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On a Diffeological Group Realization of Certain Generalized Symmetrizable Kac-Moody Lie Algebras

Joshua Leslie

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Abstract. In this paper we utilize the notion of infinite dimensional diffeological Lie groups and diffeological Lie algebras to construct a Lie group structure on the space of smooth paths into a completion of a generalized Kac-Moody Lie algebra associated to a symmetrized generalized Cartan matrix.

We then identify a large normal subgroup of this group of paths such that the quotient group has the sought-after properties of a candidate for a Lie group corresponding to the completion of the initial Kac Moody Lie algebra.

Introduction

In the case of Banach-Lie algebras, it is well known that there exists Lie algebras which are not realizable as the Lie algebras of any Lie group. Since the advent of Kac-Moody Lie algebras it has been natural to ask if there are groups associated to these Lie algebras in such a way that the representation theory of the Lie algebras are related to the representation theory of the associated groups as in the theory of finite dimensional Lie groups. Moody and Teo [16] have associated groups to Kac-Moody Lie algebras, these groups have been studied by several authors (see e.g. [5], [10], [14], [16] and [23]); however, those groups do not have some important properties which one encounters in the finite dimensional case, such as the existence of an exponential function. Goodman and Wallach [6] succeeded in associating Banach-Lie groups to affine Kac-Moody Lie algebras and used them to study the representation theory of affine Kac-Moody Lie algebras. Here, we shall give a new procedure to construct a Lie group structure on $C^\infty(I, \bar{\mathcal{G}}(\mathrm{A}))$, where $\bar{\mathcal{G}}(\mathrm{A})$ is an appropriate completion of a generalized Kac-Moody Lie algebra associated to a symmetrized generalized Cartan matrix A, where $C^\infty(\mathrm{I}, \bar{\mathcal{G}}(\mathrm{A}))$ is the space of C^∞ functions from the unit interval I into $\bar{\mathcal{G}}(\mathrm{A})$.

Further, we exhibit an exact sequence $0 \to \Omega \to C^\infty(\mathrm{I}, \bar{\mathcal{G}}(\mathrm{A})) \to \bar{\mathcal{G}}(\mathrm{A}) \to 0$ of smooth Lie algebra homomorphisms, where $\Omega = \{f \in C^\infty(I, \bar{\mathcal{G}}(\mathrm{A})) : \int \mathrm{fdt} = 0\}$; in the case of the above exact sequence, $C^\infty(I, \bar{\mathcal{G}}(\mathrm{A}))$ also designates by abuse

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of notation the Lie algebra of the above mentioned Lie group $C^\infty(I, \bar{\mathcal{G}}(\mathrm{A}))$ (the underlying topological structures are the same (see [13] for details)). There is a normal subgroup, ω, of $C^\infty(I, \bar{\mathcal{G}}(\mathrm{A}))$ having Ω as its Lie algebra. We shall show that $C^\infty(I, \bar{\mathcal{G}}(\mathrm{A}))/\omega = \Lambda(\mathrm{A})$ has many of the requisite properties for a Lie group corresponding to $\bar{\mathcal{G}}(A)$. Finally, we lay the groundwork to show that the highest weight representations of $\bar{\mathcal{G}}(\mathrm{A})$ are differentials of smooth representations of $\Lambda(\mathrm{A})$.

In section 1 we redefine diffeological spaces and reformulate some of the results on them of which we subsequently make use. Most of the verifications follow easily from the work done by Souriau, Dazord, Donato, Iglesias (e.g. see [3], [4], [7],[21]). In those cases where we state formerly unpublished results on diffeological spaces or diffeological groups we give proofs or indications of proofs when it seems appropriate. The diffeological spaces that we deal with are different from the differential spaces considered by the Polish school of Sikorski et al (see e.g. [20]); there, the generalization of manifolds studied is in terms of the commutative ring of smooth functions defined on a manifold. Here, we are looking instead at a generalization of the structure given by the smooth functions defined with domain an open subset of an arbitrary Hausdorff, complete, locally convex topological vector space and with range a fixed manifold.

In section 2, we give details on our construction of a diffeological Lie group $\Lambda(\mathrm{A})$, corresponding to a generalized symmetrized Cartan matrix A with a possibly countably infinite number of rows and columns such that the rows and columns are uniformly ℓ_2 bounded. Our definition of generalized Kac-Moody Lie algebra is more general than the one discussed by Kac in his most recent edition of Infinite Dimensional Lie Algebras, but less general than Borcherds' [1] in that we suppose that the Cartan subalgebra has a canonical Hilbert space structure.

1. Diffeological Algebraic Structures

Let Γ be the Grassmannian ring of super numbers generated by an arbitrary set $\mathcal{X} = \{x_i\}_{i\in I}$ with its topology given by the inductive limit of Γ_i, for $i \in J$, where J is the collection of finite subsets of I ordered by inclusion and Γ_J the finite dimensional subspaces of Γ generated by $x_{i_1}, ..., x_{i_{n_j}}$. . With this topology Γ is a complete locally convex topological vector space. Γ_i is a Z_2 graded commuatative (i.e. $ab = (-1)^{|a||b|}ba$ algebra, where $|a|$ designates the parity of a. Γ with this topology will be used as our base ring throughout what follows.

Let V and W be fopological graded modules over Γ. A continuous mapping

$$f : V \times \ldots \times V \to W$$

is said to be an n-*multimorphism* when f is n-multilinear with repect to the ground field K and

$$f(e_1, \ldots, e_i\gamma, e_{i+1}, \ldots, e_n) = f(e_1, \ldots, e_i, \gamma e_{i+1}, \ldots, e_n), \gamma \in \Gamma$$

and

$$f(e_1, \ldots, e_n\gamma) = f(e_1, \ldots, e_n)\gamma, \gamma \in \Gamma.$$

Now suppose $U \subseteq V$ open, a function $f : U \to W$ will be called super C^n or G^n or simply *smooth* when there exists continuous maps, which are k-multimorphisms in the k-terminal variables, for $x \in U$ fixed,

$D^k f(x; \cdots) : U \times V \times \cdots \times V \to W$, $k \leq n$, such that

$$F_k(h) = f(x+h) - f(x) - Df(x,h) - \cdots - \frac{1}{k!} D^k f(x,h,\cdots,h), \qquad 1 \leq k \leq n,$$

satisfies the property that

$$G_k(t,h) = \begin{cases} F_k(th)/t^k, & t \neq 0 \\ 0, & t = 0. \end{cases} \tag{1}$$

Let $\mathcal{M}$ be the category whose objects are super manifolds modelled on the open subsets of graded complete Hausdorff locally convex topological vector spaces (chlctvs) (see[12]); the morphisms of $\mathcal{M}$ are the G^∞ functions. A diffeological space is a set S together with a contravariant subfunctor $F_S(N) \subseteq Hom_{set}(N,S)$ such that constant maps are in $F(N)$ for each object N of $\mathcal{M}$ and each $x \in S$ and such that F restricted to the subcategory of open subsets of a fixed super manifold, N, whose morphisms are the canonical injection of open subsets of N into each other satisfies the axioms of a set valued sheaf.

When S is a G^∞ super manifold we shall suppose without explicit mention to the contrary that it has its underlying diffeology given by $F(C)$ being defined to be the set of G^∞ maps with domain N and values in S.

Given a super manifold M and a diffeological space (S,F), a mapping, f, from a subset $C \subseteq M$ to S is called smooth, when there exists an open neighborhood, U, of C and a smooth extension of f to U, $\tilde{f} : U \to S$

Given any collection of diffeological structures on a set S, F_i, we have that $\cap F_i$ is a diffeological structure, thus any assignment of functions, $G_S(U) \subset Hom_{set}(U,S), U \subset E$, E a Hausdorff locally convex topological vector space, generates a diffeology; namely, the smallest or finest diffeology containing the $G_S(U)$. For the diffeology so generated we shall call $G_S(U)$ a system of generators.

A useful notion for diffeological structures is that of the pull-back; given a diffeological stucture on a set T, (T,G), and a function, $f : S \to T$, define $f^\star G(U) = \{g \in Hom_{sets}(U,S) : f \circ g \in G(U)\}$. It is straightforward to verify that $f^\star G$ is a diffeological stucture on S.

Given a subset $S_1 \subset S_2$, where (S_2, F_2) is a diffeological space, there is a diffeological structure induced on S_1 by $F_1(C) = \{f \in F_2(C) : f(C) \subset S_1\}$. Note that $F_1 = i^\star(F_2)$, where $i : S_1 \to S_2$ is the canonical inclusion. In the rest of this paper when we consider a subset of a diffeological space as a diffeological space it will be with the above described structure unless there is an explicit mention to the contrary. When C is an open subset of a graded hlctvs we shall call f of $F_S(C)$ a plot of the diffeological structure at a point $s = f(x)$, $x \in C$.

Definition 1.1. 1 *We shall call a diffeological structure* lattice *or L type when given two plots $f : M_1 \to S, g : M_2 \to S$ at a point $f(x) = g(y)$ there always exists a third plot through which the germs of f at x and g at y factor; that is, there exists a plot $h : N \to S$ such that $f = h \circ \phi, g = h \circ \gamma$, where $\phi : \tilde{M}_1 \to N, \gamma : \tilde{M}_2 \to N$ are smooth functions such that $\phi(x) = \gamma(y)$, where $\tilde{M}_i \subset M_i$ is a neighborhood of x (resp. y).*

Given diffeological spaces (X_1, F_1) and (X_2, F_2) a function $f : X_1 \to X_2$ is called smooth or C^∞ for each $g \in F_1$, we have $f \circ g \in F_2$. We shall say that f is locally smooth at $x_0 \in X_1$ when given any smooth map, $g \in F_1$, from a neighborhood, U, of $0 \in E$,where $f(0) = x_0$ there exists a neighborhood, $U_0 \subset U$ of 0 such that $f \circ (g|U_0) \in F_2$.

Given diffeological spaces (X_1, F_1) and (X_2, F_2), and an open subset U of an graded chlctvs E a mapping $f: U \to C^\infty(X_1, X_2)$ is *smooth* when $F: U \times X_1 \to X_2$, given by $F(u, x) = f(u)(x)$ is smooth. The finest diffeology on $C^\infty(X_1, X_2)$ admitting these functions as generating plots will be called the function space diffeology on $C^\infty(X_1, X_2)$.

From the definitions, using the notion of the pull-back, it follows that

Lemma 1.2. *Given diffeological spaces (X_1, F_1) and (X_2, F_2) and a set of generators $G_S(U)$ of F_1, if the function $f : X_1 \to X_2$ satisfies $f(G_S(U)) \subset F_2(U)$, then f is smooth.*

Suppose that (S, F) is a diffeological space of $L - type$; consider the equivalence relation generated between germs of one dimensional plots at $s \in S$ C_1 and C_2 are open intervals containing 0 as follows: let f and g be one dimensional plots at 0 with domains open intervals C_1 and C_2 containing 0, we write $f_0 \equiv g_0$, when $f(0) = g(0) = s$ and there exists a plot k:U$\to S$ through which the germs of f and g factor at 0 and we have that $D_0(h \circ f) = D_0(h \circ g)$, where h is any smooth real valued function defined on the image of k. The equivalence classes will be called tangent vectors at s, we shall designate the set of tangents at s by T_sS. It is immediate that a smooth map $f : S \to W$ defines a function $Tf : T_sS \to T_{f(s)}W$. When M is a manifold modelled on a locally convex topological vector space, this definition is equivalent to the classical one. Given a plot $f : U \to S$, we use $Tf : TU \to TS$ to define plots on TS and thus a diffeology on TS, in what follows TS will be considered as a diffeological space with the finest diffeology admitting such Tf's as plots.

Given diffeological spaces (X_1, F_1) and (X_2, F_2) the Cartesian product diffeology is defined by $f \in F_1 \times F_2$ if and only if $pr_1 \circ f \in F_1$ and $pr_2 \circ f \in F_2$. A group G with a diffeology $\mathcal{F}$ on its underlying set will be called a diffeological group when multiplication and inversion define smooth maps $G \times G \to G$ and $G \to G$; in a similar vein we define the notions of diffeological vector space and diffeological Lie algebra.

Definition 1.3. *A diffeological vector space E will be called integral when there exists a smooth linear map $\int : C^\infty(I, E) \to E$, such that given any smooth real valued linear function $H : E \to R$, we have $H(\int(f)) = \int H(f(t))dt$ and such that given any $v \in E$ we have $\int(f(t)v) = (\int f(t)dt)v$.*

A subspace, K, of an integral diffeological vector space, E, will be called *closed* when $\int(C^\infty(I, K)) \subseteq K$. One readily verifies that each diffeological group is of $L - type$.

Given a plot at the identity, $e \in G, f : U \to G, x \in U$, $U \subset E$, E a locally convex topological vector space, U open, define $Df(x; \alpha) = [f(x + t\alpha]_{t=0}, x \in U$. The definitions imply

Lemma 1.4. *Let (X, G) and (Y, K) be diffeological spaces and suppose that $f : X \to Y$ is a smooth map. Then $Tf : TX \to TY$ is a smooth map.*

Lemma 1.5. *The canonical map $\Pi : TG \to G$ is a smooth map.*

Proof. Let $f : W \to TG$ be a plot, by definition there exists a neighborhood, U, at each point $x \in W$ and an open subset V of a hlctvsp so that $f|U$ factors smoothly through TV; that is, $f = Th \circ \phi, h : V \to G$ and $\phi : W \to TV$ are plots. Now, $\pi_G \circ Th \circ \phi = h \circ \pi_V \circ \phi$, which is clearly smooth being a composition of smooth maps. ∎

Proposition 1.6. *If G is a diffeological group, then T_eG is a diffeological vector space.*

Proof. Given two vectors $\alpha = [[f]_0], \beta = [[g]_0]$ at $e \in G$, define $\alpha + \beta = [[f \times g]_0]$, and scalar multiplication by $r \times \alpha = [[f(rt)]_0]$. It is straigtforward to verify that T_eG becomes a well-defined vector space. Lemma 1.2 implies that vector addition in T_eG is smooth. Lemma 1.4 below implies that scalar multiplication is smooth. ∎

Lemma 1.7. *Let U, X and Y be Ltype diffeological spaces. If $f : U \times X \to Y$ is a smooth map, then the partial in U (resp. X), $T_1f : TU \times X \to TY$ (resp. $T_2f : U \times TX \to TY$) is smooth.*

Proof. It suffices to observe that the canonical map $U \times TX \to T(U \times X)$ sending (u, α) to $(0_u, \alpha)$ (respectively, $TU \times X \to T(U \times X)$ sending (α, x) to $(\alpha, 0_x)$ is smooth. ∎

Corollary 1.8. *Given a diffeological group G and a plot $f : U \to G$, set $F(x; \alpha) = R_{f(x)^{-1}} \circ Df(x; \alpha)$, where $U \subset E$ is an open subset of a graded chlctvs and R_x is induced by right multiplication by x. Then $F : U \times E \to T_eG$ is smooth.*

We use the diffeomorphism $TR_x : TG \to TG$ induced by right multiplication by x to transport the vector space structure of T_eX to T_xG, with these definitions $Df(x; \alpha) : E \to T_{f(x)}G$ is linear in α. Note that given any smooth map $f : G \to H$ we have that $Tf : T_xG \to T_{f(x)}H$ is linear, when G and H are diffeological groups.

Proposition 1.9. *Let G be a diffeological group, then TG is a diffeological group, where the group operation is given by $TP \circ \sigma^{-1}$, where $\sigma : T(G \times G) \to TG \times TG$ is the canonical map, and $P : G \times G \to G$ is the diffeological group operation.*

Proof. As in the classical case the product map in TG can be expressed as

$$\alpha \star \beta = R_{\pi(\alpha)\pi(\beta)}(R_{\pi(\alpha)^{-1}}(\alpha)) + \mathrm{Ad}_{\pi(\alpha)}(R_{\pi(\beta)^{-1}}(\beta)),$$

where $\mathrm{Ad}_x : TG \to TG$ is the bundle map over the identity induced by the adjoint action on G by x, and $R_x : TG \to TG$ is the map induced by right multiplication by x. From this formula it follows immediately that "$\star$" defines a smooth map $TG \times TG \to TG$.

Again as in the classical case the inverse is given by

$$\alpha \to -L_{\pi(\alpha)^{-1}}(R_{\pi(\alpha)^{-1}}(\alpha)).$$

The above lemmas now imply that indeed TG is a diffeological group. ∎

Remark 1.10. Let G be a diffeological group and N a normal subgroup of G, then one has that G/N is a diffeological group.

Before proceeding we shall adapt what we need of Iglesias's treatment [7] of diffeological fiber bundles to our context.

Given diffeological spaces (X, G) and (Y, K) and a smooth map $f : X \to Y$ consider the diffeological groupoid of automorphisms of f, $\mathcal{G}_f$, it has as objects the elements of Y, its morphisms, $\mathrm{Morph}_f(y, y')$, are the diffeomorphisms from $f^{-1}(y)$ to $f^{-1}(y')$. Let $s : \mathrm{Morph}_f \to Y$ (resp. $t : \mathrm{Morph}_f \to Y$ be the source map (resp. target map); that is, given $h \in \mathrm{Morph}_f(y, y')$, $s(h) = y$ (resp. $t(h) = y'$). We now define a diffeology on $\mathcal{G}_f$ as follows: given an open subset U of a graded hlctps E, we say that $g : U \to \mathrm{Morph}_f$ is a plot if and only if we have that

i) $s \times t \circ g$ is smooth,

ii) with $X_s :\equiv \{(x, y) : x \in U, y \in y_x\} \subset U \times X$, the mapping $\rho : X_s \to X$, given by $\rho(x, y) = g(x)(y)$, is smooth, and

iii) setting $X_t :\equiv \{(x, y) : x \in U, y \in y'_x)\} \subset U \times X$, we have that the mapping $\mu : X_t \to X$ given by $\mu(x, y) = g(x)^{-1}(y)$ is smooth.

We take on $\mathcal{G}_f$ the finest diffeology which induces the given diffeology on Y and admits the above maps to the morphisms of $\mathcal{G}_f$ as smooth maps. This diffeology on $\mathcal{G}_f$ will be called the *standard diffeology*.

It follows in a straightforward manner from the definitions that

Proposition 1.11. *The standard diffeology endows $\mathcal{G}_f$ with the structure of a diffeological groupoid; that is, setting $S_{def} :\equiv \{(x, y) \in \mathrm{Morph} \times \mathrm{Morph} : t(x) = s(y)\} \subset \mathrm{Morph} \times \mathrm{Morph}, \sigma(x, y) :\equiv y \circ x$ defines a smooth map $\sigma : S_{def} \to \mathrm{Morph}$, the inverse operation in the groupoid defines a smooth map $\mathrm{Morph} \to \mathrm{Morph}$, and $\iota(m) = i_m : f^{-1}(m) \to f^{-1}(m)$ defines a diffeomorphism from the objects of $\mathcal{G}_f$ to the identities of $\mathcal{G}_f$.*

Theorem 1.12. *Let G be a diffeological group and let $\Pi : TG \to G$ be the canonical map, then $s \times t : \mathrm{Morph}_\Pi \to G \times G$ satisfies the local lifting property; that is, given any open subset of a graded hlctvs $U \subset E, x \in U$ and smooth map $f : U \to G \times G$, there exists an open neighborhood $U_0 \subset U$ of x and smooth map $F : U_0 \to \mathrm{Morph}_\Pi$ such that $s \times t \circ F = f$.*

Proof. Let $f : U \to G \times G$ be a plot, such that $f(u) = (f_1(u), f_2(u)) \in G \times G$, set $F(u) = T_{f_1(u)} L_{f_2(u) f_1(u)^{-1}} : T_{f_1(u)} G \to T_{f_2(u)} G$. $F : U \to \mathrm{Morph}_{\mathcal{G}_\diamond}$, where $\mathcal{G}_\diamond$ is the groupoid of automorphisms of the canonical map $\Pi : TG \to G$. It follows from the definitions and Lemma 1.4 that F is smooth, by definition $t \times s \circ F = f$. ■

Definition 1.13. *A diffeological group G will be called a diffeological Lie group when the tangent space at the identity, T_eG, is a diffeological vector space which*

i) admits for every non zero $\alpha \in T_eG$ a smooth real valued linear map $T : T_eG \to R$ such that $T(\alpha) \neq 0$.

ii) the linear plots of T_eG are cofinal in the sense that every plot of T_eG factors smoothly through a smooth linear map of a complete Hausdorff locally convex topological vector space into T_eG.

Theorem 1.14. *Let G be a diffeological Lie group, then T_eG admits the structure of a Lie algebra such that the bracket operation defines a smooth linear map $\bigtriangledown_X : T_eG \to T_eG$, where $\bigtriangledown_X(Y) = [X, Y]$.*

Proof. Proposition 1.6 shows that T_eG is a vector space. We shall now verify that T_eG admits the structure of a Lie algebra. Given two vectors $\alpha, \beta \in T_eG$,let a_1, a_2, b be paths on open intervals containing 0, such that $[a_1]_0 = [a_2]_0 = \alpha$, $[b]_0 = \beta$. Now $Y_1(t) = a_1(t)\beta a(t)_1^{-1}$ is a smooth path which therefore factors through a linear plot, $H_1 : E_1 \to T_eG$ at β; that is, $Y_1(t) = H_1 \circ y_1(t)$, where $y_1 : (-\epsilon, \epsilon) \to E, \epsilon > 0$. Suppose that $\phi : U \to G$ is a plot at e through which a_1, a_2, and b factor with $D_{t=0}(k \circ a_1) = D_{t=0}(k \circ a_2)$, where k is any smooth real valued function defined on the image of ϕ. Since T_eG is of lattice type we may also suppose without loss of generality that H_1 factors through Tϕ; in paricular, $Y_1(t)$ and $a_2(t)\beta a_2(t)^{-1}$ factor through $T_0\phi$. Define $[\alpha, \beta] = H(D_{t=0} y_1(t))$.

To verify that $[\alpha, \beta]$ is well-defined suppose $Y_2(t) = a_2(t)\beta a_2^{-1}(t) = H_2 \circ y_2(t)$, where $H_2 : E_2 \to T_eG$ is a linear plot; let $k : T_eG \to R$ be a smooth linear real valued function. Considering the smooth real valued function on $\phi(U)$ given by $W_k(z) = k(z\beta z^{-1})$ we now have that $(k \circ H_2)(D_{t=0}(y_2(t))) = D_{t=0}(k \circ H_2(y_2(t))) = D_{t=0}(W_k(a_2(t))) = D_{t=0}(W_k(a_1(t))) = D_{t=0}(k \circ H_1(y_1(t))) = (k \circ H_1)(D_{t=0}(y_1(t)))$; since k is an arbitrary smooth linear functional we have that $[\alpha, \beta]$ is well-defined.

As in the classical case we prove that $[\alpha, \beta]$ is linear in β, $[\alpha, \beta]$ is anti-commutative, and satisfies the Jacobi identity; for example, for anticommutativity we choose an arbitrary plot at e through which the germs of a, b, and $f(s,t) = a(s)b(t)a(s)^{-1}$ at 0 factor, $K : W \to G$, let $\lambda : K(W) \to R$ at e be a smooth real valued function on that plot, and we consider $d\lambda([\alpha, \beta]) = D_{t=0}D_{s=0}(\lambda Y_2(\tilde{f}(s,t))) = d\lambda(Y_2(D_{s=0}D_{t=0}\tilde{f}(s,t)))$. It is straightforward to verify that $d\lambda([\alpha, \beta])$ satisfies anti-commutativity for each λ. The Jacobi identity is proved in a similar manner. Given a smooth function $\beta : U \to T_eG$ we consider the smooth function $g(t,u) = a(t)\beta(u)a(t)^{-1}$, which we now factor linearly and smoothly through a graded hclctvs E, H:E$\to$ T_eG so that $g(t,u) = H \circ k(t,u)$. The smoothness of $D_t k$ shows that $\bigtriangledown_X$ is smooth. ■

Let E be an Hausdorff, sequentially complete, locally convex topological vector space, we suppose E furnished with the canonical diffeology, and let $\mathrm{Aut}(E)$ be the group of linear diffeomorphisms of E. Given an arbitrary locally convex topological vector space V and an open subset $U \subset V$, a mapping $f : U \to \mathrm{Aut}(E)$ is a plot if and only if

i) $F : U \times E \to E$ defined by $F(x, y) = f(x)(y)$ is smooth and

ii) $G : U \times E \to E$ defined by $G(x, y) = f(x)^{-1}(y)$ is smooth.

The finest diffeology on $\mathrm{Aut}(E)$ admitting the above plots of $\mathrm{Aut}(E)$ will be called the normal diffeology on Aut(E).

Theorem 1.15. $\mathrm{Aut}(E)$ *with the normal diffeology is a diffeological group.*

Proof. To show that multplication is a smooth map from $\mathrm{Aut}(E) \times \mathrm{Aut}(E)$ to $\mathrm{Aut}(E)$ it suffices to consider smooth maps $f, g : U \to \mathrm{Aut}(E)$ and apply the chain rule to the composition $g\dot{f}(x, e) = g(x)(f(x)(e))$. As the composition of plots is smooth and the inversion of plots is by definition smooth it follows that $\mathrm{Aut}(E)$ under composition with the normal diffeology is a diffeological group. Designate by $L(E, E)$ the diffeological vector space of all smooth linear endomorphisms of E with $f : U \to E$ a plot if and only if $F : U \times E \to E$ given by $F(u, e) = f(u)(e)$ is smooth. It is straightforward to verify that $L(E, E)$ is indeed a diffeological vector space such that the canonical injection $\mathrm{Aut}(E) \to L(E)$ is a smooth map. ■

A diffeological Lie group G will be called *regular* when the logarithmic derivative $f'(t)\dot{f}(t)^{-1}$ defines a diffeological isomorphism from the diffeological set of smooth mappings from the unit interval into G which map zero to the identity, $e \in G$, $C_0^\infty(I, G)$, to the diffeological set of smooth mapppings from the unit interval into T_eG, $\chi : C_0^\infty(I, G) \to C^\infty(I, T_eG)$. We recall that a map from the unit interval into a diffeological space is called smooth when there exists an open interval containing the unit interval to which the map can be smoothly extended.

Proposition 1.16. *Suppose that G is a regular diffeological Lie group and suppose that $g \in G$, then $\chi_g : C_0^\infty(I, (G, g)) \to C^\infty(I, T_eG)$ is a diffeomorphism, where $C_0^\infty(I, (G, g)) = \{f \in C^\infty(I, G) : f(0) = g\}$ and $\chi_g = f'(t)\dot{f}(t)^{-1}$.*

Proof. $\chi_g = \chi \circ R_{g^{-1}}$. ■

Proposition 1.16 implies

Theorem 1.17. *Let G be a regular diffeological Lie group, then there exists a smooth function $exp : T_eG \to G$ such that $exp((t + s)) = exp(t\xi) \times exp(s\xi)$ and $[exp(t\xi)]_t = R_{exp(t\xi)}(\xi)$.*

Corollary 1.18. *If G is a regular diffeological Lie group, then T_eG admits the structure of a diffeological Lie algebra.*

Definition 1.19. *Let $\mathcal{L}$ be an integral diffeological Lie algebra. A closed ideal, $\mathcal{I}$, will be called pre-integrable when for each $a(t) \in C^\infty(I,\mathcal{L})$ the differential equation, $(\star) y' = [a(t), y]$, admits a smooth flow, $\Phi(a(t), s, l)$ such that $D_s\Phi(a(t), s, l) = [a(s), \Phi(a(t), s, l)]$, $\Phi(a(t), 0, l) = l$ and such that Φ defines by an abuse of notation a smooth map*

i) $\Phi : C^\infty(I, \mathcal{I}) \times \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, *such that*

ii) $\Phi(a(t), s, \cdot) : \mathcal{I} \to \mathcal{I}$ *is a diffeomorphism which induces a diffeomorphism $\Xi : C^\infty(I, \mathcal{I}) \to \mathcal{C}^\infty(\mathcal{I}, \mathcal{I})$, where $\Xi(f)(t) = \Phi(a(t), t, f(t))$ and where $C^\infty(I, \mathcal{I})$ has the functional diffeology.*

Theorem 1.20. *Let G be a regular Lie group and $\mathcal{H}$ a pre-integrable diffeological Lie subalgebra of T_eG with an ideal, $\mathcal{K}$, as a diffeological vector space complement; that is, a mapping, $f : U \to T_eG = H \times K$ is smooth if and only if $\pi_H \circ f$ and $\pi_K \circ f$ are smooth. Then there exists a regular diffeological Lie subgroup, H, of G such that the canonical injection, $i : H \to G$ induces an isomorphism of diffeological Lie algebras, $T(i) : T_eH \to \mathcal{H} \subset T_eG$.*

Proof. Define $H :\equiv \{g \in G : \exists$ a smooth path $g : [0,1] \to G$ such that $g(0) = e, g(1) = g$, and $D_t g(t)\dot{g}(t)^{-1} \in \mathcal{H}$ for all $t \in [0,1]\}$. Translating the product in $C_0^\infty(I, G)$ to $C^\infty(I, T_eG)$ by means of χ allows one to establish that H is indeed a subgroup of G. The regularity of H is an immediate consequence of the regularity of G. ∎

Theorem 1.21. *Let G be a simply connected, regular, diffeological Lie group with canonical diffeomorphism $\chi : C_0^\infty(I, G) \to C^\infty(I, T_eG)$ and suppose that H is a connected normal subgroup of G such that there exists a pre-integrable Lie ideal, $\mathcal{H} \subseteq T_eH \subseteq T_eG = \mathcal{G}$, of the Lie algebra, $\mathcal{G}$, of G with a diffeological vector space complement, $\mathcal{K}$, satisfying*

i) $\chi^{-1}(C^\infty(I, \mathcal{H})) \subseteq C_0^\infty(I, H)$;

ii) *given $h \in H$ there exists a smooth path $f : [0,1] \to H$ such that $f(0) = e$, $f(1) = h$, and $\chi(f) \in C^\infty(I, \mathcal{H})$;*

iii) *given any $k \in \mathcal{G}/\mathcal{H}$, there exists a smooth Lie algebra homomorpism ϕ, of $\mathcal{G}/\mathcal{H}$ into a the Lie algebra $\mathcal{S}$ of a regular diffeological Lie group S such that $\phi(k) \neq 0$.*

Then G/H is a diffeological Lie group with Lie algebra $\mathcal{G}/\mathcal{H}$.

Proof. By definition of the diffeology on G/H the canoniacal map $\pi : T_eG \to T_eG/H$ is onto; iii) implies that the canonical map restricted to $\mathcal{K}$ determines a smooth one-one linear isomorphism onto $\mathcal{G}/\mathcal{H}$. From the definition of the diffeology on $\mathcal{G}/\mathcal{H}$, we have that $f : U \to \mathcal{G}/\mathcal{H}$ is smooth if and only if there exists a smooth $F : U_x \to \mathcal{G}$ where x is an arbitrary element of U and U_x is an open neighborhood of x in U such that $\pi \circ F = f|U_x$. Now $\pi_{\mathcal{K}} \circ F$ is a smooth mapping which depends only on f. The preceeding suffices to show that the canonical map restricted to $\mathcal{K}$ is a linear diffeomorphism onto $\mathcal{G}/\mathcal{H}$. ∎

Lemma 1.22. *Under the hypotheses of* Theorem 1.21, *given* $u(t), v(t) \in C_0^\infty(I, G)$ *suppose that* $\chi(u) - \chi(v) \in \mathcal{H}$. *Then, there exists* $w(t) \in C_0^\infty(I, H)$ *such that* $v(t) = u(t)\dot{w}(t)$.

Proof. We have as in the classical case that

$$\chi(u\dot{v})(s) = \chi(u)(s) + \Phi(\chi(v), s, \chi(u)(s)).$$

For a verification of this equality that is applicable in this case see [13]. The preceding equality and the fact that $\mathcal{H}$ is a pre-integrable ideal, which implies that $\Phi(\chi(v), s, \cdot)$ restricted to $C^\infty(I, \mathcal{H})$ is a diffeomorphism onto $C^\infty(I, \mathcal{H})$ which imply $u(t)^{-1}\dot{v}(t) \in C^\infty(I, H)$. ∎

Lemma 1.22 implies that

Theorem 1.23. *Under the hypotheses of* Theorem 1.21, *H and G/H are regular diffeological Lie groups.*

2. On the Integrability of some Generalized Kac-Moody Lie Algebras

Inspired by Borcherds we define a generalized Kac Moody Lie algebra by the given:

1) A real Hilbert space $\{H, \langle\cdot,\cdot\rangle\}$ together with a continuous nondegenerate symmetric bilinear form $(\cdot,\cdot)$.

2) Suppose given a countable set of elements $h_i \in \mathrm{H}$ such that $(h_i, h_j) \leq 0$ if $i \neq j$ and such that if (h_i, h_i) is positive, then $2(h_i, h_j)/(h_i, h_i)$ is an integer; $A = ((h_i, h_j))$ is called a generalized symmetrized Cartan matrix of real numbers.

The generalized Kac-Moody algebra $\mathcal{G} = \{\mathcal{G}(\mathrm{A}), \mathrm{H}\}$ is the Lie algebra generated by the vector space $\mathcal{H}$ and symbols e_i and f_i, with defining relations:

$$[\mathcal{H}, \mathcal{H}] = 0; \qquad [e_i, f_j] = \delta_{ij} h_i; \qquad [h, e_i] = (h_i, h)e_i, \qquad [h, f_i] = -(h_i, h)f_i.$$

If $a_{ii} > 0$, then $\mathrm{ad}(e_i)^{1-2a_{ij}/a_{ii}}(e_j) = 0 = \mathrm{ad}(f_i)^{1-2a_{ij}/a_{ii}}(f_j)$.

We have the root space decomposition of $\mathcal{G}(\mathrm{A}) = \sum_{\alpha \in \mathcal{H}^\star} \mathcal{G}_\alpha$, where $\mathcal{G}_\alpha = \{x \in \mathcal{G}(\mathrm{A}) | [\mathrm{h}, \mathrm{x}] = \alpha(\mathrm{h})\mathrm{x} \text{ for all } h \in \mathcal{H}\}$. Let's recall that the height of a root is defined by $ht(\sum_{i=1}^n z_i\alpha_i) = \sum_{i=1}^n |z_i|$, where $\alpha_i(h) = (h_i, h)$.

The non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on H uniquely extends to a non-degenerate invariant symmetric bilinear form on $\mathcal{G}(\mathcal{A})$ satisfying $(e_i, f_j) = \delta_{ij}$ called the the *canonical* form.. Now define $\omega : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ by $\omega(e_i) = -f_i, \omega(f_i) = -e_i, \omega(h) = -h, h \in \mathcal{H}$ and set $(x, y)_0 = -(x, \omega(y))$.

We now define on $\mathcal{G}(\mathrm{A})$ the positive definite inner product

$$(\alpha_+ + h + \alpha_-, \beta_+ + k + \beta_-)_1 = (\alpha+, \beta_+)_0 + < h, k > + (\alpha_-, \beta_-)_0.$$

We shall suppose that $\sum_i |a_{ij}|^2 < E < \infty, \forall j$. We have [5] $\exists C > 0$ so that

$$\|[z, z']\|_1 \leq C(\|d(z)\|_1 \|z'\|_1 + \|z\|_1 \|d(z')\|_1) \qquad (\star)$$

where d is a derivation of $\mathcal{G}(\mathrm{A})$ satifying $d(x) = ht(\alpha)x$ for $x \in \mathcal{G}_\alpha(\mathrm{A})$.

Now consider the subspace $\bar{\mathcal{G}}(\mathrm{A}) \subset \prod \mathcal{G}_\alpha(\mathrm{A})$, where α ranges over all the roots of $\mathcal{G}(\mathrm{A})$, characterized by

$$\{M_\alpha\} \in \bar{\mathcal{G}(\mathrm{A})}$$

if and only if given any $t \geq 1$ there exists a constant K_t so that

$$\sum_{\alpha, ht(\alpha)=n} \|M_\alpha\| \leq K_t e^{-t|n|}.$$

On the vector space $\bar{\mathcal{G}}(\mathrm{A})$, consider the Hausdorff locally convex topology generated by the following fundamental system of neighborhoods of $0 \in \bar{\mathcal{G}}(\mathrm{A})$: to each pair of positive real numbers (t, k), let

$$U(t,k) = \left\{ \{M_\alpha\} \in \bar{\mathcal{G}}(\mathrm{A}) : \sum_{\alpha, \mathrm{ht}(\alpha)=\mathrm{n}} \|\mathrm{M}_\alpha\| < \mathrm{k}\mathrm{e}^{-\mathrm{t}|\mathrm{n}|} \right\}$$

With this topology $\bar{\mathcal{G}}(\mathrm{A})$ is a Frechet space. In this topology, the bounded sets are characterized by

Lemma 2.1. *$B \subset \bar{\mathcal{G}}(\mathrm{A})$ is bounded if and only if for any positive real number t there exists a constant B_t such that $\{M_\alpha\} \in B$ implies $\| \sum_{\alpha, ht(\alpha)=n} M_\alpha \| \leq B_t e^{-tn}$.*

It is not difficult to verify that all bounded sets are relatively compact.

Proposition 2.2. *Given $\{g_\alpha\}, \{h_\alpha\} \in \bar{\mathcal{G}}(\mathrm{A})$ then $[\{g_\alpha\}, \{h_\alpha\}] = \{k_\alpha\} \in \bar{\mathcal{G}}(\mathrm{A})$; further, $[\{g_\alpha\}, \{h_\alpha\}]$ determines a topological Lie algebra structure on $\bar{\mathcal{G}}(\mathrm{A})$.*

Proof. Given $\{g_\alpha\}, \{h_\alpha\} \in \bar{\mathcal{G}(\mathrm{A})}$, let G^t (resp. H^t) be so that

$$\sum_{\alpha, ht(\alpha)=n} \|g_\alpha\| \leq G_t e^{-t|n|}, \qquad \sum_{\alpha, ht(\alpha)=n} \|h_\alpha\| \leq H_t e^{-t|n|}.$$

Setting $g_n = \sum_{ht(\alpha)=n} g_\alpha$ (resp. $h_n = \sum_{ht(\alpha)=n} h_\alpha$),

$$\begin{aligned}
\sum_{n \in Z} \|[g_{n+e}, h_{-n}]\| &\leq C\{\sum_{n \in Z} (|n+e| + |n|) G_t H_t e^{-t(|n+e|+|n|)} \\
&\leq CG_t H_t \sum_{n \in Z} (|n+e| + |n|) e^{-(t-s)(|n+e|+|n|)} \cdot e^{-s|e|} \\
&\leq CG_t H_t \sum_{n \in Z} 2|n| e^{-(t-s)|n|} \cdot e^{-s|e|},
\end{aligned}$$

where C is the constant of $(\star)$ above and $s < t$. Setting $K_{t,s} = \sum_{n \in Z} 2|n| e^{-(t-s)(|n|)}$, the inequality

$$\sum_{n \in Z} \|[g_{n+e}, h_{-n}]\| \leq CG_t H_t K_{t,s} e^{-s|e|} \tag{$\star\star$}$$

implies that the above bracket gives to $\bar{\mathcal{G}}(\mathrm{A})$ the structure of a topological algebra which contains the Kac-Moody Lie algebra $\mathcal{G}(\mathrm{A})$ as a dense subalgebra, thus $\bar{\mathcal{G}}(\mathrm{A})$ is a topological Lie algebra. ∎

We recall that a Hausdorff, sequentialy complete, locally convex topological vector space V is called strongly bornological (resp. bornological) when any subset (resp. convex subset) absorbing all the bounded subsets of V is a neighborhood of the origin. In general in this paper the topological vector spaces with which we deal will be strongly bornological. Note that metrisable locally convex topological vector spaces are strongly bornological as are countable inductive limits of strongly bornological spaces.

In what follows we shall make use of a sequence of functions defined iteratively as follows:

$$\begin{aligned} F_0(t,q) &= \sum_{n\in Z}(|n+q|+|n|)e^{-t(|n+q|+|n|)}, \\ F_{k+1}(t,q) &= \sum_n(|n+q|+|n|)F_k(t,n+q)e^{-t|n|}, \qquad t>0, q\in Z. \end{aligned}$$

One readily verifies the following lemmas:

Lemma 2.3. *$F_0(t,q)$ is bounded by some constant D for $1\le t_0<s<t, q\in Z$ and satisfies $F_0(t,q)\le F_0(t-s,q)\cdot e^{-s|q|}$.*

Corollary 2.4. $F_k(t,q)\le F_k(t-s,q)\cdot e^{-s|q|}$.

Lemma 2.5. *There exists a constant $\kappa\ge 1$ such that*

$$\sum_{n\in Z}(|n+q|+|n|)e^{-t(|n|)}<\kappa|q| \text{ for all } t>t_0\ge 1,$$

where κ is independent of t.

Corollary 2.6. $F_n(t,q)\le D(\kappa|q|)^n$.

Lemma 2.7. *Define $\phi(t,q)=\sum_{n\ge 0}\frac{F_n(t,q)}{n!}$, then $|\phi(t,q)|\le De^{\kappa|q|}$.*

Lemma 2.8. *Given $\xi=\{\xi_\alpha\}$, $\eta=\{\eta_\alpha\}\in\mathcal{G}(\bar{\mathcal{A}})$ such that $\|\xi_n\|\le X_te^{-tn}$ and $\|\eta_n\|\le Y_te^{-t|n|}$, where $\xi_n=\sum_{\alpha,ht(\alpha)=n}\xi_\alpha,\eta_n=\sum_{\alpha,ht(\alpha)=n}\eta_\alpha$. Then $\|[\xi,\eta]\|\le CX_tY_tF_0(t,q)$, where C is the constant of $(\star)$.*

Lemma 2.9. *Given the hypotheses of* Lemma 2.8*, then*

$$\|[[\xi,\eta_1],\eta_2]\|\le C^2X_tY_t^2F_1(t,q),$$

where η_1,η_2 satisfy the conditions of Lemma 2.8.

By iteration, we obtain

Lemma 2.10. *Given the hypotheses of* Lemma 2.8*, then $\|\mathrm{ad}_{\eta_k}\circ\ldots\circ\mathrm{ad}_{\eta_1}(\xi)\|\le C^kX_tY_t^kF_{k-1}(t,q)$, where $\eta_k,\ldots,\eta_1$, and ξ satisfy the conditions of* Lemma 2.8.

We have from [13]:

Theorem 2.11. *Let $\mathcal{A}$ be a graded Hausdorff complete locally convex topological Lie algebra with a strongly bornological underlying vector space topology such that given any bounded set $B \subset \mathcal{A}$ there exists a balanced convex bounded set C such that $\sum_{n=k}^{\infty}(1/n!)B^n \to 0$ as $k \to \infty$ in $\mathcal{A}_{\mathcal{C}} = \bigcup_{\setminus} \backslash \mathcal{C}$ with the $C-$gauge norm topology; that is, $|\alpha|_C = inf\{\lambda > 0 : \alpha \in \lambda C\}$, where $B^{n+1} = [B, B^n]$. Then the canonical diffeological structure on $\mathcal{A}$ is pre-integrable.*

Lemmas 2.3–2.10 imply

Theorem 2.12. *If $\mathcal{G}(\mathrm{A})$ is a generalized Kac-Moody Lie algebra, then $\bar{\mathcal{G}}(\mathrm{A})$ is an integral pre-integrable diffeological Lie algebra.*

Definition 2.13. *Given two smooth paths $f, g : I \to \mathcal{L}$ into a diffeological Lie algebra, $\mathcal{L}, \{,\} : \mathcal{I} \to \mathcal{L}$ we say then are Lie homotopic when there exists smooth maps from the square $V, W : I \times I \to \mathcal{L}$ such that $V(t, 0) = f$, $V(t, 1) = g, W(0, s) \equiv 0$, $W(1, s) \equiv 0$, $V_s - W_t = [V, W]$.*

Now let $E \subset H$, where H is a Hilbert space be an Hausdorff, sequentially complete, locally convex topological vector space with a topology finer than the induced topology from the Hilbert space, we suppose E furnished with the canonical diffeology from its locally convex topology, and let $\mathrm{End}(E)$ be the algebra of smooth linear endomorphisms of E. We suppose that $\mathrm{End}(E)$ has the diffeology induced from $F(E, E)$.

Proposition 2.14. $\mathrm{End}(E)$ *is a diffeological algebra.*

Proof. To show that multiplication is a smooth map from $\mathrm{End}(E) \times \mathrm{End}(E)$ to $\mathrm{End}(E)$ it suffices to consider smooth maps $f, g : U \to \mathrm{End}(E)$ and apply the chain rule to the composition $g\dot{f}(x, e) = g(x)(f(x)(e))$. As the composition of plots is smooth and the addition of plots is by definition smooth it follows that End(E) under composition with the normal diffeology is a diffeological algebra. ■

Lemma 2.15. *Given $a > 0$, let $p : (-a, a) \to \mathrm{End}(E)$ be a path through the identity, $p(0) = id \in \mathrm{End}(E)$; if $[p] = 0 \in T_x(\mathrm{End}(E))$, then $D_{t=0}p(t)(x) = 0$ for each $x \in$ E.*

Proof. Suppose $[p] = 0 \in T_e(\mathrm{End}(E))$, then for every smooth linear map functional $F : E \to R$, we have $0 = D_{t=0}(F \circ p) = F \circ D_{t=0}(p)$; therefore, $D_{t=0}p(t)(x) = 0$ for each $x \in E$, by the Hahn-Banach theorem. ■

Remark 2.16. Lemma 2.5 implies that there exists a canonical homomorphism

$$\kappa : T_x(\mathrm{End}(E)) \to \mathrm{End}(E)$$

Now we shall give conditions which imply that a representation of the Lie algebra, $\mathcal{L}$, of a simply connected regular diffeological Lie group L,

$$\mathcal{L} \to \mathrm{End}(E)$$

is the derivative of a multiplicative homomorphism of $L \to$ End(E), where $\mathrm{End}(E)$ has the canonical Lie algebra structure: $[A, B] = AB - BA$.

Notation: Given a diffeological vector space, E, and a smooth map, $f : I \to E$, where $I = [0, 1]$, designate by $\hat{f}$ the convex hull of $f(I) \subset E$.

Definition 2.17. *Let $E \subset H$, where H is a Hilbert space, be an Hausdorff, sequentially complete, bornological, locally convex topological vector space with a topology rendering continuous the canonical injection into H, and suppose that L is a regular diffeological Lie group with diffeological Lie algebra, $\mathcal{L}$. A smooth representation $F : \mathcal{L} \to \mathrm{End(E)}$, where $\mathrm{End(E)}$ has the diffeology induced by the function space diffeology, is called regular when given any smooth map, $f : I \to \mathcal{L}$ and any bounded subset, $B \subset E$, we have that $\sum_{n=k}^{\infty}(1/n!)L(\hat{f})^n(B) \to 0$ as $k \to \infty$ in $A_C = \bigcup_n nC$ with the C − gauge norm topology; that is, $|\alpha|_C = \inf\{\lambda > 0 : \alpha \in \lambda C\}$, where C is some bounded convex subset of E.*

We are able to define the fundamental group of a diffeological space by means of the connected components, given by smooth arcs, of the diffeological space $\Omega(X, x_0) \equiv \{f \in F(R, X) : f(t) = x_0, t < \epsilon, t > 1 - \epsilon, 0 < \epsilon < 1/2\}$. By iteration one defines the higher homotopy groups and can show that an exact sequence of homotopy groups for diffelogical fibrations exists. Iglesias [7] has shown that for a connected diffeological space a universal covering space exists; that is, he establishes a unique principal fibration with discrete fiber isomorphic to the fundamental group. By a straightforward generalisation of classical strategies one can show that a connected diffeological group has a unique universal covering diffeological group such that the covering map is a smooth homomorphism.

As an direct consequence of the theory of linear differential equations on bornological spaces [13] we have

Lemma 2.18. *Let L be a simply connected regular diffeological Lie group, with Lie algebra $\mathcal{L}$, $H : \mathcal{L} \to \mathrm{End}(\mathcal{E})$ a smooth regular representation of Lie algebras, and $f : I \to \mathcal{L}$ a smooth function, then there exists an unique smooth path $k : I \to \mathrm{End}(E)$ such that $k'(t) = H(f(t))(k(t))$, such that $k(0) = id \in \mathrm{End(E)}$.*

Theorem 2.19. *Let L be a regular simply connected diffeological Lie group, under the hypotheses of* Lemma 6, *there exists an unique multiplicative smooth homomorphism $\phi{:}L{\to}\mathrm{End}(E)$ such that $\kappa \circ T_e(\phi) = H$, where $\kappa : T_e(\mathrm{End}(E)) \to \mathrm{End}(E)$ is the homomorphism of the remark after* Lemma 8.

Proof. Given the identity $e \in L$ let $f : I \to L$ be a smooth path such that $f(0) = e \in L$ and $f(1) = \ell$ and designate by $F : I \to \mathcal{L}$ the logarithmic derivative of f. Set $\phi(\ell) = k(1)$, where $k : I \to \mathrm{End(E)}$ is the unique smooth path in $\mathrm{End(E)}$ satisfying

$$k'(t) = H(F(t))(k(t)) \qquad (\star\star\star)$$

such that $k(0) = id \in \mathrm{End(E)}$. The simple connectivity of L and the assumption that E has a topology finer than the induced topology from a Hilbert space implies that ϕ is unique. That ϕ is smooth follows from the existence and uniqueness theorem for differential equations with parameters on the class of bornological spaces in which bounded sets are compact (see [13]).

To prove that ϕ is multiplicative, suppose that $f, g : I \to L$ are smooth paths from the identity in L to g and h respectively, then the logarithmic derivative of $f \times g : I \to L$ is given $v + \phi_v(t, w(t))$ where v (resp. w) is the logarithmic derivative of f (resp. g) and ϕ_v is the unique flow of the differential equation

$D_t(\phi_v(t,w)) = [v, \phi_v(t,w)]$ satisfying the initial condition $\phi_v(0,w) = w$.To prove that ϕ is multiplicative, It suffices to show that when $k_i(t)$, $i = 1,2$ satisfy $k_1'(t) = H(v(t))(k_1(t))$ and $k_2'(t) = H(w(t))(k_2(t))$, then $k(t) = k_1(t) \times (k_2(t))$ satisfies $H(v + \phi_v(t, w(t)))(k(t)) = k'(t)$. From the theory of linear differential equations on bornological topological vector spaces it follows that (see [13]) solutions to $(\star)$ define unique invertible linear maps in $\mathrm{End}(E)$. It then becomes straightforward to show that $H(v + \phi_v(t, w(t)))(k(t)) = k'(t)$. ■

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Joshua Leslie,
Mathematics Department,
Howard University
Washington, DC
USA
jleslie@howard.edu

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