenient to think in terms of such multivalued functions when doing practical calculations, but of course, the precise definition is behind this. We also recall the more informal definition of G_2 as follows:

Take again $Z \in T_\Omega$ and $g \in Sp(r, \mathbb{R})$ such that $\mathsf{C}^{-1}\, g\, \mathsf{C} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. A determination on T_Ω of the square root $\mathrm{Det}(CZ+D)^{-\frac{1}{2}}$ is completely determined by its value on $Z = iI$. For each $g \in Sp(n, \mathbb{R})$ we choose a determination of $\mathrm{Det}(CZ+D)^{-\frac{1}{2}}$ as in Principle 1, noting that this is a global determination. We consider here Z as a variable, since the group (and indeed all contractions) acts on the tube domain, and we consider the function

$$Z \longmapsto \omega(g^{\mathsf{C}}, Z) := \mathrm{Det}(CZ + D)^{-\frac{1}{2}} \tag{5.2}$$

from T_Ω into $\mathbb{C} \setminus \{0\}$, where $g^{\mathsf{C}} = \mathsf{C}^{-1}\, g\, \mathsf{C}$. We read ω as "a holomorphic choice of square root of the determinant". It follows from Principle 1 that ω may be viewed as a cocycle for G_2, and it gives a choice of square root at the product of two elements as follows:

$$\omega(g_1^{\mathsf{C}} g_2^{\mathsf{C}}, Z) = \omega(g_1^{\mathsf{C}}, g_2^{\mathsf{C}} \cdot Z)\omega(g_2^{\mathsf{C}}, Z).$$

This equation is to be understood as an equation for the two-valued function ω; it does not hold for any single-valued function. More generally, assuming the determinant to be non-zero, we let

$$\omega_2(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z) := \mathrm{Det}(CZ+D)^{-1}$$

and correspondingly $\omega(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z)$ a choice of one of the two square roots of this, either (as here) global and holomorphic, or (as below) local, i.e. at the fixed point Z. Then we may consider our double covering group to be

$$G_2 := \{\widetilde{g} := (g, \omega(g^{\mathsf{C}}, \cdot)) \quad | \quad g \in Sp(r, \mathbb{R})\},$$

endowed with the group law

$$(5.3) \qquad \big(g_1, \omega(g_1^{\mathsf{C}}, Z)\big)\,\big(g_2, \omega(g_2^{\mathsf{C}}, Z)\big) = \big(g_1 g_2, \omega(g_1^{\mathsf{C}}, g_2^{\mathsf{C}} \cdot Z)\omega(g_2^{\mathsf{C}}, Z)\big).$$

G_2 is a two–sheeted covering group of G, since we are considering both choices of square root. G_2 is called the *metaplectic group.*

Now we wish to apply Principle 2 to give another (and by Lemma 5.2 isomorphic) version of the double covering construction. Here N will be the open semigroup, realized as a subset of the tube domain as in Proposition 2.3. For $Z \in T_\Omega \setminus \Sigma$ and for a choice of a local determination of $\mathrm{Det}(Z+iJ)^{-\frac{1}{2}}$ we note that up to a constant

$$\mathrm{Det}(Z+iJ)^{-\frac{1}{2}} = \omega(\mathsf{C}, Z).$$

This is again an identity between two-valued functions. Hence at each fixed point Z we make a choice between the two possible values of the square root, so here the notation does not consider Z as a variable. Note that we may extend our cocycle to the complexified group in the natural way. Therefore, the complex manifold

$$\Gamma(C)_2^\circ := \{\widetilde{\gamma} = (\gamma, \omega(\mathsf{C}, Z)) \quad | \quad \gamma \in \Gamma(C)^\circ,\ \gamma = \mathsf{C}(Z),\ Z \in T_\Omega \setminus \Sigma\},$$

is a two–sheeted covering of the semigroup $\Gamma(C)^\circ$. As before, we consider both choices of square root here, and corresponding to the modern point of view, the more precise definition is as follows:

$$\Gamma(C)_2^\circ = \{\widetilde{\gamma} = (\gamma, w) \in \Gamma(C)^\circ \times \mathbb{C} \quad | \quad \gamma = \mathsf{C}(Z),\ Z \in T_\Omega \setminus \Sigma,\ w^2 = \mathrm{Det}(Z+iJ)^{-1}\}.$$

In particular, w is just a complex number.

Lemma 5.1. *The group G_2 acts on the right on the manifold $\Gamma(C)_2^\circ$*

Proof. We define the action at the same time: Indeed, letting Z' satisfy $g^{-1}\gamma = \mathsf{C} \cdot Z'$, which implies that $Z = g^{\mathsf{C}} \cdot Z'$,

$$\begin{aligned}
\big(\gamma, \omega(\mathsf{C}, Z)\big) \cdot \big(g, \omega(g^{\mathsf{C}}, \cdot)\big) &= \big(g^{-1}\gamma, \omega(\mathsf{C}, Z)\omega(g^{\mathsf{C}}, Z')\big) \quad \text{by definition} \\
&= \big(g^{-1}\gamma, \omega(\mathsf{C}\, g^{\mathsf{C}}, Z')\big) \\
&= \big(g^{-1}\gamma, \omega(g\, \mathsf{C}, Z')\big) \\
&= \big(g^{-1}\gamma, \omega(g, \mathsf{C} \cdot Z')\omega(\mathsf{C}, Z')\big) \\
&= \big(g^{-1}\gamma, \omega(\mathsf{C}, Z')\big)
\end{aligned}$$

because $\omega_2(g, \mathsf{C} \cdot Z') = 1$. ∎

In this calculation, Z and Z' are fixed points, and the cocycles indicate choices of square roots at the relevant points. Indeed, the same calculation holds for ω_2 instead, and the ω then is a complex number with the relevant square. To see that we have indeed defined a group action, we note that the two sides of the equation checking that we have an action both have as ω part

$$\omega(\mathsf{C}, Z)\omega(g_1^{\mathsf{C}}, g_2^{\mathsf{C}} \cdot Z'')\omega(g_2^{\mathsf{C}}, Z'')$$

where $g_2^{\mathsf{C}} \cdot Z'' = Z'$ and $Z = g_1^{\mathsf{C}} \cdot Z'$. To show that $\Gamma(C)_2^\circ$ is a semigroup we consider just like for G_2 the following manifold

$$\Gamma(C)_2^{\circ\prime} := \{(\gamma, \omega(\gamma^{\mathsf{C}}, \cdot)) \quad | \quad \gamma \in \Gamma(C)^\circ\}.$$

It is clear that $\Gamma(C)_2^{\circ\prime}$ is a double covering of $\Gamma(C)^\circ$ and has a semigroup structure with respect to the law (5.3). See Principle 1 and the remark following (5.3).

Consider the map φ from $\Gamma(C)_2^{\circ\prime}$ to $\Gamma(C)_2^\circ$ defined by

$$(\gamma, \omega(\gamma^{\mathsf{C}}, \cdot)) \longmapsto (\gamma, \omega(\mathsf{C}, Z)), \quad \text{where } \gamma = \mathsf{C}(Z) \in \Gamma(C)^\circ.$$

This is to be understood as using the relation (up to a constant)

$$\omega_2((-\gamma)^{\mathsf{C}}, Z_o) = \mathrm{Det}(I - \gamma)^{-1}$$

where $Z_o = 0$ and $\gamma \in \Gamma(C)^\circ$. Hence a choice of global square root corresponds to a choice of local square root in a one-to-one way, since a global choice is determined by its value at a single point. Thus we have

Lemma 5.2. *φ is a homeomorphism from $\Gamma(C)_2^{\circ\prime}$ onto $\Gamma(C)_2^\circ$.* ■

Remark 5.3. The semigroup $\Gamma(C)_2^\circ$ is isomorphic to the interior of the *metaplectic semigroup* or the *Howe oscillator semigroup.* We call it the *open metaplectic semigroup.*

The Hardy space $H^2(\Gamma(C)_2)$ on the *metaplectic semigroup* $\Gamma(C)_2$ is the space of holomorphic functions $F \in \mathcal{O}(\Gamma(C)_2^\circ)$ such that

$$\sup_{\widetilde{\gamma} \in \Gamma(C)_2^\circ} \int_{G_2} |F(\widetilde{\gamma}\widetilde{g})|^2 \, d\widetilde{g} < \infty.$$

The maximal compact subgroup K of $G = Sp(r, \mathbb{R})$ is isomorphic to $U(r)$ and the maximal split abelian subalgebra

$$\mathfrak{t}_{\mathbb{R}} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \in \mathcal{M}(r \times r, \mathbb{R}) \quad | \quad X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{pmatrix} \right\},$$

can be identified with $\mathbb{R}^r$. Let $\epsilon_1, \ldots, \epsilon_r$ be the canonical basis of $\mathfrak{t}_{\mathbb{R}}^* = \mathfrak{t}_{\mathbb{R}} = \mathbb{R}^r$. Then the root system $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is of type C_r :

$$\begin{aligned}
\Delta &= \{\pm(\epsilon_i \pm \epsilon_j)\ (1 \leq i < j \leq r)\ ,\ \pm 2\epsilon_i\ (1 \leq i \leq r)\} \\
\Delta^+ &= \{\epsilon_i \pm \epsilon_j\ (1 \leq i < j \leq r)\ ,\ 2\epsilon_i\ (1 \leq i \leq r)\} \\
\Delta_{\mathfrak{k}}^+ &= \{\epsilon_i - \epsilon_j\ (1 \leq i < j \leq r)\} \\
\Delta_{\mathfrak{p}}^+ &= \{\epsilon_i + \epsilon_j\ (1 \leq i \leq j \leq r)\} \\
\rho &= r\epsilon_1 + (r-1)\epsilon_2 + \ldots + \epsilon_r \\
&\simeq (r, r-1, \ldots, 1).
\end{aligned}$$

Furthermore $\mathcal{P}$ is the lattice $\mathbb{Z}^r$, the set of highest weights relative to $\Delta_{\mathfrak{k}}^+$ is given by

$$\mathcal{R} = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \quad | \quad \lambda_1 \geq \ldots \geq \lambda_r\},$$

and $\lambda \in \mathcal{R}$ satisfies the Harish–Chandra condition if

$$-r > \lambda_1 \geq \ldots \geq \lambda_r,$$

which gives the set $\mathcal{R}'$.

Let $K_2 \subset G_2$, resp. $T_2 \subset K_2$ be the corresponding covering of K and T. Then the corresponding $\mathcal{P}_2$, $\mathcal{R}_{\in}$ and $\mathcal{R}'_{\in}$ are given by

$$\begin{aligned}
\mathcal{P}_2 &= \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) = \mathcal{P} \cup (\mathcal{P} + \frac{1}{2}) = \mathcal{P}_{2,\text{even}} \cup \mathcal{P}_{2,\text{odd}}, \\
\mathcal{R}_2 &= \{\lambda \in \mathcal{P}_2 \quad | \quad (\forall \alpha \in \Delta_{\mathfrak{k}}^+)\ \langle \lambda, \alpha \rangle \geq 0\} \\
&= \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) \quad | \quad \lambda_1 \geq \ldots \geq \lambda_r\}, \\
&= \mathcal{R}_{2,\text{even}} \cup \mathcal{R}_{2,\text{odd}} \\
\mathcal{R}'_{\in} &= \{\lambda \in \mathcal{R}_2 \quad | \quad \langle \lambda + \rho, \beta \rangle < 0,\ \ \forall \beta \in \Delta_{\mathfrak{p}}^+\}, \\
&= \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) \quad | \quad -r > \lambda_1 \geq \ldots \geq \lambda_r\}, \\
&= \mathcal{R}'_{2,\text{even}} \cup \mathcal{R}'_{2,\text{odd}},
\end{aligned}$$

where $\frac{1}{2}$ stands for the tuple $(\frac{1}{2}, \ldots, \frac{1}{2})$. The holomorphic discrete series representations for the metaplectic group G_2 are those irreducible unitary representations π_λ of G_2 that are square–integrable with a highest weight $\lambda \in \mathcal{R}'_2 = \mathcal{R}'_{2,\text{even}} \cup \mathcal{R}'_{2,\text{odd}}$. Therefore

$$H^2(\Gamma(C)_2) = \bigoplus_{\lambda \in (C^* \cap \mathfrak{t}_{\mathbb{R}}) \cap \mathcal{R}'_2} \pi_\lambda \otimes \pi_\lambda^*. \tag{5.4}$$

The cone C is the minimal one in $i\mathfrak{g}$, so the above summation is over $\mathcal{R}'_2$ and the Hardy space $H^2(\Gamma(C)_2)$ splits into two parts, namely, even and odd part,

$$\begin{aligned}
H^2(\Gamma(C)_2) &= H^2_{\text{even}}(\Gamma(C)_2) \oplus H^2_{\text{odd}}(\Gamma(C)_2) \\
&= \Big(\bigoplus_{\lambda \in \mathcal{R}'_{2,\text{even}}} \pi_\lambda \otimes \pi_\lambda^* \Big) \oplus \Big(\bigoplus_{\lambda \in \mathcal{R}'_{2,\text{odd}}} \pi_\lambda \otimes \pi_\lambda^* \Big).
\end{aligned}$$

Remark 5.4. The even part

$$
\begin{aligned}
H^2(\Gamma(C)_2)_{\mathrm{even}} &= \bigoplus_{\lambda \in \mathcal{R}'_{2,\mathrm{even}}} \pi_\lambda \otimes \pi^*_\lambda \\
&= \bigoplus_{\substack{(\lambda_1,\dots,\lambda_r)\in\mathbb{Z}^r \\ -r>\lambda_1\geq\dots\geq\lambda_r}} \pi_\lambda \otimes \pi^*_\lambda
\end{aligned}
$$

coincides with the Hardy space $H^2(\Gamma(C))$ on the semigroup $\Gamma(C)$. For this space Olshanskiĭ (*cf.* [**23**]) gives an explicit formula for the Cauchy–Szegö kernel using a Littlewood-type combinatorial formula : for $\gamma \in \Gamma(C)^\circ$ with eigenvalues $x_1,\dots,x_r,x_1^{-1},\dots,x_r^{-1}$ where $|x_1|,\dots,|x_r|<1$,

$$
L(\gamma) = K(\gamma, e) = \prod_{i=1}^{r} \frac{x_i^{r+1}}{(1+x_i)(1-x_i)^{2r+1}}.
$$

Our goal in this section is to identify the odd part

$$
\begin{aligned}
H^2(\Gamma(C)_2)_{\mathrm{odd}} &= \bigoplus_{\lambda \in \mathcal{R}'_{2,\mathrm{odd}}} \pi_\lambda \otimes \pi^*_\lambda \\
&= \bigoplus_{\substack{(\lambda_1,\dots,\lambda_r)\in\mathbb{Z}^r+\frac{1}{2} \\ -(r+\frac{1}{2})\geq\lambda_1\geq\dots\geq\lambda_r}} \pi_\lambda \otimes \pi^*_\lambda
\end{aligned}
$$

with the classical Hardy space $H^2(Sp(2r,\mathbb{R})/U(2r))$.

Theorem 5.5. *The operator $\mathsf{C}_{r+\frac{1}{2}}$ given by (5.1) induces a unitary isomorphism*

$$
H^2(\Gamma(C)_2)_{\mathrm{odd}} \simeq H^2(Sp(2r,\mathbb{R})/U(2r)).
$$

Proof. From (5.1), $\mathsf{C}^{-1}_{r+\frac{1}{2}}$ imbeds the classical Hardy space into $H^2(\Gamma(C)_2)_{\mathrm{odd}}$. This is because the multiplier factor in the transformation is chosen as the square root of the Radon-Nikodym derivative, i.e. the L^2 norms on the respective Shilov boundaries agree. The image is inside the odd functions because $\mathrm{Det}(Z+iJ)^{-1/2}$ is a well-defined odd function (with respect to the group $\mathbb{Z}_2$ of deck transformations) and $F(\gamma)$ is even – hence the product is an odd function. It suffices then to prove that $\mathsf{C}^{-1}_{r+\frac{1}{2}}$ is onto. For this we will show that for every representation of the odd part of the holomorphic discrete series for G_2 the corresponding matrix coefficients extend to the tube domain T_Ω.

We shall use Lemma 2.4 to find the necessary and sufficient condition for the removal of singularities i.e. to estimate the growth of the function f pulled back from a matrix coefficient. This turns out below as exactly Harish-Chandra's condition for the holomorphic discrete series.

Let π_λ be among these representations with the highest weight $\lambda = (\lambda_1,\dots,\lambda_r) \in \mathbb{Z}^r + \frac{1}{2}$, $\lambda_1 \geq \dots \geq \lambda_r$ and let v_λ be a corresponding normalized highest weight vector. Take

$\widetilde{\gamma} \in \Gamma(C)_2$, with $\gamma \in \exp(C_{\min} \cap \mathfrak{t}_{\mathbb{R}})$. Then γ is of the form

$$\gamma = \begin{pmatrix} e^{s_1} & & \\ & \ddots & \\ & & e^{s_r} \end{pmatrix}, \qquad s_i \geq 0$$

and

$$F(\widetilde{\gamma}) := (\pi_\lambda(\widetilde{\gamma})v_\lambda, v_\lambda) = \widetilde{\gamma}^\lambda = e^{s_1\lambda_1} \cdots e^{s_r\lambda_r}.$$

Then the corresponding function on T_Ω is given (up to a constant factor) by

$$\begin{aligned} f(Z) &= \mathrm{Det}(I-\gamma)^{r+\frac{1}{2}} F(\widetilde{\gamma}) \\ &= \Big((1-e^{s_1})^{r+\frac{1}{2}} \cdots (1-e^{s_r})^{r+\frac{1}{2}}\Big)\Big(e^{s_1\lambda_1} \cdots e^{s_r\lambda_r}\Big) \\ &= e^{s_1(\lambda_1+r+\frac{1}{2})} \cdots e^{s_r(\lambda_r+r+\frac{1}{2})} + \cdots \\ &\underset{s_1,\ldots,s_r \to +\infty}{\sim} e^{s_1(\lambda_1+r+\frac{1}{2})}. \end{aligned}$$

Thus, to remove the singularity of the holomorphic function on T_Ω we need to take

$$-(r+\frac{1}{2}) \geq \lambda_1$$

and this is exactly the Harish-Chandra condition for G_2 we gave above. The sufficiency of this condition comes from Lemma 2.4. ∎

Corollary 5.6. *Under the action of $Mp(r,\mathbb{R}) \times Mp(r,\mathbb{R})$ the Hardy space $H^2(T_\Omega)$ can be decomposed into a direct sum of the 'odd' holomorphic discrete series representations of $Mp(r,\mathbb{R})$, i.e.*

$$H^2(Sp(2r,\mathbb{R})/U(2r))_{|_{Mp(r,\mathbb{R})\times Mp(r,\mathbb{R})}} = \bigoplus_{\lambda \in \mathcal{R}'_{2,\mathrm{odd}}} \pi_\lambda \otimes \pi_\lambda^*.$$

∎

Corollary 5.7. *Let K_{odd} be the kernel corresponding to $H^2(\Gamma(C)_2)_{\mathrm{odd}}$. Then for every $\widetilde{\gamma_1}, \widetilde{\gamma_2} \in \Gamma(C)_2$*

$$K_{\mathrm{odd}}(\widetilde{\gamma_1}, \widetilde{\gamma_2}) = \mathrm{Det}(J - \gamma_2^* J \gamma_1)^{-(r+1/2)}.$$

Proof. This is a simple application of (3.1) and the isomorphism in Theorem 5.5. ∎

Corollary 5.8. *On the interior of the metaplectic semigroup the holomorphic function $\mathrm{Det}(I-\gamma)^{-(r+1/2)}$ has the following expansion*

$$\mathrm{Det}(I-\gamma)^{-(r+1/2)} = \sum_{\lambda \in \mathcal{R}'_{2,\mathrm{odd}}} d_\lambda \mathrm{tr}(\pi_\lambda(\gamma)),$$

where d_λ is the formal dimension of π_λ. ∎

The Bergman space on $\Gamma(C)$ is $\mathcal{H}_{2r+1}(\Gamma(C))$ and its reproducing kernel is given by

$$K_B(\gamma_1, \gamma_2) = \mathrm{Det}(J - \gamma_2^* J \gamma_1)^{-(2r+1)}.$$

Corollary 5.9. *The Bergman kernel K_B on the semigroup $\Gamma(C)$ is the square of the odd part K_{odd} of the Cauchy-Szegö kernel for $\Gamma(C)_2$.* ■

6. The case of $G = SO^*(2l)$

Let $G = SO^*(2l)$ realized as a subgroup of $U(l,l)$,

$$G = \{g \in SO^*(2l,\mathbb{C}) \quad | \quad g^*Jg = J\}, \;\; J = \begin{pmatrix} -I_l & 0 \\ 0 & I_l \end{pmatrix}.$$

The Hardy parameter in this case is $N/R = l(2l-1)/l = 2l-1$ and the Koecher norm "det" is the square root of the usual determinant "Det" ($\det = \mathrm{Det}^{1/2}$). Thus the operator $\mathsf{C}_{\frac{N}{R}} = \mathsf{C}_{2l-1}$,

$$f = \mathsf{C}_{2l-1}(F) \;\; : \;\; f(Z) = \mathrm{Det}(Z+iJ)^{-(l-1/2)}F(\gamma) \tag{6.1}$$

provides an equivariant embedding of the classical Hardy space $H^2(SO^*(4l)/U(2l))$ into the odd part of the Hardy space $H^2(\Gamma(C)_2)_{\text{odd}}$ on the double covering semigroup $\Gamma(C)_2$ of the minimal semigroup

$$\Gamma(C) = \{\gamma \in SO^*(2l,\mathbb{C}) \quad | \quad J - \gamma^*J\gamma \geq 0\}$$

(because of the square root in $\mathrm{Det}(Z+iJ)^{(l-1/2)}$). This is just like the symplectic case; again the power of the determinant is chosen so that L^2 norms are preserved. We will identify the maximal compact subgroup with $U(l)$ as in the above section. The determinanat factor is again exactly the Jacobian to a power such that we have preservation of L^2-norms on the respective boundaries. Then $\mathfrak{t}_\mathbb{R}$ is given by the same formula as in $Sp(r,\mathbb{R})$ case. Let $\epsilon_1,\epsilon_2,\ldots,\epsilon_l$ be the canonical basis of $\mathfrak{t}^*_\mathbb{R} = \mathfrak{t}_\mathbb{R} = \mathbb{R}^l$. The root system $\Delta = \Delta(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ is of type D_l :

$$\begin{aligned}
\Delta &= \{\pm\epsilon_i \pm \epsilon_j \quad | \quad 1 \leq i < j \leq l\},\\
\Delta^+ &= \{\epsilon_i \pm \epsilon_j \quad | \quad 1 \leq i < j \leq l\},\\
\Delta^+_\mathfrak{k} &= \{\epsilon_i - \epsilon_j \quad | \quad 1 \leq i < j \leq l\},\\
\Delta^+_\mathfrak{p} &= \{\epsilon_i + \epsilon_j \quad | \quad 1 \leq i < j \leq l\},\\
\rho &= (l-1)\epsilon_1 + (l-2)\epsilon_2 + \ldots + \epsilon_{l-1}\\
&\simeq (l-1, l-2, \ldots, 1, 0).
\end{aligned}$$

The set of highest weights relative to the positive roots of $SO^*(2l)$ is

$$\mathcal{R} = \{\lambda = (\lambda_1 \,\ldots,\lambda_l) \in \mathbb{Z}^l \quad | \quad \lambda_1 \geq \ldots \geq \lambda_l\},$$

and $\lambda \in \mathcal{R}$ satisfies to the Harish-Chandra condition if and only if

$$-2l+3 > \lambda_1 + \lambda_2. \tag{6.2}$$

Therefore, the odd holomorphic discrete serie representations of the double covering group G_2 of $SO^*(2l)$ are those irreducible unitary representations π_λ, square-integrable with a highest weight $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l - \frac{1}{2}$ such that $0 \geq \lambda_1 \geq \ldots \geq \lambda_l$ and satisfying

$$-2l + 2 \geq \lambda_1 + \lambda_2. \tag{6.3}$$

Let $\mathcal{R}'_{2,\mathrm{odd}}$ denotes the set of these λ's. The Hardy space $H^2(\Gamma(C)_2)$ on the minimal cone $\Gamma(C)_2$ has then the following decomposition

$$H^2(\Gamma(C)_2)_{\mathrm{odd}} = \bigoplus_{\lambda \in \mathcal{R}'_{2,\mathrm{odd}}} \pi_\lambda \otimes \pi^*_\lambda. \tag{6.4}$$

Theorem 6.1. *The classical Hardy space $H^2(SO^*(4l)/U(2l))$ is a proper invariant subspace of the "non-classical" Hardy space $H^2(\Gamma(C)_2)_{\mathrm{odd}} \subset H^2(\Gamma(C)_2)$.*

Proof. With the same arguments as in the proof of Theorem 5.5, we prove that a unitary irreducible representation π_λ of G_2 continue to the tube domain $SO^*(4l)/U(2l)$ if and only if $-(l-1/2) \geq \lambda_1$, which is different from (6.3). ∎

Corollary 6.2. *The representation of $SO^*(2l) \times SO^*(2l)$ in the Hardy space $H^2(\Gamma(C))$ cannot be obtained by a restriction of a representation of the holomorphic discrete series of $SO^*(4l)$ nor any continuation of this, such as the Hardy space.* ∎

Let $\mathcal{H}^2(\Gamma(C))$ the conformal image of $H^2(SO^*(4l)/U(2l))$ via the operator C_{2l-1}.

Corollary 6.3. *$\mathcal{H}^2(\Gamma(C))$ is a reproducing kernel Hilbert space and its reproducing kernel K is the pre-image of the Cauchy–Szegö kernel of*

$$H^2(SO^*(4l)/U(2l)), \quad i.e., K(\gamma_1, \gamma_2) = \mathrm{Det}(J - \gamma_2^* J \gamma_1)^{-(l-1/2)}. \quad ∎$$

Corollary 6.4. *On $\Gamma(C)^\circ$ the holomorphic function $\mathrm{Det}(I-\gamma)^{-(l-1/2)}$ has the following expansion*

$$\mathrm{Det}(I-\gamma)^{-(l-1/2)} = \sum_{-l+1/2 \geq \lambda_1 \geq \ldots \geq \lambda_l} d_\lambda \mathrm{tr}(\pi_\lambda(\gamma)),$$

where d_λ is the formal dimension of π_λ. ∎

7. The case of $G = U(p,q)$

In this section we fix $G = U(p,q)$ realized by

$$G = U(p,q) = \{g \in GL(n,\mathbb{C}) \ | \ g^* J g = J\},\ J = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$$

where $n = p+q$. In this case the Hardy parameter is $N/R = n^2/n = n$ and the Koecher norm "det" is the usual determinant "Det". Therefore the operator $\mathsf{C}_{\frac{N}{R}}$ given by

$$f = \mathsf{C}_n(F) \ : \ f(Z) = \mathrm{Det}(Z + iJ)^{-n} F(\gamma) \tag{7.1}$$

may be an intertwining operator between the Hardy space

$$H^2(SU(n,n)/S(U(n)\times U(n)) \text{ and the Hardy space } H^2(\Gamma(C))$$

over the semigroup

$$\Gamma(C)=\{\gamma\in GL(n,\mathbb{C}) \quad | \quad J-\gamma^*J\gamma\geq 0\}.$$

To study unitary representations of $G=U(p,q)$ we identify it with $(U(1)\times SU(p,q))/\mathbb{Z}_{p+q}$. Thus the unitary irreducible representations of G are those of $U(1)\times SU(p,q)$ that are trivial on $(\zeta,\zeta^{-1}I_n)$ as in Remark 2.1, where I_n is the identity matrix and $\zeta^n=1$. Therefore, the holomorphic discrete series representations of G that we are interested in are

$$\pi_{\lambda,k}(e^{i\theta}g)=e^{ik\theta}\pi_\lambda(g)\ ,\ g\in SU(p,q),\ \theta\in\mathbb{R} \tag{7.2}$$

where $k\in\mathbb{Z}$ and π_λ are the holomorphic discrete series representations of $SU(p,q)$, realized on $\mathcal{D}=SU(p,q)/S(U(p)\times U(q))$, for example in the scalar case:

$$\big(\pi_\lambda(g)f\big)(Z)=\mathrm{Det}(CZ+D)^{-\lambda}f\big((AZ+B)(CZ+D)^{-1}\big),\ g^{-1}=\begin{pmatrix}A & B\\ C & D\end{pmatrix},$$

with λ an integer, and in general $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n$. There will an underlying parity condition to make the representations trivial on $\mathbb{Z}_n$ as above; for example in the scalar case we must have that $k-q\lambda$ is divisible by n.

Let $\mathfrak{t}\subset\mathfrak{k}$ be a Cartan subalgebra consisting of diagonal matrices with purely imaginary values and $\mathfrak{t}_\mathbb{R}=i\mathfrak{t}$. Then the root system $\Delta=\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ is of type A_{n-1} :

$$\begin{aligned}
\Delta&=\{\epsilon_i-\epsilon_j \quad | \quad 1\leq i\neq j\leq n\},\\
\Delta^+&=\{\epsilon_i-\epsilon_j \quad | \quad 1\leq i<j\leq n\},\\
\Delta^+_\mathfrak{k}&=\{\epsilon_i-\epsilon_j \quad | \quad 1\leq i<j\leq p \text{ or } p+1\leq i<j\leq n\},\\
\Delta^+_\mathfrak{p}&=\{\epsilon_i-\epsilon_j \quad | \quad 1\leq i\leq p \text{ and } p+1\leq j\leq n\},\\
2\rho&=(n-1)\epsilon_1+(n-3)\epsilon_2+\ldots-(n-3)\epsilon_{n-1}-(n-1)\epsilon_n\\
&\simeq(n-1,n-3,n-5,\dots,-n+3,-n+1),
\end{aligned}$$

where $\epsilon_1,\dots,\epsilon_n$ is the canonical basis of $\mathfrak{t}^*_\mathbb{R}\simeq\mathfrak{t}_\mathbb{R}\simeq\mathbb{R}^n$. Then the holomorphic discrete series representations of $SU(p,q)$ are the above representations π_λ with $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n$ satisfying

$$\lambda_i-\lambda_{i+1}\geq 0,\ \ i\neq p,\ \ 1\leq i\leq n-1,$$

and the Harish-Chandra condition

$$\lambda_n-\lambda_1>n-1.$$

Let $\pi_{\lambda,k}$ be an irreducible unitary representation of G with highest weight (λ, k), $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then according to (7.2), its highest weight character is given by

$$\chi_{\lambda,k}(e^{i\theta}g) = e^{ik\theta}\chi_\lambda(g)$$

where χ_λ is the character of the representation π_λ of $SU(p,q)$. The cone $C \cap \mathfrak{t}_{\mathbb{R}}$ consists of all matrices of rank $(n \times n)$,

$$\begin{pmatrix} -T & 0 \\ 0 & S \end{pmatrix}$$

where T is the diagonal matrix with diagonal entries $t_1 \leq 0, \ldots, t_p \leq 0$ and S the diagonal matrix with diagonal entries $s_{p+1} \leq 0, \ldots, s_n \leq 0$. Take $\gamma \in \exp(C \cap \mathfrak{t}_{\mathbb{R}})$. Then γ is of the form $\gamma = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^S \end{pmatrix}$ with $\begin{pmatrix} -T & 0 \\ 0 & S \end{pmatrix} \in C \cap \mathfrak{t}_{\mathbb{R}}$. Therefore, if $\mathrm{Det}(\gamma) = 1$,

$$\chi_{\lambda,k}(\gamma) = \chi_\lambda(\gamma) = e^{-t_1\lambda_1} \ldots e^{-t_p\lambda_p} e^{s_{p+1}\lambda_{p+1}} \ldots e^{s_{n-1}\lambda_{n-1}}.$$

Thus in this case the $C-$dissipativity condition is

$$0 \geq \lambda_1 \geq \ldots \geq \lambda_p, \quad \lambda_{p+1} \geq \ldots \geq \lambda_n \geq 0.$$

On the other hand if $\mathrm{Det}(\gamma) \neq 1$ we put

$$c = (\mathrm{Det}(\gamma))^{1/n} = e^{-(t_1+\ldots+t_p)/n} e^{(s_{p+1}+\ldots+s_n)/n},$$

hence

$$\begin{aligned}
\chi_{\lambda,k}(\gamma) &= \chi_{\lambda,k}\Big(c\begin{pmatrix} e^{-T}/c & 0 \\ 0 & e^S/c \end{pmatrix}\Big), \\
&= c^k \chi_\lambda\Big(\begin{pmatrix} e^{-T}/c & 0 \\ 0 & e^S/c \end{pmatrix}\Big), \\
&= e^{-k(t_1+\ldots+t_p)/n} e^{k(s_{p+1}+\ldots+s_n)/n} \\
&\qquad\qquad \cdot e^{\lambda_1\left(-t_1+(t_1+\ldots+t_p)/n-(s_{p+1}+\ldots+s_n)/n\right)} \\
&\qquad\qquad \vdots \\
&\qquad\qquad \cdot e^{\lambda_p\left(-t_p+(t_1+\ldots+t_p)/n-(s_{p+1}+\ldots+s_n)/n\right)} \\
&\qquad\qquad \cdot e^{\lambda_{p+1}\left(s_{p+1}+(t_1+\ldots+t_p)/n-(s_{p+1}+\ldots+s_n)/n\right)} \\
&\qquad\qquad \vdots \\
&\qquad\qquad \cdot e^{\lambda_n\left(s_n+(t_1+\ldots+t_p)/n-(s_{p+1}+\ldots+s_n)/n\right)}, \\
&= e^{t_1\left(\frac{[\lambda]-k}{n}-\lambda_1\right)} \ldots e^{t_p\left(\frac{[\lambda]-k}{n}-\lambda_p\right)} e^{s_{p+1}\left(\frac{k-[\lambda]}{n}+\lambda_{p+1}\right)} \ldots e^{s_n\left(\frac{k-[\lambda]}{n}+\lambda_n\right)},
\end{aligned}$$

where $[\lambda] = \lambda_1 + \ldots + \lambda_n$. So, in this case, the $C-$dissipativity condition is

$$[\lambda] - n\lambda_n \leq k \leq [\lambda] - n\lambda_1. \tag{7.3}$$

Therefore, $\pi_{\lambda,k}$ is a $C-$dissipative representation of the holomorphic discrete series if and only if (λ, k) belongs to the set $\mathcal{R}_{\text{diss}}$

$$\mathcal{R}_{\text{diss}} = \{(\lambda_1, \dots, \lambda_n, k) \in \mathbb{Z}^{n+1} \quad | \quad \begin{cases} \lambda_n - \lambda_1 > n-1 \\ 0 \geq \lambda_1 \geq \dots \geq \lambda_p,\ \lambda_{p+1} \geq \dots \geq \lambda_n \geq 0 \\ [\lambda] - n\lambda_n \leq k \leq [\lambda] - n\lambda_1 \end{cases} \}.$$

Hence the Hardy space on the semigroup $\Gamma(C)$ has the following decomposition

$$H^2(\Gamma(C)) = \bigoplus_{(\lambda,k)\in\mathcal{R}_{\text{diss}}} \pi_{\lambda,k} \otimes \pi^*_{\lambda,k}. \tag{7.4}$$

Theorem 7.1. *The classical Hardy space $H^2\big(SU(n,n)/S\big(U(n)\times U(n)\big)\big)$ is a proper invariant subspace of the "non-classical" Hardy space $H^2(\Gamma(C))$.*

Proof. We just compare those two Hardy spaces according to the transformation C_n given in (7.1). Let $\pi_{\lambda,k}$ be a representation of the holomorphic discrete series of $G = U(p,q)$. The matrix coefficient $\chi_{\lambda,k}(\gamma) = (\pi_{\lambda,k}(\gamma)v_{\lambda,k}, v_{\lambda,k})$ is a holomorphic function on the semigroup $\Gamma(C)^\circ$ ($v_{\lambda,k}$ is the highest weight vector of highest weight (λ,k)). Put $F := \text{Det}(I-\gamma)^n\chi_{\lambda,k}(\gamma)$. Again, we use Lemma 2.4 - recall the argument: F is holomorphic on $\Gamma(C)^\circ$, so it is determined by its restriction to $\exp(C^\circ)$. Since each $X \in C^\circ$ is conjugate to an element of $\mathfrak{t}_{\mathbb{R}}$ (see Vinberg [**25**]), to find the decay condition at infinity in the semigroup to ensure a removal of the singularities on Σ of the holomorphic function F it suffices to check it on $\exp(C^\circ \cap \mathfrak{t}_{\mathbb{R}})$. This is because any other matrix coefficient will have a better decay at the singularity than the highest one above, and by conjugation it is enough to check the removal of singularity on the compact Cartan subspace. Let $\gamma \begin{pmatrix} e^{-T} & 0 \\ 0 & e^S \end{pmatrix} \in \exp(C^\circ \cap \mathfrak{t}_{\mathbb{R}})$, then

$$\begin{aligned}\text{Det}(I-\gamma)^n\chi_{\lambda,k}(\gamma) &= (1-e^{-t_1})^n \dots (1-e^{-t_p})^n(1-e^{s_{p+1}})^n \dots (1-e^{s_n})^n \\ &\quad \cdot e^{t_1\left(\frac{[\lambda]-k}{n}-\lambda_1\right)} \dots e^{t_p\left(\frac{[\lambda]-k}{n}-\lambda_p\right)} e^{s_{p+1}\left(\frac{k-[\lambda]}{n}+\lambda_{p+1}\right)} \dots e^{s_n\left(\frac{k-[\lambda]}{n}+\lambda_n\right)} \\ &\underset{t_1,\dots,s_n\to-\infty}{\sim} e^{t_1(-n-\lambda_1+\frac{[\lambda]-k}{n})} \dots e^{t_p(-n-\lambda_p+\frac{[\lambda]-k}{n})} \\ &\quad \cdot e^{s_{p+1}(n+\lambda_{p+1}+\frac{k-[\lambda]}{n})} \dots e^{s_n(n+\lambda_n+\frac{k-[\lambda]}{n})}.\end{aligned}$$

Therefore the decay condition is

$$[\lambda] - n(\lambda_n + n) \leq k \leq [\lambda] - n(\lambda_1 + n),$$

which is different from the dissipativity condition in (7.3). ∎

Corollary 7.2. *The representation of $S\big(U(p,q)\times U(p,q)\big)$ in the Hardy space $H^2(\Gamma(C))$ cannot be obtained by a restriction of a representation of the holomorphic discrete series of $SU(n,n)$ nor any analytic continuation of this, such as the Hardy space.* ∎

Note that this should be taken in the strong sense, that for no choice of cone C do the spaces coincide. This is seen by noting that a larger cone will impose dissipativity conditions which only make the spectral overlap smaller. Let $\mathcal{H}^2(\Gamma(C))$ be the conformal image of $H^2\big(SU(n,n)/S\big(U(n)\times U(n)\big)\big)$ via the operator C_n.

Corollary 7.3. *$\mathcal{H}^2(\Gamma(C))$ is a reproducing Hilbert space and its reproducing kernel K is the pre-image of the Cauchy-Szegö kernel of $H^2\big(SU(n,n)/S\big(U(n)\times U(n)\big)\big)$, i.e.*

$$K(\gamma_1,\gamma_2)=\operatorname{Det}(J-\gamma_2^*J\gamma_1)^{-n}.$$

■

Corollary 7.4. *On $\Gamma(C)^\circ$ the holomorphic function $\operatorname{Det}(I-\gamma)^{-n}$ has the following expansion*

$$\operatorname{Det}(I-\gamma)^{-n}=\sum_{(\lambda,k)\in\mathcal{R}_{\mathrm{decay}}}d_{\lambda,k}\operatorname{tr}(\pi_{\lambda,k}(\gamma)),$$

where $d_{\lambda,k}$ is the formal dimension of $\pi_{\lambda,k}$ and

$$\mathcal{R}_{\mathrm{decay}}=\{(\lambda_1,\ldots,\lambda_n,k)\in\mathbb{Z}^{n+1}\quad|\quad\begin{cases}\lambda_n-\lambda_1>n-1\\ 0\geq\lambda_1\geq\ldots\geq\lambda_p,\ \lambda_{p+1}\geq\ldots\geq\lambda_n\geq 0\\ [\lambda]-n(\lambda_n+n)\leq k\leq[\lambda]-n(\lambda_1+n)\end{cases}\}$$ ■

8. Remarks and conjectures

Our geometric construction of the double cover semigroup could naturally be extended to the $n-$fold cover by choosing n'th roots of the Jacobian ; similarly a choice of logarithm could yield the universal cover. Hence we could compare, again via the Cayley transform, modules in the analytic continuation of the scalar holomorphic discrete series for $G^\flat$ with function spaces on the covering semigroup. For example we saw in [**18**] how the wave equation representation of $SU(2,2)$ naturally lives on $U(1,1)$ (and the semigroup), solving the wave equation here.

Another aspect for the further work is to extend to the remaining groups and symmetric spaces of Hermitian type. We conjecture that in many cases will the non–commutative Hardy spaces be different from their natural classical counterpart, and that in some cases, the double cover will be necessary and indeed natural. We also conjecture that in the bad case the difference between the two Hardy spaces may be a (sum of) non–commutative Hardy space(s) of lower rank. For example in [**18**], we prove, using the Cauchy–Szegö kernel, that the difference between the classical Hardy space of $SU(2,2)$ and the non–commutative Hardy space of $U(1,1)$ is the non–commutative Hardy space of $SU(1,1)$ studied by Gel'fand and Gindikin in [**6**]. Hence several of the goals aimed in this paper, and especially that of calculating Cauchy–Szegö kernels, remain open problems in general.

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