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PBW and Duality Theorems for Quantum Groups and Quantum Current Algebras

Benjamin Enriquez

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Abstract. We give proofs of the PBW and duality theorems for the quantum Kac-Moody algebras and quantum current algebras, relying on Lie bialgebra duality. We also show that the classical limit of the quantum current algebras associated with an untwisted affine Cartan matrix is the enveloping algebra of a quotient of the corresponding toroidal algebra; this quotient is trivial in all cases except the $A_1^{(1)}$ case.

Introduction

The purpose of "quantum algebra" is to study deformations of various commutative (or cocommutative) algebras (or Hopf algebras). They are usually presented by generators and relations. Then a natural problem arises: to show that the deformed algebras have the same size as their undeformed counterparts. The first instance of such a result is the Poincaré-Birkhoff-Witt (PBW) theorem, which says that if $\mathfrak{g}$ is a Lie algebra over a field of characteristic zero, then the symmetrization map $S^\bullet(\mathfrak{g}) \to U(\mathfrak{g})$ is a linear isomorphism. In that case, $U(\mathfrak{g})$ is viewed as a quantization of the Poisson algebra $S^\bullet(\mathfrak{g})$.

The problem of comparing the size of an algebra presented by generators and relations with that of the classical algebra is called with PBW problem. A useful tool for solving this problem is Bergman's combinatorial "diamond" lemma ([4]). It was used in the quantum situation by several authors (notably Berger in [3] and Rosso in [26]). Another approach to the PBW problem involves constructing "enough" representations of the deformed algebra. For instance, Lusztig's PBW result (Corollary 33.1.5 of [22]) relies on the study of integrable modules over quantized Kac-Moody algebras. A last, direct approach involves realizing the deformed algebra structure on some model space. The original PBW result can be proved in this way, by explicitly constructing a star-product on $S^\bullet(\mathfrak{g})$ ([5]). This is also the approach followed in the present paper in the case of quantum groups and quantum current algebras. In this situation, these algebras are realized as

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(functional) shuffle algebras. We also use in an essential way the Hopf algebra structures.

1. Outline of results

1.1. Quantum Kac-Moody algebras.

Let $A = (a_{ij})_{1\le i,j\le n}$ be a symmetrizable Cartan matrix. Let $(d_i)_{1\le i\le n}$ be the coprime positive integers such that the matrix $(d_i a_{ij})_{1\le i,j\le n}$ is symmetric. Let r be the rank of A; we assume that the matrix $(a_{ij})_{n-r+1\le i,j\le n}$ is nondegenerate.

Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated with A; let $\mathfrak{n}_+$ be its positive pro-nilpotent subalgebra and $(\bar{e}_i)_{1\le i\le n}$ be the generators of $\mathfrak{n}_+$ corresponding to the simple roots of $\mathfrak{g}$.

Let $\mathbb{C}[[\hbar]]$ be the formal series ring in $\hbar$. Let $U_\hbar\mathfrak{n}_+$ be the quotient of the free algebra with n generators $\mathbb{C}[[\hbar]]\langle e_i, i = 1,\ldots,n\rangle$ by the two-sided ideal generated by the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^k e_j e_i^{1-a_{ij}-k} = 0, \tag{1}$$

where $\begin{bmatrix} m \\ p \end{bmatrix}_q = \frac{[m]_q^!}{[p]_q^![m-p]_q^!}$, $[k]_q^! = [1]_q\cdots[k]_q$, $[k]_q = \frac{q^k - q^{-k}}{q-q^{-1}}$, and $q = e^\hbar$ ([9, 17]).

We will show:

Theorem 1.1. *$U_\hbar\mathfrak{n}_+$ is a free $\mathbb{C}[[\hbar]]$-module, and the map $e_i \mapsto \bar{e}_i$ defines an algebra isomorphism of $U_\hbar\mathfrak{n}_+/\hbar U_\hbar\mathfrak{n}_+$ with $U\mathfrak{n}_+$.*

This Theorem may be derived from the Poincaré-Birkhoff-Witt (PBW) results of Lusztig's book [22]; in the case $\mathfrak{g} = \mathfrak{sl}_n$, it can also be derived from those of Rosso ([26]), and in the cases when $\mathfrak{g}$ is semisimple or untwisted affine, from those of [21].

The proof presented here is based on the comparison of $U_\hbar\mathfrak{n}_+$ with a quantum shuffle algebra, Lie bialgebra duality and the Deodhar-Gabber-Kac theorem.

As a corollary of this proof, we show

Corollary 1.1. *The map $p_\hbar$ defined in Lemma 2.3 is an algebra isomorphism from $U_\hbar\mathfrak{n}_+$ to the subalgebra $\langle V\rangle$ of the shuffle algebra* $\mathrm{Sh}(V)$ *defined in sect. 2..*

This result was proved in [27]; it can also be derived from the results of [28]. Rosso's proof uses the nondegeneracy of the pairing between opposite Borel quantum algebras ([22], Corollary 33.1.5; see also Theorem 1.2). Schauenburg shows that $\langle V\rangle$ is isomorphic to the quotient of the free algebra generated by the e_i by the radical of a braided Hopf pairing. Together with [22], Corollary 33.1.5, this implies Corollary 1.1.

Define $U_\hbar\mathfrak{n}_-$ as the algebra with generators f_i, $i = 1,\ldots,n$, and the same defining relations as $U_\hbar\mathfrak{n}_+$ (with e_i replaced by f_i). Define a grading on $U_\hbar\mathfrak{n}_\pm$ by $(\pm\mathbb{N})^n$ by $\deg(e_i) = \epsilon_i$, $\deg(f_i) = -\epsilon_i$, where ϵ_i is the ith basis vector of $\mathbb{N}^n$, and define the braided tensor products $U_\hbar\mathfrak{n}_\pm\bar{\otimes}U_\hbar\mathfrak{n}_\pm$ as the algebras isomorphic to $U_\hbar\mathfrak{n}_\pm \otimes_{\mathbb{C}[[\hbar]]} U_\hbar\mathfrak{n}_\pm$ as $\mathbb{C}[[\hbar]]$-modules, with mutiplication rule

$$(x\otimes y)(x'\otimes y') = q^{-\langle \deg(x'),\deg(y)\rangle}(xx'\otimes yy'); \tag{2}$$

we set $\langle \epsilon_i, \epsilon_j \rangle = d_i a_{ij}$.

Then $U_\hbar \mathfrak{n}_\pm$ are endowed with braided Hopf algebra structures, defined by $\Delta_+(e_i) = e_i \otimes 1 + 1 \otimes e_i$, and $\Delta_-(f_i) = f_i \otimes 1 + 1 \otimes f_i$.

In [8], Drinfeld showed that there exists a unique pairing $\langle\ ,\ \rangle_{U_\hbar \mathfrak{n}_\pm}$ of $U_\hbar \mathfrak{n}_+$ and $U_\hbar \mathfrak{n}_-$ with values in $\mathbb{C}((\hbar)) = \mathbb{C}[[\hbar]][\hbar^{-1}]$, defined by

$$\langle e_i, f_{i'} \rangle_{U_\hbar \mathfrak{n}_\pm} = \frac{1}{\hbar} d_i^{-1} \delta_{ii'}, \tag{3}$$

$$\langle x, yy' \rangle_{U_\hbar \mathfrak{n}_\pm} = \sum \langle x^{(1)}, y \rangle_{U_\hbar \mathfrak{n}_\pm} \langle x^{(2)}, y' \rangle_{U_\hbar \mathfrak{n}_\pm}, \tag{4}$$

and

$$\langle xx', y \rangle_{U_\hbar \mathfrak{n}_\pm} = \sum \langle x, y^{(1)} \rangle_{U_\hbar \mathfrak{n}_\pm} \langle x', y^{(2)} \rangle_{U_\hbar \mathfrak{n}_\pm} \tag{5}$$

for x, x' in $U_\hbar \mathfrak{n}_+$ and y, y' in $U_\hbar \mathfrak{n}_-$, and $\Delta_\pm(z) = \sum z^{(1)} \otimes z^{(2)}$ (braided Hopf pairing axioms).

As a direct consequence of Corollary 1.1, we show:

Theorem 1.2. *The pairing $\langle\ ,\ \rangle_{U_\hbar \mathfrak{n}_\pm}$ between $U_\hbar \mathfrak{n}_+$ and $U_\hbar \mathfrak{n}_-$ is nondegenerate.*

This result can be found in Lusztig's book ([22], Corollary 33.1.5, Def. 3.1.1 and Proposition 3.2.4); it relies on the construction of dual PBW bases. Another argument using Lusztig' results on integrable modules is in [29], and an argument using irreducible Verma modules is in [25].

We also show:

Proposition 1.1. *For any α in $\mathbb{N}^n$, let $U_\hbar \mathfrak{n}_\pm[\pm\alpha]$ be the part of $U_\hbar \mathfrak{n}_\pm$ of degree $\pm\alpha$, and let $P[\alpha]$ be the element of $U_\hbar \mathfrak{n}_+[\alpha] \otimes U_\hbar \mathfrak{n}_-[-\alpha]$ induced by $\langle\ ,\ \rangle_{U_\hbar \mathfrak{n}_\pm}$. Let Δ_+ be the set of positive roots of $\mathfrak{g}$ (the ϵ_i are the simple roots). Let $(\bar{e}_{\alpha,i})_{\alpha \in \Delta_+}$ and $(\bar{f}_{\alpha,i})_{\alpha \in \Delta_+}$ be dual Cartan-Weyl bases of $\mathfrak{n}_+$ and $\mathfrak{n}_-$, and let $e_{\alpha,i}, f_{\alpha,i}$ be lifts of the $\bar{e}_{\alpha,i}, \bar{f}_{\alpha,i}$ to $U_\hbar \mathfrak{n}_\pm$. Then, if k is the integer such that α belongs to $k\Delta_+ \setminus (k-1)\Delta_+$, we have*

$$P[\alpha] = \frac{\hbar^k}{k!} \sum_{\alpha_1, \ldots, \alpha_k \in \Delta_+ | \sum_i \alpha_i = \alpha; i_j} e_{\alpha_1, i_1} \cdots e_{\alpha_k, i_k} \otimes f_{\alpha_1, i_1} \cdots f_{\alpha_k, i_k} + o(\hbar^k).$$

The fact that $P[\alpha]$ has $\hbar$-adic valuation equal to k was stated by Drinfeld in [8].

• *The case of a generic deformation parameter*

It is easy to derive from the above results, PBW and nondegeneracy results in the case where the parameter $q = e^\hbar$ is generic.

Corollary 1.2. *Let q' be an indeterminate, and let $U_{q'} \mathfrak{n}_+$ be the algebra over $\mathbb{C}(q')$ with generators e'_i, $i = 1, \ldots, r$, and relations (1), with e_i and $q = e^\hbar$ replaced by e'_i and q'. We have for any α in $\mathbb{N}^n$*

$$\dim_{\mathbb{C}(q')} U_{q'} \mathfrak{n}_+[\alpha] = \dim_{\mathbb{C}(q')} U_{q'} \mathfrak{n}_+[\alpha].$$

Let $(\bar{e}_\nu)_{\nu\in I}$ *be a basis of homogeneous elements of* $\mathfrak{n}_+$. *Let*

$$\bar{e}_\nu = \sum_{i_j\in\{1,\dots,n\}} C_{\nu;i_1,\dots,i_k}\bar{e}_{i_1}\cdots\bar{e}_{i_k}$$

be expressions of the $\bar{e}_\nu$ *in terms of the generators* $\bar{e}_1,\dots,\bar{e}_n$.

Let $C_{\nu;i_1,\dots,i_k}(q')$ *be rational functions of* q', *such that*

$$C_{\nu;i_1,\dots,i_k}(1) = C_{\nu;i_1,\dots,i_k},$$

and set $e'_\nu = \sum_{i_j\in\{1,\dots,n\}} C_{\nu;i_1,\dots,i_k}(q')e_{i_1}\cdots e_{i_k}$. *Then the family* $(\prod_\nu e_\nu^{n_\nu})$, *where the* n_ν *are in* $\mathbb{N}$ *and vanish except for a finite number of them, forms a basis of* $U_{q'}\mathfrak{n}_+$ *(over* $\mathbb{C}(q')$*).*

Corollary 1.3. *Let* $U_{q'}\mathfrak{n}_-$ *be the* $\mathbb{C}(q')$*-algebra with generators* f'_i, $1\le i\le n$, *and relations (1), with* e_i *replaced by* f'_i. *Define the braided tensor squares* $U_{q'}\mathfrak{n}_\pm\bar{\otimes}U_{q'}\mathfrak{n}_\pm$ *using (2). We have a braided Hopf pairing* $\langle\ ,\ \rangle_{\mathbb{C}(q')}$ *between* $U_{q'}\mathfrak{n}_+$ *and* $U_{q'}\mathfrak{n}_-$, *defined by (3), (4) and (5). The pairing* $\langle\ ,\ \rangle_{\mathbb{C}(q')}$ *is nondegenerate.*

1.2. Quantum current and Feigin-Odesskii algebras.

Our next results deal with quantum current algebras. Assume that the Cartan matrix A is of finite type. Let $L\mathfrak{n}_+$ be the current Lie algebra $\mathfrak{n}_+\otimes\mathbb{C}[t,t^{-1}]$, endowed with the bracket $[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{n+m}$.

• *Quantum affine algebras*

Let $\mathcal{A}$ be the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle e_i[k], i=1,\dots,n, k\in\mathbb{Z}\rangle$ by the two-sided ideal generated by the coefficients of monomials in the formal series identities

$$(q^{d_ia_{ij}}z-w)e_i(z)e_j(w) = (z-q^{d_ia_{ij}}w)e_j(w)e_i(z), \tag{6}$$

$$\mathrm{Sym}_{z_1,\dots,z_{1-a_{ij}}}\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q^{d_i}} e_i(z_1)\dots e_i(z_k)e_j(w)e_i(z_{k+1})\dots e_i(z_{1-a_{ij}}) = 0, \tag{7}$$

where $e_i(z)$ is the generating series $e_i(z) = \sum_{k\in z} e_i[k]z^{-k}$, and $q=e^\hbar$.

Let $U_\hbar L\mathfrak{n}_+$ be the quotient $\mathcal{A}/(\cap_{N>0}\hbar^N\mathcal{A})$.

Define $\widetilde{A}$ as the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle e_i[k]^{\widetilde{\mathcal{A}}}, i=1,\dots,nk\in\mathbb{Z}\rangle$ by the two-sided ideal geberated by the coefficients of monimials of (6) and

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q^{d_i}}(e_i[0]^{\widetilde{\mathcal{A}}})^k e_j[l]^{\widetilde{\mathcal{A}}}(e_i[0]^{\widetilde{\mathcal{A}}})^{1-a_{ij}-k} = 0, \tag{8}$$

for any $i,j=1,\dots,n$ and l integer. Define $\widetilde{U}_\hbar L\mathfrak{n}_+$ as the quotient $\widetilde{\mathcal{A}}/(\cap_{N>0}\hbar^N\widetilde{\mathcal{A}})$.

Theorem 1.3. *1)* $U_\hbar L\mathfrak{n}_+$ *is a free* $\mathbb{C}[[\hbar]]$*-module, and the map* $e_i[n]\mapsto\bar{e}_i\otimes t^n$ *defines an algebra isomorphism from* $U_\hbar L\mathfrak{n}_+/\hbar U_\hbar L\mathfrak{n}_+$ *to* $UL\mathfrak{n}_+$.

2) Let $U_\hbar L\mathfrak{n}_+^{top}$ *be the quotient of* $\mathbb{C}\langle e_i[k], i=1,\dots,n, k\in\mathbb{Z}\rangle[[\hbar]]$ *by the* $\hbar$*-adically closed two-sided ideal generated by the coefficients of monomials in relations (6) and (7). Then* $U_\hbar L\mathfrak{n}_+^{top}$ *is a topologically free* $\mathbb{C}[[\hbar]]$*-module; it is* *naturally the $\hbar$-adic completion of $U_\hbar L\mathfrak{n}_+$, and the map $e_i[n] \mapsto \bar{e}_i \otimes t^n$ defines an algebra isomorphism from $U_\hbar L\mathfrak{n}_+^{top}/\hbar U_\hbar L\mathfrak{n}_+^{top}$ to $UL\mathfrak{n}_+$.*

3) There is a unique algebra map from $\widetilde{U}_\hbar L\mathfrak{n}_+$ to $U_\hbar L\mathfrak{n}_+$, sending each $e_i[k]^{\widetilde{\mathcal{A}}}$ to $e_i[k]$; it is an algebra isomorphism.

4) Let $\widetilde{U}_\hbar L\mathfrak{n}_+^{top}$ be the quotient of $\mathbb{C}\langle e_i[k]^{\widetilde{\mathcal{A}}}, i = 1, \ldots, n, k \in \mathbb{Z}\rangle[[\hbar]]$ by the $\hbar$- adically closed two-sided ideal generated by the coefficients of monomials in relations (6) and (8). Then $e_i[k]^{\widetilde{\mathcal{A}}} \mapsto e_i[k]$ defines an algebra isomorphism between $\widetilde{U}_\hbar L\mathfrak{n}_+^{top}$ and $U_\hbar L\mathfrak{n}_+^{top}$.

The statements 1) and 2) of this Theorem can be derived from the results of [2].

In [13, 14], Feigin and Odesskii defined the algebra FO, which may be viewed as a functional version of the shuffle algebra. FO is defined as

$$\mathrm{FO} = \oplus_{\mathbf{k}\in\mathbb{N}^n} \mathrm{FO}_{\mathbf{k}}, \tag{9}$$

where if $\mathbf{k} = (k_i)_{1\le i\le n}$, we set

$$\mathrm{FO}_{\mathbf{k}} = \frac{1}{\prod_{\substack{i<j \\ 1\le\alpha\le k_i \\ 1\le\beta\le k_j}} (t_\alpha^{(i)} - t_\beta^{(j)})} \mathbb{C}[[\hbar]][(t_j^{(i)})^{\pm 1}, i = 1, \ldots, n, j = 1, \ldots, k_i]^{\mathfrak{S}_{k_1}\times\cdots\times\mathfrak{S}_{k_n}},$$

where the product of symmetric groups acts by permutation of variables of each group of variables $(t_j^{(i)})_{1\le j\le k_i}$. $\mathrm{FO}_{\mathbf{k}}$ therefore consists of rational functions in the $t_j^{(i)}$, symmetric in each group $(t_j^{(i)})_{1\le j\le k_i}$, regular except for poles when the variables go to 0 or infinity, and simple poles when variables of different "colors" collide. (9) defines a grading of FO by $\mathbb{N}^n$. The product on FO is also graded, and we have, for f in $\mathrm{FO}_{\mathbf{k}}$ and g in $\mathrm{FO}_{\mathbf{l}}$ ($\mathbf{l} = (l_i)_{1\le i\le n}$) ,

$$(f * g)(t_j^{(i)})_{1\le i\le n, 1\le j\le k_i+l_i} = \mathrm{Sym}_{\{t_j^{(1)}\}} \ldots \mathrm{Sym}_{\{t_j^{(n)}\}} \tag{10}$$
$$\Big(\prod_{1\le i\le N} \prod_{N+1\le j\le N+M} \frac{q^{\langle\epsilon(i),\epsilon(j)\rangle} t_i - t_j}{t_i - t_j} f(t_1, \ldots, t_N) g(t_{N+1}, \ldots, t_{N+M})\Big),$$

where $N = \sum_{i=1}^n k_i$ and $M = \sum_{i=1}^n l_i$; we set for any s,

$$t_{k_1+\cdots+k_{s-1}+1} = t_1^{(s)}, \ldots, t_{k_1+\cdots+k_s} = t_{k_s}^{(s)}, \quad t_{N+l_1+\cdots+l_{s-1}+1} = t_{k_s+1}^{(s)}, \ldots,$$
$$t_{N+l_1+\cdots+l_s} = t_{k_s+l_s}^{(s)};$$

we also define $\epsilon(i) = \epsilon_k$ if $t_i = t_l^{(k)}$ for some l; as before, $\langle\epsilon_i, \epsilon_j\rangle = d_i a_{ij}$ for $i, j = 1, \ldots, n$.

In the right side of (10), each symmetrization can be replaced by a sum over shuffles, since the argument in symmetric in each group of variables $(t_1^{(s)}, \ldots, t_{k_s}^{(s)})$ and $(t_{k_s+1}^{(s)}, \ldots, t_{k_s+l_s}^{(s)})$.

In Proposition 3.2, we define a topological braided Hopf structure on FO.

In [11], we showed:

Proposition 1.2. *There is a unique algebra morphism $i_\hbar$ from $U_\hbar L\mathfrak{n}_+$ to* FO*, such that $i_\hbar(e_i[n])$ is the element $(t_1^{(i)})^n$ of FO_{ϵ_i}.*

Let us denote by LV the direct sum $\oplus_{i=1}^n \mathrm{FO}_{\epsilon_i}$ and let $\langle LV\rangle$ be the sub-$\mathbb{C}[[\hbar]]$-algebra of FO generated by LV. As a corollary of the proof of Theorem 1.3, we prove:

Corollary 1.4. *$i_\hbar$ is an algebra isomorphism between $U_\hbar L\mathfrak{n}_+$ and $\langle LV\rangle$.*

$U_\hbar L\mathfrak{n}_+$ is also endowed with a topological braided Hopf structure (the Drinfeld comultiplication); it is then easy to see that $i_\hbar$ is compatible with both Hopf structures.

Define $T(LV_\pm)$ as the free algebras $\mathbb{C}[[\hbar]]\langle e_i[k]^{(T)}, 1\le i\le n, k\in\mathbb{Z}\rangle$ and $\mathbb{C}[[\hbar]]\langle f_i[k]^{(T)}, 1\le i\le n, k\in\mathbb{Z}\rangle$. We have a pairing $\langle\ ,\ \rangle_{T(LV_\pm)}$ between $T(LV_+)$ and $T(LV_-)$ defined by

$$\begin{aligned}&\langle e_{i_1}[k_1]^{(T)}\cdots e_{i_p}[k_p]^{(T)}, f_{j_1}[l_1]^{(T)}\cdots f_{j_{p'}}[l_{p'}]^{(T)}\rangle_{T(LV_\pm)}\\ &=\delta_{pp'}\frac{1}{\hbar^p}\sum_{\sigma\in\mathfrak{S}_p}\mathrm{res}_{z_1=0}\cdots\mathrm{res}_{z_p=0}\\ &\Big(\prod_{s>t,\sigma^{-1}(s)<\sigma^{-1}(t)}\frac{q^{(\epsilon_{i_s},\epsilon_{i_t})}z_s-z_t}{z_s-q^{(\epsilon_{i_s},\epsilon_{i_t})}z_t}\prod_{s=1}^p\frac{1}{d_{j_s}}\delta_{i_{\sigma(s)},j_s}\prod_{s=1}^p z_i^{k_s+l_s}\prod_{s=1}^p\frac{dz_s}{z_s}\Big)\end{aligned}\tag{11}$$

where the ratios $\frac{q^{(\epsilon_{i_s},\epsilon_{i_t})}z_s-z_t}{z_s-q^{(\epsilon_{i_s},\epsilon_{i_t})}z_t}$ are expanded for $z_t\ll z_s$.

Let $U_\hbar L\mathfrak{n}_-$ be the quotient of $T(LV_-)$ by the homomorphic image of the ideal defining $U_{-\hbar}L\mathfrak{n}_+$ by the map $e_i[k]^{(T)}\mapsto f_i[k]^{(T)}$.

Proposition 1.3. *(see [11]) This pairing induces a pairing $\langle\ ,\ \rangle_{U_\hbar L\mathfrak{n}_\pm}$ between $U_\hbar L\mathfrak{n}_+$ and $U_\hbar L\mathfrak{n}_-$.*

We then prove:

Theorem 1.4. *The pairing $\langle\ ,\ \rangle_{U_\hbar L\mathfrak{n}_\pm}$ is nondegenerate.*

• *The form of the R-matrix*

Let us set $A_\pm = U_\hbar L\mathfrak{n}_\pm$. Let a and b be two integers. Define $I^+_{\ge a}$ and $I^+_{\le a}$, as the right, resp. left ideals of A_+ generated by the $e_i[k], k\ge a$, resp. the $e_i[k], k\le a$. The ideals $I^+_{\ge a}$ and $I^+_{\le a}$ are graded; for α in $(\pm\mathbb{N})^n$, denote by $I^+_{\ge a}[\alpha]$ and $I^+_{\le a}[\alpha]$ their component of degree α.

Proposition 1.4. *For any α in $\mathbb{N}^n$, for any integers a and b, $(I^+_{\le a}+I^+_{\ge b})^\perp[-\alpha]$ and $[A_+/(I^+_{\le a}+I^+_{\ge b})^{\perp\perp}][\alpha]$ are free finite-dimensional $\mathbb{C}[[\hbar]]$-modules. The pairing between A_+ and A_- induces a nondegenerate pairing between them. Moreover, the intersection $\cap_{a,b}(I^+_{\le a}+I^+_{\ge b})^{\perp\perp}$ is zero.*

Denote by $P_{a,b}[\alpha]$ the corresponding element of

$$[A_+/(I^+_{\le a}+I^+_{\ge b})^{\perp\perp}][\alpha]\otimes(I^+_{\le a}+I^+_{\ge b})^\perp[-\alpha][\hbar^{-1}].$$

$P_{a,b}[\alpha]$ defines an element of $\lim_{\leftarrow a,b}A_+/(I^+_{\le a}+I^+_{\ge b})^{\perp\perp}\otimes_{\mathbb{C}[[\hbar]]}A_-[\hbar^{-1}]$.

Let $(\bar e_{\beta,i})_{\beta\in\Delta_+}$ and $(\bar f_{\beta,i})_{\beta\in\Delta_+}$ be dual Cartan-Weyl bases of $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Let $e_{\beta,i}[p]$ and $f_{\beta,i}[p]$ be lifts to $U_\hbar L\mathfrak{n}_\pm$ of $\bar e_{\beta,i}\otimes t^p$ and $\bar f_{\beta,i}\otimes t^p$. Then if α belongs to $k\Delta_+ - (k-1)\Delta_+$, $P[\alpha]$ has the form

$$P[\alpha] = \frac{\hbar^k}{k!} \sum_{\alpha_1,\ldots,\alpha_k\in\Delta_+,\sum_i\alpha_i=\alpha;i_j} \sum_{p_1,\ldots,p_k\in\mathbb{Z}}$$
$$e_{\alpha_1,i_1}[p_1]\ldots e_{\alpha_k,i_k}[p_k]\otimes f_{\alpha_1,i_1}[-p_1]\ldots f_{\alpha_k,i_1}[-p_k] + o(\hbar^k)$$

(*all but a finite number of elements of this sum belong to* $(I^+_{\leq a} + I^+_{\geq b})^{\perp\perp}\otimes_{\mathbb{C}[[\hbar]]} A_-[\hbar^{-1}]$.)

Let $(h'_i)_{i=1,\ldots,n}$ be the basis of $\mathfrak{h}$, dual to $(h_i)_{i=1,\ldots,n}$. Set

$$\mathcal{K} = \exp(\sum_{i=1}^n h_i[0]\otimes h'_i[0] + \sum_{p>0} h_i[p]\otimes h'_i[-p]).$$

Then the elements $\mathcal{R}[\alpha] = \mathcal{K}P[\alpha]$ of $\lim_{\leftarrow N}(U_\hbar L\mathfrak{b}_+\otimes U_\hbar L\mathfrak{b}_-)/I_N^{\mathfrak{b}_\pm,(2)}$, where $I_N^{\mathfrak{b}_\pm,(2)}$ is the ideal generated by the $h_i[p]\otimes 1, e_i[p]\otimes 1, p\geq N$, and the $1\otimes f_i[p], p\geq N$, satisfy the R-matrix identity

$$\sum_{\gamma\in\mathbb{N}^n,\beta\in(\pm\mathbb{N})^n,\beta+\gamma=\lambda} \mathcal{R}[\gamma]\Delta(x)_{(\beta,\alpha-\beta)} = \sum_{\gamma\in\mathbb{N}^n,\beta\in(\pm\mathbb{N})^n,\beta+\gamma=\lambda} \Delta'(x)_{(\beta,\alpha-\beta)}\mathcal{R}[\gamma],$$

for any $\lambda\in\mathbb{Z}^r$ and x in the double $U_\hbar L\mathfrak{g}$ of $U_\hbar L\mathfrak{b}_+$ of degree α (the sums over the root lattice are obviously finite, and each product makes sense in $\lim_{\leftarrow N}(U_\hbar L\mathfrak{g}\otimes U_\hbar L\mathfrak{g})/I_N^{\mathfrak{g},(2)}$, where $I_N^{\mathfrak{g},(2)}$ is the left ideal generated by the $x[p]\otimes 1$ and $1\otimes x[p], p\geq N$, $x = e_i, h_i, f_i$).

• *Yangians*

Let us describe how the above results are modified in the case of Yangians. Let $\mathcal{A}^{rat}$ be the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle e_i[k]^{rat}, i = 1,\ldots,n, k\in\mathbb{Z}\rangle$ by the two-sided ideal generated by the coefficients of the relations

$$(z-w+\hbar a_{ij})e_i(z)^{rat}e_j(w)^{rat} = (z-w-\hbar a_{ij})e_j(w)^{rat}e_i(z)^{rat}, \qquad (12)$$

$$\mathrm{Sym}_{z_1,\ldots,z_{1-a_{ij}}}\mathrm{ad}(e_i(z_1)^{rat})\cdots\mathrm{ad}(e_i(z_{1-a_{ij}})^{rat})(e_j(w)^{rat}) = 0, \qquad (13)$$

where we set $e_i(z)^{rat} = \sum_{k\in\mathbb{Z}} e_i[k]^{rat}z^{-k-1}$, and let us set $U_\hbar^{rat}L\mathfrak{n}_+ = \mathcal{A}^{rat}/\cap_{N>0}\hbar^N\mathcal{A}^{rat}$.

Define also $\widetilde{\mathcal{A}}^{rat}$ as the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle e_i[k]^{\widetilde{\mathcal{A}}^{rat}}, i = 1,\ldots,n, k\in\mathbb{Z}\rangle$ by the two-sided ideal generated by the coefficients of the relations (12) and

$$\mathrm{ad}(e_i[0]^{\widetilde{\mathcal{A}}^{rat}})^{1-a_{ij}}e_j[l]^{\widetilde{\mathcal{A}}^{rat}} = 0, \qquad (14)$$

for any $i,j = 1,\ldots,n$ and integer l.

Theorem 1.5. *1) $U_\hbar L\mathfrak{n}_+$ is a free $\mathbb{C}[[\hbar]]$-module. There is a unique algebra isomorphism from $U_\hbar^{rat}L\mathfrak{n}_+/\hbar U_\hbar^{rat}L\mathfrak{n}_+$ to $UL\mathfrak{n}_+$, sending the class of $e_i[k]^{rat}$ to $\bar e_i\otimes t^k$.*

2) There is a unique algebra morphism from $\widetilde{U}_\hbar L\mathfrak{n}_+$ to $U_\hbar L\mathfrak{n}_+$, sending $e_i[k]^{rat}$ to $e_i[k]^{\widetilde{\mathcal{A}}^{rat}}$; it is an isomorphism between these algebras.

3) Let $U_\hbar L\mathfrak{n}_+^{rat,top}$ and $\widetilde{U}_\hbar L\mathfrak{n}_+^{rat,top}$ be the quotients of

$$\mathbb{C}\langle e_i[k]^{rat}, i=1,\ldots,n, k\in\mathbb{Z}\rangle[[\hbar]] \text{ and } \mathbb{C}\langle e_i[k]^{\widetilde{\mathcal{A}}^{rat}}, i=1,\ldots,n, k\in\mathbb{Z}\rangle[[\hbar]]$$

by the $\hbar$-adically closed two-sided ideals generated by the coefficients of monomials in relations (12) and (13), resp. (12) and (14). Then $e_i[k]^{\widetilde{\mathcal{A}}^{rat}} \mapsto e_i[k]^{rat}$ defines an algebra isomorphism between $U_\hbar L\mathfrak{n}_+^{rat,top}$ and $\widetilde{U}_\hbar L\mathfrak{n}_+^{rat,top}$. $U_\hbar L\mathfrak{n}_+^{rat,top}$ is the $\hbar$-adic completion of $U_\hbar L\mathfrak{n}_+^{rat}$; it is a topologically free $\mathbb{C}[[\hbar]]$-module.

Define FO^{rat} as the graded space FO, endowed with the product obtained from (10) by the replacement of each $q^\lambda z - q^\mu w$ by $z-w+\hbar(\lambda-\mu)$. FO^{rat} is an associative algebra and we have

Theorem 1.6. *There is a unique algebra map $i_\hbar^{rat}$ from $U_\hbar L\mathfrak{n}_+^{rat}$ to FO^{rat}, sending each $e_i[k]^{rat}$ to $e_i\otimes t^k$. $i_\hbar$ is an isomorphism between $U_\hbar L\mathfrak{n}_+^{rat}$ and its subalgebra $\langle LV\rangle^{rat}$ generated by the degree one elements.*

Define a pairing $\langle\ ,\ \rangle_{T(LV_\pm),rat}$ between $T(LV_+)$ and $T(LV_-)$ by the formula (11), where each $q^\lambda z - q^\mu w$ is replaced by $z-w+\hbar(\lambda-\mu)$. Let $U_\hbar L\mathfrak{n}_-^{rat}$ be the quotient $T(LV_-)$ by the homomorphic image of the ideal defining $U_\hbar L\mathfrak{n}_+^{rat}$ by the map $e_i[k]^{(T)}\mapsto f_i[k]^{(T)}$.

Theorem 1.7. *$\langle\ ,\ \rangle_{T(LV_\pm),rat}$ induces a nondegenerate pairing $\langle\ ,\ \rangle_{U_\hbar L\mathfrak{n}_\pm,rat}$ between $U_\hbar L\mathfrak{n}_\pm^{rat}$. Define $I_a^{\pm,rat}$ as the right, resp. left ideals of $U_\hbar L\mathfrak{n}_\pm^{rat}$ generated by the $e_i[k]^{rat}$, $k\geq a$, resp. by the $f_i[k]^{rat}$, $k\geq a$. Then for any α in $\mathbb{N}^n$, $I_a^{+,rat}[\alpha]\cap(I_b^{-,rat}[-\alpha])^\perp$ is a space of finite codimension in $(I_b^{-,rat}[-\alpha])^\perp$. Let $P[\alpha]^{rat}$ be the corresponding element of*

$$\lim_{\leftarrow a,b}(U_\hbar L\mathfrak{n}_+^{rat}/I_a^{+,rat})[\alpha]\otimes(U_\hbar L\mathfrak{n}_-^{rat}/I_b^{-,rat})[-\alpha][\hbar^{-1}].$$

Let $e_{\beta,i}[p]^{rat}$ be lifts to $U_\hbar L\mathfrak{n}_\pm$ of the $\bar e_{\beta,i}[p]$. Then if α belongs to $k\Delta_+-(k-1)\Delta_+$, $P[\alpha]^{rat}$ has the form

$$P[\alpha]^{rat} = \frac{\hbar^k}{k!}\sum_{\alpha_1,\ldots,\alpha_k\in\Delta_+,\sum_i\alpha_i=\alpha;i_j}\ \sum_{p_1,\ldots,p_k\in\mathbb{Z}}$$
$$e_{\alpha_1,i_1}[p_1]^{rat}\ldots e_{\alpha_k,i_k}[p_k]^{rat}\otimes f_{\alpha_1,i_1}[-p_1-1]^{rat}\ldots f_{\alpha_k,i_1}[-p_k-1]^{rat}+o(\hbar^k).$$

The proofs of the statements of this section are analogous to those of the quantum affine case and will be omitted.

1.3. Quantum current algebras of affine type (toroidal algebras) .

Assume that A is an arbitrary symmetrizable Cartan matrix. Define $U_\hbar L\mathfrak{n}_+$ and $\widetilde{U}_\hbar L\mathfrak{n}_+$ as in sect. 1..

Proposition 1.5. *1) Let $\widetilde{F}_+$ be the Lie algebra with generators $\tilde{x}_i^+[k]$, $1\leq i\leq n$, $k\in\mathbb{Z}$, and relations given by the coefficients of monomials in*

$$(z-w)[\tilde{x}_i^+(z),\tilde{x}_j^+(w)]=0,\quad \mathrm{ad}(\tilde{x}_i^+(z_1))\cdots\mathrm{ad}(\tilde{x}_i^+(z_{1-a_{ij}}))(\tilde{x}_j^+(w))=0,$$

where for x any $\tilde{x}_i^+$, $x(z)$ is the generating series $\sum_{k\in\mathbb{Z}} x[k]z^{-k}$. If we give degree ϵ_i to $\tilde{x}_i^+[k]$, $\widetilde{F}_+$ is graded by set Δ_+ of the roots of $\mathfrak{n}_+$.

Then $U_\hbar L\mathfrak{n}_+/\hbar U_\hbar L\mathfrak{n}_+$ and $\widetilde{U}_\hbar L\mathfrak{n}_+/\hbar\widetilde{U}_\hbar L\mathfrak{n}_+$ are both isomorphic to the enveloping algebra $U(\widetilde{F}_+)$.

2) There is a unique Lie algebra morphism $j_+ : \widetilde{F}_+ \to L\mathfrak{n}_+$, such that $j_+(\tilde{e}_i^+[k]) = \bar{e}_i \otimes t^k$. j_+ is graded and surjective. The kernel of j_+ is contained in $\oplus_{\alpha\in\Delta_+,\alpha \text{ imaginary}}\widetilde{F}_+[\alpha]$.

Let us assume now that A is untwisted affine. $\mathfrak{n}_+$ is the isomorphic to a subalgebra of the loop algebra $\bar{\mathfrak{g}}[\lambda,\lambda^{-1}]$, with $\bar{\mathfrak{g}}$ semisimple. Define $\mathfrak{t}_+$ as the direct sum $L\mathfrak{n}_+ \oplus (\oplus_{k>0,l\in\mathbb{Z}}\mathbb{C}K_{k\delta}[l])$, and endow it with the bracket such that the $K_{k\delta}[l]$ are central, and

$$[(x\otimes t^l,0),(y\otimes t^m,0)] = ([x\otimes t^l, y\otimes t^m], \langle\bar{x},\bar{y}\rangle_{\bar{\mathfrak{g}}}(lk''-mk')K_{(k'+k'')\delta}[l+m])$$

for $x\mapsto \bar{x}\otimes\lambda^{k'}, y\mapsto\bar{y}\otimes\lambda^{k''}$ by the inclusion $\mathfrak{n}_+\subset\bar{\mathfrak{g}}[\lambda,\lambda^{-1}]$, where $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}}$ is an invariant scalar product on $\bar{\mathfrak{g}}$.

Then $\mathfrak{t}_+$ is a Lie subalgebra of the toroidal algebra $\mathfrak{t}$, which is the universal central extension of $L\mathfrak{g}$ ([20, 23]). In what follows, we will set $x[k]^{\mathfrak{t}} = (x\otimes t^k, 0)$.

Proposition 1.6. *1) When A is of affine Kac-Moody type, the kernel of j_+ is equal to the center of $\widetilde{F}_+$, so that $\widetilde{F}_+$ is a central extension of $L\mathfrak{n}_+$.*

2) We have a unique Lie algebra map j' from $\mathfrak{t}_+$ to $\widetilde{F}_+$ such that $j'(\bar{e}_i\otimes t^n) = \tilde{e}_i[n]$. This map is an isomorphism iff A is not of type $A_1^{(1)}$. If A is of type $A_1^{(1)}$, j' is surjective, and its kernel is $\oplus_{n\in\mathbb{Z}}\mathbb{C}K_\delta[n]$.

In Remark 4.3, we discuss possible generalizations of Theorem 1.3 to the case of affine quantum current algebras, and the connection of Proposition 1.5 with the results of [15].

The basic idea of the constructions of the two first parts of this work is to compare the quantized algebras defined by generators and relations with quantum shuffles algebras. The idea to use shuffle algebras to provide examples of Hopf algebras dates back to Nichols ([24]). Later, Schauenburg ([28]) and Rosso ([27]) showed that the positive part $U_\hbar\mathfrak{n}_+$ of the Drinfeld-Jimbo quantized enveloping algebras are isomorphic to the subalgebra $\mathrm{Sh}(V)$ of quantum shuffle Hopf algebras generated in degree 1. Their results rely on Lusztig's PBW or duality (nondegeneracy of Drinfeld's pairing) results. A nonabelian generalization of Schauenburg's result can be found in [1].

In sect. 2., we show that applying Drinfeld's theory of Lie bialgebras to $\mathrm{Sh}(V)$ yields at the same time proofs of these results (PBW for $U_\hbar\mathfrak{n}_+$ and isomorphism of $U_\hbar\mathfrak{n}_+$ with $\mathrm{Sh}(V)$, and nondegeneracy of the pairing as a simple consequence), when the deformation parameter is formal or generic.

In sect. 3., we apply the same idea to quantum current algebras. These algebras, also know as "new realizations" algebras, depend on the datum of a Cartan matrix. In that situation, the proper replacement of shuffle algebras are the functional shuffle algebras introduced by Feigin and Odesskii ([13, 14]). We show that when the Cartan matrix is of finite type, the ideas of sect. 2. allow to complete the results of [11] on comparison of the quantum current algebras and the Feigin-Odesskii algebras. However, there are still some open problems in this direction, see Remark 3.16. We hope that the ideas of this section will help generalize the results of [12] from $\mathfrak{sl}_2$ to arbitrary semisimple Lie algebras. For this one should, in particular, find analogues of the quantum Serre relations for the algebras in genus ≥ 1.

In sect. 4., we consider the classical limit of the quantum current algebras in the case of an affine Kac-Moody Cartan matrix. We show that this classical limit is the enveloping algebra of a Lie algebra $\widetilde{F}_+$, which is a central extension of the Lie algebra $L\mathfrak{n}_+$ of loops with values in the positive subalgebra $\mathfrak{n}_+$ of the affine Kac-Moody algebra. $\widetilde{F}_+$ is graded by the roots of $\mathfrak{n}_+$, and its center is contained in the part of imaginary degrees. We show that in all affine untwisted cases, except the $A_1^{(1)}$ case, $\widetilde{F}_+$ is isomorphic to a subalgebra $\mathfrak{t}_+$ of the toroidal algebra $\mathfrak{t}$ introduced and studied in [23, 16, 30]. In the $A_1^{(1)}$ case, we identify $\widetilde{F}_+$ with a quotient of $\mathfrak{t}_+$.

In the quantum case, the center of $U_\hbar L\mathfrak{n}_+$ seems closely connected with the central part of the affine elliptic algebras constructed in the recent work of Feigin and Odesskii ([15]). We hope that a better understanding of this center will enable to extend to toroidal algebras the results of sect. 3..

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2. Quantum Kac-Moody algebras (proofs of the results of sect. 1.)

2.1. PBW theorem and comparison with shuffle algebra (proofs of Theorem 1.1 and Corollary 1.1).

• *Definition of* $\mathrm{Sh}(V)$ *and* $\langle V\rangle$

Let us set $V = \oplus_{i=1}^n \mathbb{C}v_i$. Let ϵ_i be the ith basis vector of $\mathbb{N}^n$. Define the grading of V by $\mathbb{N}^n$ by $\deg(v_i) = \epsilon_i$. Let $\mathrm{Sh}(V)$ be the quantum shuffle algebra constructed from V and the braiding $V\otimes V \to V\otimes V[[\hbar]]$, $v_i\otimes v_j \mapsto q^{-d_i a_{ij}} v_j \otimes v_i$. That is, $\mathrm{Sh}(V)$ is isomorphic, as $\mathbb{C}[[\hbar]]$-module, to $\oplus_{i\geq 0} V^{\otimes i}[[\hbar]]$. Denote the element $z_1\otimes\cdots\otimes z_k$ as $[z_1|\cdots|z_k]$. The product is defined on $\mathrm{Sh}(V)$ as follows:

$$[z_1|\cdots|z_k] \quad [z_{n+1}|\cdots|z_{k+l}] = \sum_{\sigma\in\Sigma_{k,l}} q^{-\sum_{1\leq i<j\leq k+l,\sigma(i)>\sigma(j)}\langle\deg(z_i),\deg(z_j)\rangle}[z_{\sigma(1)}|\cdots|z_{\sigma(k+l)}],$$

if the z_i are homogeneous elements of V, and where $\Sigma_{k,l}$ is the subset of the symmetric group $\mathfrak{S}_{k+l}$ consisting of shuffle permutations σ such that $\sigma(i)<\sigma(j)$

if $1 \le i < j \le k$ or $k+1 \le i < j \le k+l$; the bilinear form on $\mathbb{N}^n$ is defined by form $\langle \epsilon_i, \epsilon_j \rangle = d_i a_{ij}$.

Lemma 2.1. *$\langle V \rangle$ is the direct sum of its graded components, which are free $\mathbb{C}[[\hbar]]$-modules. It follows that $\langle V \rangle$ is a free $\mathbb{C}[[\hbar]]$-module.*

Proof. That $\langle V \rangle$ is the direct sum of its graded components follows from its definition. These graded components are $\mathbb{C}[[\hbar]]$-submodules of finite-dimensional free $\mathbb{C}[[\hbar]]$-modules (the graded components of $\mathrm{Sh}(V)$). Each graded component is therefore a finite-dimensional free module over $\mathbb{C}[[\hbar]]$. The Lemma follows. ∎

• *Crossed product algebras $\mathcal{V}$ and $\mathcal{S}$*

Define linear endomorphisms $\widetilde{h}_i, i = 1, \ldots, n$ and $\widetilde{D}_j, j = 1, \ldots, n-r$ of V by the formulas

$$\widetilde{h}_i(v_j) = a_{ij} v_j, \quad \widetilde{D}_j(v_i) = \delta_{ij} v_i.$$

Extend the $\widetilde{x}$, $x \in \{h_i, D_j\}$ to linear endomorphisms of $\mathrm{Sh}(V)$ by the formulas

$$\widetilde{x}([x_1|\cdots|x_n]) = \sum_{k=1}^{n} [x_1|\cdots|\widetilde{x}(x_k)|\cdots|x_n].$$

It is clear that the $\widetilde{x}$ define derivations of $\mathrm{Sh}(V)$. These derivations preserve $\langle V \rangle$.

Define $\mathcal{V}$ and $\mathcal{S}$ as the crossed product algebras of $\langle V \rangle$ and $\mathrm{Sh}(V)$ with the derivations $\widetilde{h}_i, \widetilde{D}_j$. More precisely, $\mathcal{V}$ and $\mathcal{S}$ are isomorphic, as $\mathbb{C}[[\hbar]]$-modules, to their tensor products $\langle V \rangle \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}[h_i^{\mathcal{V}}, D_j^{\mathcal{V}}][[\hbar]]$ and $\mathrm{Sh}(V) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}[h_i^{\mathcal{V}}, D_j^{\mathcal{V}}][[\hbar]]$ with $\hbar$-adically completed polynomial algebras in $n+r$ variables. The products on $\mathcal{V}$ and $\mathcal{S}$ are then defined by the rules

$$\begin{aligned}
&(x \otimes \prod_{s=1}^{2n-r} (X_s^{\mathcal{V}})^{\alpha_s})(y \otimes \prod_{s=1}^{2n-r} (X_s^{\mathcal{V}})^{\beta_s}) \\
&= \sum_{(i_s)} \prod_{s=1}^{2n-r} \binom{\alpha_s}{i_s} \left(x \prod_{s=1}^{2n-r} \widetilde{X}_s^{i_s}(y) \right) \otimes \left(\prod_{s=1}^{2n-r} (X_s^{\mathcal{V}})^{\alpha_s+\beta_s-i_s} \right),
\end{aligned}$$

where we set $X_s = h_s$ for $s = 1, \ldots, n$, and $X_s = D_{s-n}$ for $s = n+1, \ldots, 2n-r$. In what follows, we will denote $x \otimes 1$ and $1 \otimes X_s$ simply by x and X_s, so that $x \otimes \prod_s (X_s^{\mathcal{V}})^{\alpha_s}$ will be $x \prod_s (X_s^{\mathcal{V}})^{\alpha_s}$.

$\mathcal{S}$ is then endowed with a Hopf $\mathbb{C}[[\hbar]]$-algebra structure (that is, all maps of Hopf algebra axioms are $\mathbb{C}[[\hbar]]$-module maps, and the tensor products are completed in the $\hbar$-adic topology), defined by

$$\Delta_{\mathcal{V}}(h^{\mathcal{V}}) = h^{\mathcal{V}} \otimes 1 + 1 \otimes h^{\mathcal{V}} \text{ for } h \in \{h_i, D_j\},$$

$$\Delta_{\mathcal{V}}([v_{i_1}|\cdots|v_{i_m}]) = \sum_{k=0}^{m} [v_{i_1}|\cdots|v_{i_k}] \otimes \exp(\hbar \sum_{j=1}^{k} d_{i_j} h_{i_j}^{\mathcal{V}})[v_{i_{k+1}}|\cdots|v_{i_m}]).$$

$\mathcal{V}$ is then a Hopf subalgebra of $\mathcal{S}$.

Assign degrees 0 to the elements $h_i^{\mathcal{V}}, D_j^{\mathcal{V}}$, and ϵ_i to v_i. $\mathcal{V}$ is then the direct sum of its homogeneous components, which are free finite-dimensional modules over $\mathbb{C}[h_i^{\mathcal{V}}, D_j^{\mathcal{V}}][[\hbar]]$; the grading of $\mathcal{V}$ is compatible with its algebra structure.

• *Hopf co-Poisson and Lie bialgebra structures*

Define $\mathcal{V}_0$ as $\mathcal{V}/\hbar\mathcal{V}$.

Lemma 2.2. *$\mathcal{V}_0$ is a cocommutative Hopf algebra.*

Proof. Define $\Delta'_{\mathcal{V}}$ as $\Delta_{\mathcal{V}}$ composed with the permutation of factors. We have to show that for x in $\mathcal{V}$, we have

$$(\Delta_{\mathcal{V}} - \Delta'_{\mathcal{V}})(x) \subset \hbar(\mathcal{V} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{V}). \tag{15}$$

For x one of the $\bar{h}_i, \bar{D}_j$, (15) is clearly satisfied. On the other hand, if (15) is satisfied for x and y in $\mathcal{V}$, then $(\Delta_{\mathcal{V}} - \Delta'_{\mathcal{V}})(xy)$ is equal to $(\Delta_{\mathcal{V}} - \Delta'_{\mathcal{V}})(x)\Delta_{\mathcal{V}}(y) + \Delta'_{\mathcal{V}}(x)(\Delta_{\mathcal{V}} - \Delta'_{\mathcal{V}})(y)$ and therefore belongs to $\hbar(\mathcal{V} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{V})$. It follows that (15) holds for any x in $\mathcal{V}$. ■

Lemma 2.3. *1) There exists a unique surjective Hopf algebra morphism $p_\hbar$ from $U_\hbar\mathfrak{b}_+$ to $\mathcal{V}$, such that $p_\hbar(h_i) = h_i^{\mathcal{V}}$ and $p_\hbar(x_i^+) = [v_i]$.*

2) The map $D_j \mapsto \bar{D}_j, h_i \mapsto \bar{h}_i, x_i^+ \mapsto \bar{x}_i^+$ extends to an isomorphism from $U_\hbar\mathfrak{b}_+/\hbar U_\hbar\mathfrak{b}_+$ to $U\mathfrak{b}_+$.

3) $p_\hbar$ induces a surjective cocommutative Hopf algebra morphism p from $U_\hbar\mathfrak{b}_+ = U\mathfrak{b}_+$ to $\mathcal{V}/\hbar\mathcal{V} = \mathcal{V}_0$.

Proof. That the quantum Serre relations are satisfied in $\mathrm{Sh}(V)$ by the $[v_i]$ follows from [27], Lemma 14 (the proof relies on q-binomial coefficients identities, which are proved by induction); this proves the first part of the Lemma.

Let us show that $U_\hbar\mathfrak{b}_+/\hbar U_\hbar\mathfrak{b}_+$ is isomorphic $U\mathfrak{b}_+$. $U_\hbar\mathfrak{b}_+/\hbar U_\hbar\mathfrak{b}_+$ is equal to the quotient of $\mathbb{C}\langle h_i, D_j, x_i^+, i = 1, \ldots, n, j = 1, \ldots, 2n - r\rangle[[\hbar]]$ by the sum of $\hbar\mathbb{C}\langle h_i, D_j, x_i^+\rangle[[\hbar]]$ and the closed ideal generated by the relations $[h_i, e_j] = a_{ij}e_j$, $[D_i, e_j] = \delta_{ij}e_j$ and the quantum Serre relations (1). This sum is the same as that of $\hbar\mathbb{C}\langle h_i, D_j, x_i^+\rangle[[\hbar]]$ and the closed ideal generated by $[h_i, e_j] = a_{ij}e_j$, $[D_i, e_j] = \delta_{ij}e_j$ and the classical Serre relations. The quotient of $\mathbb{C}\langle h_i, D_j, x_i^+\rangle[[\hbar]]$ by this last space is equal to $U\mathfrak{b}_+$. This proves the second part of the Lemma.

The third part is immediate. ■

Proposition 2.1. *Let $\mathfrak{l}$ be a Lie algebra and let J be a two-sided ideal of $U\mathfrak{l}$ such that $\Delta_{U\mathfrak{l}}(J) \subset U\mathfrak{l} \otimes J + J \otimes U\mathfrak{l}$. Then $\mathfrak{j} = J \cap \mathfrak{l}$ is an ideal of the Lie algebra $\mathfrak{l}$ and we have $J = (U\mathfrak{l})\mathfrak{j} = \mathfrak{j}(U\mathfrak{l})$.*

Proof. We first show:

Lemma 2.4. *Let $\mathfrak{l}$ be a Lie algebra and let J be a left ideal of $U\mathfrak{l}$ such that $\Delta_{U\mathfrak{l}}(J) \subset J \otimes U\mathfrak{l} + U\mathfrak{l} \otimes J$. Let $\mathfrak{j}$ be the intersection $\mathfrak{l} \cap J$. Then J is equal to $(U\mathfrak{l})\mathfrak{j}$.*

Proof of Lemma. Denote by $(U\mathfrak{l})_n$ the subspace of $U\mathfrak{l}$ spanned by the monomials in elements of $\mathfrak{l}$ of degree $\leq n$. Let us set $\bar{\Delta}_{U\mathfrak{l}}(x) = \Delta_{U\mathfrak{l}}(x) - x \otimes 1 - 1 \otimes x$. We have $\bar{\Delta}_{U\mathfrak{l}}((U\mathrm{L})_n) \subset \sum_{p,q>0,p+q=n}(U\mathfrak{l})_p \otimes (U\mathfrak{l})_q$. Denote by J_n the intersection $J \cap (U\mathfrak{l})_n$. Then we have $\bar{\Delta}_{U\mathfrak{l}}(J_n) \subset \sum_{p,q>0,p+q=n} J_p \otimes (U\mathfrak{l})_q + (U\mathfrak{l})_p \otimes J_q$.

Let us show by induction that J_n is contained in $(U\mathfrak{l})_{n-1}\mathfrak{j}$. This is clear if $n = 1$; assume it is true at order $n - 1$. Let x be an element of J_n. Then $\bar{\Delta}_{U\mathfrak{l}}(x)$ is contained in $\sum_{p,q>0,p+q=n}(U\mathfrak{l})_{p-1}\mathfrak{j} \otimes (U\mathfrak{l})_q + (U\mathfrak{l})_p \otimes (U\mathfrak{l})_{q-1}\mathfrak{j}$.

Let $\bar{x}$ be the image of x in $(U\mathfrak{l})_n/(U\mathfrak{l})_{n-1}$. $(U\mathfrak{l})_n/(U\mathfrak{l})_{n-1}$ is isomorphic to the nth symmetric power $S^n\mathfrak{l}$. Let $\Delta_{S\mathfrak{l}}$ be the coproduct of the symmetric algebra $S\mathfrak{l}$, for which elements of degree 1 are primitive, and set $\bar{\Delta}_{S\mathfrak{l}}(\bar{x}) = \Delta_{S\mathfrak{l}}(\bar{x}) - \bar{x}\otimes 1 - 1\otimes\bar{x}$. Then $\bar{\Delta}(\bar{x})$ is contained in $\sum_{p,q>0,p+q=n}(S^{p-1}\mathfrak{l})\mathfrak{j}\otimes S^q\mathfrak{l} + S^p\mathfrak{l}\otimes(S^{q-1}\mathfrak{l})\mathfrak{j}$. It follows that $\bar{x}$ belongs to $(S^{n-1}\mathfrak{l})\mathfrak{j}$. The difference of x with some element of $(U\mathfrak{l})_{n-1}\mathfrak{j}$ therefore belongs to $(U\mathfrak{l})_{n-1}$, so that it belongs to J_{n-1} and by hypothesis to $(U\mathfrak{l})_{n-2}\mathfrak{j}$. Therefore, x belongs to $(U\mathfrak{l})_{n-1}\mathfrak{j}$. This proves the Lemma. ■

Let us prove Proposition 2.1. Lemma 2.4 implies that $J = (U\mathfrak{l})\mathfrak{j}$ and its analogue for right Hopf ideals implies that $J = \mathfrak{j}(U\mathfrak{l})$. Therefore, we have $(U\mathfrak{l})\mathfrak{j} = \mathfrak{j}(U\mathfrak{l})$. Let us fix x in $\mathfrak{l}$ and j in $\mathfrak{j}$, then $[x,j]$ belongs J; since it also belongs to $\mathfrak{l}$, $[x,j]$ belongs to $\mathfrak{j}$. Therefore $\mathfrak{j}$ is an ideal of $\mathfrak{l}$. ■

Let J be the kernel of the cocommutative Hopf algebras morphism p defined in Lemma 2.3, 3). Let us set $\mathfrak{j} = J\cap\mathfrak{b}_+$ and $\mathfrak{a} = \mathfrak{b}_+/\mathfrak{j}$. prop. 2.1 then implies

Lemma 2.5. *The Lie algebra structure on $\mathfrak{b}_+$ induces a Lie algebra structure on $\mathfrak{a}$. Moreover, $\mathcal{V}_0$ is isomorphic with $U\mathfrak{a}$, and p can be identified with the quotient map $U\mathfrak{b}_+ \to U\mathfrak{a}$.*

Define $\delta_{\mathcal{V}_0}$ as $\frac{\Delta_{\mathcal{V}}-\Delta'_{\mathcal{V}}}{\hbar}$ mod $\hbar$. $\delta_{\mathcal{V}_0}$ is a linear map from $\mathcal{V}_0$ to the antisymmetric part of its tensor square $\wedge^2\mathcal{V}_0$. It obeys the rules

$$(\Delta_{\mathcal{V}_0}\otimes id)\circ\delta_{\mathcal{V}_0} = (\delta_{\mathcal{V}_0}^{2\to 23} + \delta_{\mathcal{V}_0}^{2\to 13})\circ\Delta_{\mathcal{V}_0}, \tag{16}$$

$$\mathrm{Alt}(\delta_{\mathcal{V}_0}\otimes id)\circ\delta_{\mathcal{V}_0}) = 0,$$

$$\delta_{\mathcal{V}_0}(xy) = \delta_{\mathcal{V}_0}(x)\Delta_{\mathcal{V}_0}(y) + \Delta_{\mathcal{V}_0}(x)\delta_{\mathcal{V}_0}(y) \text{ for } x,y \text{ in } \mathcal{V}_0,$$

if $\delta_{\mathcal{V}_0}(y) = \sum_i y'_i\otimes y''_i$, we set $\delta_{\mathcal{V}_0}^{2\to 23}(x\otimes y) = \sum_i x\otimes y'_i\otimes y''_i$, and $\delta_{\mathcal{V}_0}^{2\to 13}(x\otimes y) = \sum_i y'_i\otimes x\otimes y''_i$. These rules are the co-Leibnitz, co-Jacobi and Hopf compatibility conditions; they mean that $(\mathcal{V}_0,\delta_{\mathcal{V}_0})$ is a Hopf co-Poisson algebra (see [8]).

Lemma 2.6. *$\delta_{\mathcal{V}_0}$ maps $\mathfrak{a}$ to $\wedge^2\mathfrak{a}$.*

Proof. Let a be an element of $\mathfrak{a}$ and set $\delta_{\mathcal{V}_0}(a) = \sum_i x_i\otimes y_i$, where (y_i) is a free family. Then (16) implies that $\Delta_{\mathcal{V}_0}(x_i)\otimes y_i = \sum_i x_i\otimes 1\otimes y_i + 1\otimes x_i\otimes y_i$, so that each x_i is primitive and therefore belongs to $\mathfrak{a}$. So $\delta_{\mathcal{V}_0}(a)$ belongs to $\mathfrak{a}\otimes\mathcal{V}_0$. Since $\delta_{\mathcal{V}_0}(a)$ is also antisymmetric, it belongs to $\wedge^2\mathfrak{a}$. ■

Call $\delta_{\mathfrak{a}}$ the map from $\mathfrak{a}$ to $\wedge^2\mathfrak{a}$ defined as the restriction of $\delta_{\mathcal{V}_0}$ to $\mathfrak{a}$. $(\mathfrak{a},\delta_{\mathfrak{a}})$ is then a Lie bialgebra, which means that $\delta_{\mathfrak{a}}$ is a 1-cocyle of $\mathfrak{a}$ with values in the antisymmetric part of the tensor square of its adjoint representation, satisfying the co-Jacobi identity $\mathrm{Alt}(\delta_{\mathfrak{a}}\otimes id)\delta_{\mathfrak{a}} = 0$.

Remark 2.7. The Hopf co-Poisson algebra and Lie bialgebra axioms were introduced by Drinfeld in [8]. Drinfeld showed that the quantization of a cocommutative Hopf algebra lead to such structures. He also stated that there is an equivalence of categories between the category of Hopf co-Poisson algebras and that of Lie bialgebras. Lemma 2.6 can therefore be viewed as the proof of one part of this statement (from Hopf co-Poisson to Lie bialgebras). It is also not difficult to prove the other part (from Lie bialgebras to Hopf co-Poisson).

• *Kac-Moody Lie algebras*

Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated with A. $\mathfrak{g}$ has generators $\bar{x}_i^{\pm}, \bar{h}_i$, $i = 1, \ldots, n$ and $\bar{D}_j, j = 1, \ldots, n-r$, and relations

$$[h, h'] = 0 \text{ if } h, h' \in \{\bar{h}_i, \bar{D}_j\}, \tag{17}$$

$$[\bar{h}_i, \bar{x}_{i'}^{\pm}] = \pm a_{ii'} \bar{x}_{i'}^{\pm}, \quad [\bar{D}_j, \bar{x}_i^{\pm}] = \pm \delta_{ij} \bar{x}_i^{\pm}, \tag{18}$$

$$\mathrm{ad}(\bar{x}_i^{\pm})^{1-a_{ij}}(\bar{x}_j^{\pm}) = 0, \tag{19}$$

$$[\bar{x}_i^{+}, \bar{x}_{i'}^{-}] = \delta_{ii'} \bar{h}_i, \text{ for all } i, i' = 1, \ldots, n, j = 1, \ldots, n-r.$$

Let $\mathfrak{h}$ and $\mathfrak{n}_{\pm}$ be the subalgebras of $\mathfrak{g}$ generated by $\{\bar{h}_i, \bar{D}_j\}$ and $\{\bar{x}_i^{\pm}\}$. Then $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let us set $\mathfrak{b}_{\pm} = \mathfrak{h} \oplus \mathfrak{n}_{\pm}$. $\mathfrak{n}_{\pm}$ and $\mathfrak{b}_{\pm}$ are the Lie algebras with generators $\{\bar{x}_i^{\pm}\}$ and $\{\bar{h}_i, \bar{D}_j, \bar{x}_i^{\pm}\}$ and relations (19) for $\mathfrak{n}_{\pm}$ and (17), (18) and (19) for $\mathfrak{b}_{\pm}$. $\mathfrak{g}$ is endowed with a nondegenerate bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$, which is determined by $\langle \bar{h}_i, \bar{h}_{i'} \rangle_{\mathfrak{g}} = d_{i'}^{-1} a_{ii'}$, $\langle \bar{x}_i^{+}, \bar{x}_{i'}^{-} \rangle_{\mathfrak{g}} = d_i^{-1} \delta_{ii'}$, $\langle \bar{h}_i, \bar{D}_j \rangle_{\mathfrak{g}} = d_i^{-1} \delta_{ij}$, and that its values for all other pairs of generators is zero (see [6, 19]).

• *Hopf algebras $U_\hbar \mathfrak{b}_{\pm}$*

Define $U_\hbar \mathfrak{b}_{\pm}$ as the algebras with generators $h_i^{\pm}, D_j^{\pm}$ and $x_i^{\pm}$, and relations

$$[h_i^{\pm}, x_{i'}^{\pm}] = \pm a_{ii'} x_{i'}^{\pm}, \quad [D_j^{\pm}, x_i^{\pm}] = \pm \delta_{ij} x_i^{\pm},$$

and relations (1), with e_i replaced by $x_i^{\pm}$. It is easy to see that the maps $e_i \mapsto x_i^{+}$ and $f_i \mapsto x_i^{-}$ define algebra inclusions of $U_\hbar \mathfrak{n}_{\pm}$ in $U_\hbar \mathfrak{b}_{\pm}$. We have Hopf algebra structures on $U_\hbar \mathfrak{b}_+$ and $U_\hbar \mathfrak{b}_-$, defined by

$$\Delta_{\pm}(h^{\pm}) = h^{\pm} \otimes 1 + 1 \otimes h^{\pm} \text{ for } h \in \{\bar{h}_i, \bar{D}_j\}, \quad \Delta_{\pm}(x_i^{\pm}) = x_i^{\pm} \otimes e^{\pm \hbar d_i h_i^{\pm}} + 1 \otimes x_i^{\pm}.$$

• *Comparison lemmas*

Recall that $\mathcal{V}$ is the direct sum of its graded components. Its component of degree zero is $\mathbb{C}[h_i^{\mathcal{V}}, D_j^{\mathcal{V}}][[\hbar]]$. Let $h_i^{\mathfrak{a}}, D_j^{\mathfrak{a}}$ be the images of $h_i^{\mathcal{V}}, D_j^{\mathcal{V}}$ in $\mathcal{V}_0$. We have an inclusion of $\mathbb{C}[h_i^{\mathfrak{a}}, D_j^{\mathfrak{a}}]$ in $\mathcal{V}_0$. It follows that the $h_i^{\mathfrak{a}}$ and $D_j^{\mathfrak{a}}$ are linearly independent and commute to each other.

On the other hand, as the elements $h_i^{\mathcal{V}}$ and $D_j^{\mathcal{V}}$ are primitive in $\mathcal{V}$, the $h_i^{\mathfrak{a}}$ and $D_j^{\mathfrak{a}}$ are also primitive; it follows that they belong to $\mathfrak{a}$, and we have

$$p(\bar{h}_i) = h_i^{\mathfrak{a}}, p(\bar{D}_j) = D_j^{\mathfrak{a}}, \quad \delta_{\mathfrak{a}}(h_i^{\mathfrak{a}}) = \delta_{\mathfrak{a}}(D_j^{\mathfrak{a}}) = 0. \tag{20}$$

The degree ϵ_i component of $\mathcal{V}$ is $\mathbb{C}[h_i, D_j][[\hbar]] \cdot [v_i]$, and the map $x \mapsto x[v_i]$ is a $\mathbb{C}[[\hbar]]$-module isomorphism from $\mathbb{C}[h_i, D_j][[\hbar]]$ to this component. Therefore, $[v_i]$ has a nonzero image in $\mathcal{V}/\hbar\mathcal{V} = \mathcal{V}_0$. Since we have

$$\Delta_{\mathcal{V}}([v_i]) = [v_i] \otimes e^{\hbar d_i h_i^{\mathcal{V}}} + 1 \otimes [v_i],$$

$[v_i]$ *mod* $\hbar$ is primitive in $\mathcal{V}_0$. Therefore, $[v_i]$ *mod* $\hbar$ belongs to $\mathfrak{a}$; call $v_i^{\mathfrak{a}}$ this element of $\mathfrak{a}$. It is clear that

$$\delta_{\mathfrak{a}}(v_i^{\mathfrak{a}}) = d_i v_i^{\mathfrak{a}} \wedge h_i^{\mathfrak{a}}. \tag{21}$$

On the other hand, $p_\hbar(\bar x_i^+) = [v_i]$ implies that

$$p(\bar x_i^+) = v_i^{\mathfrak a}. \tag{22}$$

Recall that we have a Lie bialgebra structure on $\mathfrak b_+$; it consists in a map $\delta_{\mathfrak b_+}$ from $\mathfrak b_+$ to $\wedge^2 \mathfrak b_+$, which is uniquely determined by the conditions $\delta_{\mathfrak b_+}(\bar x_i^+) = d_i \bar x_i^+ \wedge \bar h_i$, $\delta_{\mathfrak b_+}(\bar h_i) = \delta_{\mathfrak b_+}(\bar D_j) = 0$ and that it satisfies the 1-cocycle identity.

Lemma 2.8. *p is also a Lie bialgebra morphism from $(\mathfrak b_+, \delta_{\mathfrak b_+})$ to $(\mathfrak a, \delta_{\mathfrak a})$.*

Proof. This means that

$$(\wedge^2 p) \circ \delta_{\mathfrak b_+}(x) = \delta_{\mathfrak a} \circ p(x), \text{ for } x \text{ in } \mathfrak b_+. \tag{23}$$

For x equal to $\bar h_i$ and $\bar D_j$, (23) follows from (20). It follows from (21) and (22) that

$$\delta_{\mathfrak a} \circ p(\bar x_i^+) = \delta_{\mathfrak a}(v_i^{\mathfrak a}) = d_i v_i^{\mathfrak a} \wedge h_i^{\mathfrak a} = (\wedge^2 p)(d_i \bar x_i^+ \wedge \bar h_i) = \wedge^2 p(\delta_{\mathfrak a}(\bar x_i^+)),$$

so that (23) is also true for $x = \bar x_i^+$.

Since p is a Lie algebra morphism, both sides of (23) are 1-cocycles of $\mathfrak b_+$ with values in the antisymmetric part of the tensor square of $(\mathfrak a, \mathrm{ad}_{\mathfrak a} \circ p)$. Therefore, (23) holds on the subalgebra of $\mathfrak b_+$ generated by the $\bar h_i, \bar D_j$ and $\bar x_i^+$, which is $\mathfrak b_+$ itself. ∎

Denote by $\mathfrak h_{\mathfrak a}$ the subspace of $\mathfrak a$ spanned by the $h_i^{\mathfrak a}$ and $D_j^{\mathfrak a}$; it forms an abelian Lie subalgebra of $\mathfrak a$.

Since $p_\hbar(h_i) = h_i^\nu$ and $p_\hbar(D_j) = D_j^\nu$, the restriction of p to the Cartan subalgebra $\mathfrak h$ of $\mathfrak b_+$ is a Lie algebra isomorphism from $\mathfrak h$ to $\mathfrak h_{\mathfrak a}$.

Define, for α in $\mathfrak h^*$, the root subspace $\mathfrak a[\alpha]$ associated with α by

$$\mathfrak a[\alpha] = \{x \in \mathfrak a | [p(h), x] = \alpha(h)x, \text{ for all } h \text{ in } \mathfrak h\},$$

and as usual

$$\mathfrak b_+[\alpha] = \{x \in \mathfrak b_+ | [h, x] = \alpha(h)x, \text{ for all } h \text{ in } \mathfrak h\}.$$

Lemma 2.9. *$\mathfrak a$ is the direct sum of its root subspaces $\mathfrak a[\alpha]$, where α belongs to the set $\Delta_+ \cup \{0\}$ of roots of $\mathfrak b_+$. Each $\mathfrak a[\alpha]$ is finite dimensional. $\delta_{\mathfrak a}$ is a graded map from $\mathfrak a$ to $\wedge^2 \mathfrak a$, therefore the graded dual $\mathfrak a^*$ of $\mathfrak a$, defined as $\oplus_\alpha \mathfrak a[\alpha]^*$, has a Lie bialgebra structure.*

Proof. For any α in $\mathfrak h^*$, p maps $\mathfrak b_+[\alpha]$ to $\mathfrak a[\alpha]$. It follows that $\mathfrak a$ is the sum of the root subspaces $\mathfrak a[\alpha]$, where α belongs to the set of roots of $\mathfrak b_+$. That this sum is direct is proved as in the case of $\mathfrak b_+$: let α_i be the root such that $\bar x_i^+$ belongs to $\mathfrak b_+[\alpha_i]$. Then $\alpha_1, \ldots, \alpha_n$ form a basis of $\mathfrak h^*$ (they are the simple roots of $\mathfrak g$). Let $(\bar H_1, \ldots, \bar H_n)$ be the basis of $\mathfrak h$ dual to $(\alpha_1, \ldots, \alpha_n)$. Then for any family x_α of $\mathfrak a[\alpha]$ such that $\sum_\alpha x_\alpha = 0$, we have, by applying $\mathrm{ad}(p(\bar H_1))^k$ to this equality, $\sum_{(n_i) \in \mathbb N^n} n_1^k x_{\sum_i n_i \alpha_i} = 0$, which gives for any integer a_1, $\sum_{(n_i) \in \mathbb N^n | n_1 = a_1} x_{\sum_i n_i \alpha_i} = 0$; applying $\mathrm{ad}(p(\bar H_2))^k$, we get $\sum_{(n_i) \in \mathbb N^n | n_1 = a_1, n_2 = a_2} x_{\sum_i n_i \alpha_i} = 0$; finally each x_α vanishes.

That $\delta_{\mathfrak a}$ is graded follows from the fact that its restriction to $\mathfrak h_{\mathfrak a}$ vanishes and from the cocycle identity. ∎

Lemma 2.10. *p is a Lie bialgebra isomorphism.*

Proof. It follows from Lemma 2.8 that p^* is an injective Lie bialgebra morphism from $\mathfrak{a}^*$ in the graded dual $\mathfrak{b}_+^*$ of $\mathfrak{b}_+$. Recall that $\mathfrak{b}_+^*$ is isomorphic, as a Lie algebra, to $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$; this relies on the nondegeneracy of the invariant pairing between $\mathfrak{b}_+$ and $\mathfrak{b}_-$, itself a consequence of [6] (in what follows, we will denote by $\mathfrak{h}_-$ the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{b}_-$). We will show that the image of p^* contains a generating family of $\mathfrak{b}_-$.

Let us denote by $\mathfrak{h}_{\mathfrak{a}^*}$ the space of forms on $\mathfrak{a}$, which vanish on all the $\mathfrak{a}[\alpha], \alpha \neq 0$. The duality beween $\mathfrak{b}_+$ and $\mathfrak{b}_-$ identifies $\mathfrak{h}_-$ with the space of the forms on $\mathfrak{b}_+$ which vanish on $\mathfrak{n}_+ = \oplus_{\alpha\neq 0}\mathfrak{b}_+[\alpha]$. We have $p(\mathfrak{b}_+[\alpha]) \subset \mathfrak{a}[\alpha]$ for any α, therefore

$$p^*[(\oplus_{\alpha\neq 0}\mathfrak{a}[\alpha])^\perp] \subset (\oplus_{\alpha\neq 0}\mathfrak{b}_+[\alpha])^\perp,$$

which means that $p^*(\mathfrak{h}_{\mathfrak{a}^*}) \subset \mathfrak{h}_-$. Since p^* is injective and $\mathfrak{h}_{\mathfrak{a}^*}$ and $\mathfrak{h}_-$ have the same dimension, p^* induces an isomorphism between $\mathfrak{h}_{\mathfrak{a}^*}$ and $\mathfrak{h}_-$. It follows that the image of p^* contains $\mathfrak{h}_-$.

Since $\bar{x}_i^+$ belongs to $\mathfrak{b}_+[\alpha_i]$, the element $v_i^{\mathfrak{a}}$ of $\mathfrak{a}$ defined before Lemma 2.8 belongs to $\mathfrak{a}[\alpha_i]$. We have seen that $v_i^{\mathfrak{a}}$ is nonzero.

Let ξ_i be the element of $\mathfrak{a}^*$ which is 1 on $v_i^{\mathfrak{a}}$ and zero on each $\mathfrak{a}[\alpha]$, $\alpha \neq \alpha_i$. For x in $\mathfrak{b}_+[\alpha]$, $\alpha \neq \alpha_i$, $\langle p^*(\xi_i), x\rangle_{\mathfrak{b}_+\times\mathfrak{b}_-} = \langle \xi_i, p(x)\rangle_{\mathfrak{a}^*\times\mathfrak{a}} = 0$ because ξ_i vanishes on $\mathfrak{a}[\alpha]$. It follows that $p^*(\xi_i)$ has weight $-\alpha_i$ in $\mathfrak{b}_-$. On the other hand, $p^*(\xi_i)$ is nonzero, because p^* is injective, so it is a nonzero constant times $\bar{x}_i^-$.

Since the image of p^* contains $\mathfrak{h}_-$ and the $\bar{x}_i^-$, p^* is an isomorphism. ■

Lemma 2.11. *$p_\hbar$ mod $\hbar$ restricts to an isomorphism of $\mathbb{N}^n$-graded algebras from $U\mathfrak{n}_+$ to $\langle V\rangle/\hbar\langle V\rangle$.*

Proof. It follows from Lemma 2.10 that $p_\hbar$ mod $\hbar$ induces an isomorphism from $U\mathfrak{b}_+ = U_\hbar\mathfrak{b}_+/\hbar U_\hbar\mathfrak{b}_+$ to $\mathcal{V}_0 = \mathcal{V}/\hbar\mathcal{V}$. Therefore it induces an isomorphism from $U\mathfrak{n}_+$ to its image in $\mathcal{V}_0$. Since $U\mathfrak{n}_+$ coincides with the image of $U_\hbar\mathfrak{n}_+$ by the projection $U_\hbar\mathfrak{b}_+ \to U_\hbar\mathfrak{b}_+/\hbar U_\hbar\mathfrak{b}_+ = U\mathfrak{b}_+$, this image coincides with that of the composed map

$$U_\hbar\mathfrak{n}_+ \to U_\hbar\mathfrak{b}_+ \to \mathcal{V} \to \mathcal{V}_0. \tag{24}$$

The image of the composed map $U_\hbar\mathfrak{n}_+ \to U_\hbar\mathfrak{b}_+ \to \mathcal{V}$ is equal to $\langle V\rangle$. We have a $\mathbb{C}[[\hbar]]$-module isomorphism of $\mathcal{V}$ with $\langle V\rangle \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}[h_i^{\mathcal{V}}, D_j^{\mathcal{V}}][[\hbar]]$, so that $\hbar\mathcal{V} \cap \langle V\rangle = \hbar\langle V\rangle$. It follows that the image of $\langle V\rangle$ by $\mathcal{V} \to \mathcal{V}_0$ is $\langle V\rangle/\hbar\langle V\rangle$. Therefore the image of (24) is $\langle V\rangle/\hbar\langle V\rangle$. ■

Proof of Theorem 1.1. Assign degree ϵ_i to the generator e_i of $U_\hbar\mathfrak{n}_+$. Then $U_\hbar\mathfrak{n}_+$ is the direct sum of its homogeneous components $(U_\hbar\mathfrak{n}_+)[\alpha]$, $\alpha \in \mathbb{N}^n$, which are finitely generated $\mathbb{C}[[\hbar]]$-modules. As a $\mathbb{C}[[\hbar]]$-module, $(U_\hbar\mathfrak{n}_+)[\alpha]$ is therefore isomorphic to the direct sum $\oplus_{i=1}^{q_\alpha}\mathbb{C}[[\hbar]]/(\hbar^{n_i^{(\alpha)}}) \oplus \mathbb{C}[[\hbar]]^{p_\alpha}$ of its torsion part with a free module (see Lemma A1).

Lemma 2.3, 2) implies that $U_\hbar\mathfrak{n}_+[\alpha]/\hbar U_\hbar\mathfrak{n}_+[\alpha] = U\mathfrak{n}_+[\alpha]$ so

$$p_\alpha + q_\alpha = \mathrm{dim} U\mathfrak{n}_+[\alpha]. \tag{25}$$

On the other hand, $\langle V\rangle[\alpha]$ is a free finite dimensional module over $\mathbb{C}[[\hbar]]$, by Lemma 2.1 above, so it is isomorphic, as a $\mathbb{C}[[\hbar]]$-module, to $\mathbb{C}[[\hbar]]^{p'_\alpha}$. $p_\hbar$ restricts to a surjective $\mathbb{C}[[\hbar]]$-module morphism from $U_\hbar\mathfrak{n}_+[\alpha]$ to $\langle V\rangle[\alpha]$, therefore $p_\hbar$ maps the torsion part of $U_\hbar\mathfrak{n}_+[\alpha]$ to zero and

$$p_\alpha \geq p'_\alpha. \tag{26}$$

Moreover,

$$p'_\alpha = \mathrm{dim} U\mathfrak{n}_+[\alpha] \tag{27}$$

by Lemma 2.11. It follows from (25), (26) and (27) that $p_\alpha = p'_\alpha$ and $q_\alpha = 0$.

This means that $U_\hbar\mathfrak{n}_+$ has no torsion, and is isomorphic to $\langle V\rangle$. In view of Lemma 2.3, 2), this proves Theorem 1.1. This also proves Corollary 1.1. ■

2.2. Nondegeneracy of Hopf pairing (proof of Theorem 1.2).

Let (v_i^*) be the basis of V^* such that $\langle v_i^*, v_j\rangle = d_i^{-1}\delta_{ij}$. Assign to v_i^* the degree $-\epsilon_i$. Let $T(V^*)$ be the tensor algebra $\oplus_i (V^*)^{\otimes i}[[\hbar]]$. Define the braided tensor product strucutre on the tensor square of $T(V^*)$ according to (2). $T(V^*)$ is endowed with the braided Hopf structure defined by $\Delta_{T(V^*)}(v_i^*) = v_i^* \otimes 1 + 1 \otimes v_i^*$, for any $i = 1, \ldots, n$. We have a surjective braided Hopf algebra morphism from $T(V^*)$ to $U_\hbar\mathfrak{n}_-$, defined by $v_i^* \mapsto f_i$, for $i = 1, \ldots, n$.

Then we have a braided Hopf pairing

$$\langle \, , \, \rangle_{\mathrm{Sh}(V)\times T(V^*)} : \mathrm{Sh}(V) \times T(V^*) \to \mathbb{C}((\hbar)),$$

defined by the rules

$$\langle [v_{i_1}|\cdots|v_{i_k}], \xi_{i'_1}\cdots\xi_{i'_{k'}}\rangle_{\mathcal{S}\times U_\hbar\widetilde{\mathfrak{b}}_-} = \frac{1}{\hbar}\delta_{kk'}\prod_{j=1}^{k}\langle v_{i_j}, \xi_{i'_j}\rangle_{V\times V^*}. \tag{28}$$

The ideal of $T(V^*)$ generated by the quantum Serre relations is in the radical of this pairing (see e.g. [22], chap. 1; this is a consequence of q-binomial identities).

It follows that $\langle \, , \, \rangle_{\mathrm{Sh}(V)\times T(V^*)}$ induces a braided Hopf pairing

$$\langle \, , \, \rangle_{\mathrm{Sh}(V)\times U_\hbar\mathfrak{n}_-} : \mathrm{Sh}(V) \times U_\hbar\mathfrak{n}_- \to \mathbb{C}((\hbar)).$$

By Theorem 1.1, $U_\hbar\mathfrak{n}_+$ is a braided Hopf subalgebra of $\mathrm{Sh}(V)$. The restriction of $\langle \, , \, \rangle_{\mathrm{Sh}(V)\times U_\hbar\mathfrak{n}_-}$ to $U_\hbar\mathfrak{n}_+ \times U_\hbar\mathfrak{n}_-$ therefore induces a braided Hopf pairing between $U_\hbar\mathfrak{n}_+$ and $U_\hbar\mathfrak{n}_-$; since it coincides on generators with $\langle \, , \, \rangle_{U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-}$, it is equal to $\langle \, , \, \rangle_{U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-}$.

View $V^{\otimes k}$ as a subspace of $\mathrm{Sh}(V)$. Assign degree 1 to each element of V^* in $T(V^*)$; then $T(V^*)$ is a graded algebra; we denote by $T(V^*)^{(k)}$ is homogeneous component of degree k. The restriction of $\langle \, , \, \rangle_{\mathrm{Sh}(V)\times T(V^*)}$ to $V^{\otimes k}\times T(V^*)^{(k)}$ can be identified with the natural pairing of $V^{\otimes k}$ with $(V^*)^{\otimes k}$, which is nondegenerate. Therefore the annihilator of $T(V^*)$ in $\mathrm{Sh}(V)$ for $\langle \, , \, \rangle_{\mathrm{Sh}(V)\times T(V^*)}$ is zero. By Theorem 1.1, it follows that the annihilator of $U_\hbar\mathfrak{n}_-$ in $U_\hbar\mathfrak{n}_+$ for $\langle \, , \, \rangle_{U_\hbar\mathfrak{n}_\pm}$ is zero.

Since the pairing $\langle \, , \, \rangle_{U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-}$ is graded and the graded components of $U_\hbar\mathfrak{n}_+$ and $U_\hbar\mathfrak{n}_-$ have the same dimensions (as $\mathbb{C}[[\hbar]]$-modules), the pairing

$$\langle \, , \, \rangle_{U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-}$$

is nondegenerate. ■

2.3. The form of the R-matrix (proof of Proposition 1.1).

Let us endow $U_\hbar\mathfrak{g} = U_\hbar\mathfrak{b}_+ \otimes U_\hbar\mathfrak{n}_-$ with the double algebra structure such that $U_\hbar\mathfrak{b}_+ \to U_\hbar\mathfrak{g}$, $x_+ \mapsto x_+ \otimes 1$, and $U_\hbar\mathfrak{n}_- \to U_\hbar\mathfrak{g}$, $x_- \mapsto 1 \otimes x_-$ are algebra morphisms and, if $e_i^{\mathfrak{g}} = e_i \otimes 1$, $f_i^{\mathfrak{g}} = 1 \otimes f_i$, $h_i^{\mathfrak{g}} = h_i \otimes 1$ and $D_j^{\mathfrak{g}} = D_j \otimes 1$,

$$[e_i^{\mathfrak{g}}, f_j^{\mathfrak{g}}] = \delta_{ij}\frac{q^{d_i h_i^{\mathfrak{g}}} - q^{-d_i h_i^{\mathfrak{g}}}}{q^{d_i} - q^{-d_i}},$$

and

$$[h_i^{\mathfrak{g}}, f_j^{\mathfrak{g}}] = -a_{ij} f_j^{\mathfrak{g}}, \quad [D_j^{\mathfrak{g}}, f_j^{\mathfrak{g}}] = -\delta_{ij} f_j^{\mathfrak{g}}.$$

$U_\hbar\mathfrak{g}$ is endowed with a topological Hopf algebra structure $\Delta : U_\hbar\mathfrak{g} \mapsto U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g} = \lim_{\leftarrow N}(U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g})/\hbar^N(U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g})$, extending Δ_+ and Δ_- ([8]).

Let t_0 be the element of $\mathfrak{h} \otimes \mathfrak{h}$ corresponding to the restriction of the invariant pairing of $\mathfrak{g}$ to $\mathfrak{h}$ and let $\mathcal{R}[\alpha]$ be the element $\exp(\hbar t_0)P[\alpha]$ of $[\lim_{\leftarrow N}(U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g})/\hbar^N(U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g})][\hbar^{-1}]$. Then we have the equalities

$$\mathcal{R}[\alpha - \alpha_i](e_i^{\mathfrak{g}} \otimes q^{d_i h_i^{\mathfrak{g}}}) + \mathcal{R}[\alpha](1 \otimes e_i^{\mathfrak{g}}) = (q^{d_i h_i^{\mathfrak{g}}} \otimes e_i^{\mathfrak{g}})\mathcal{R}[\alpha] + (e_i^{\mathfrak{g}} \otimes 1)\mathcal{R}[\alpha - \alpha_i], \quad (29)$$

for any $i = 1, \dots, n$.

Lemma 2.12. *For any nonzero α in $\mathbb{N}^n$, $P[\alpha]$ belongs to $\hbar U_\hbar\mathfrak{n}_+ \otimes U_\hbar\mathfrak{n}_-$.*

Proof. Let us show this by induction on the height of α (we say that the height of $\alpha = (\alpha_i)_{1\le i\le n}$ is $\sum_{i=1}^n \alpha_i$). If α is a simple root α_i, $P[\alpha] = \hbar e_i^{\mathfrak{g}} \otimes f_i^{\mathfrak{g}}$, so that the statement holds when $\deg(\alpha) = 1$.

Assume that we know that $P[\alpha]$ belongs to $\hbar U_\hbar\mathfrak{n}_+ \otimes U_\hbar\mathfrak{n}_-$ for any α of height $< \nu$. Let α be of height ν. Let v be the $\hbar$-adic valuation of $P[\alpha]$, and assume that $v \le 0$. $\mathcal{R}[\alpha]$ belongs to $\hbar^v(U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g})$, and since $v \le 0$ the equality (29) takes place in $\hbar^v(U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g})$. Let us set $R_\alpha = \hbar^{-v}\mathcal{R}[\alpha]$ mod $\hbar$; R_α is an element of $U\mathfrak{b}_+ \otimes U\mathfrak{b}_-$. Since $\hbar^{-v}\mathcal{R}[\alpha - \alpha_i]$ is zero mod $\hbar$, (29) implies that R_α commutes with each $1 \otimes \bar{e}_i$.

Lemma 1.5 of [19] says that if a belongs to $\mathfrak{n}_-$ and commutes with each e_i, then a is zero. It follows that if x belongs to $U\mathfrak{n}_-$ and commutes with each $\bar{e}_i$, x is scalar; and if in addition x has nonzero degree, x is zero. Therefore R_α is zero.

It follows that $v \ge 1$, which proves the induction. ∎

It follows from [8] that the $\mathcal{R}[\alpha]$ satisfy the quasi-triangular identities

$$(\Delta \otimes id)\mathcal{R}[\alpha] = \sum_{\beta,\gamma\in\mathbb{N}^n, \beta+\gamma=\alpha} \mathcal{R}[\beta]^{(13)}\mathcal{R}[\gamma]^{(23)}, \quad (30)$$

$$(id \otimes \Delta)\mathcal{R}[\alpha] = \sum_{\beta,\gamma\in\mathbb{N}^n, \beta+\gamma=\alpha} \mathcal{R}[\beta]^{(13)}\mathcal{R}[\gamma]^{(12)}. \quad (31)$$

Let us set, for $\alpha \ne 0$, $r[\alpha] = R[\alpha]/\hbar$ mod $\hbar$; $r[\alpha]$ belongs to $U\mathfrak{b}_+ \otimes U\mathfrak{b}_-$. Dividing the equalities (30) and (31) by $\hbar$, we get $(\Delta_{U\mathfrak{b}_+} \otimes id)r[\alpha] = r[\alpha]^{(13)} + r[\alpha]^{(23)}$ and $(id \otimes \Delta_{U\mathfrak{b}_+})r[\alpha] = r[\alpha]^{(12)} + r[\alpha]^{(13)}$. Therefore $r[\alpha]$ belongs to $\mathfrak{b}_+ \otimes \mathfrak{b}_-$.

Moreover, (29) implies the identity

$$\delta(x)_{(\beta,\alpha-\beta)} = [r[\beta - \alpha], x \otimes 1] + +[r[\beta], 1 \otimes x]$$

for x in $\mathfrak{g}[\alpha]$, where we set $(\sum_i x_i \otimes y_i)_{(\alpha,\beta)} = \sum_i (x_i)_{(\alpha)} \otimes (y_i)_{(\beta)}$ and we denote by $x_{(\alpha)}$ the degree α component of an element x of $U\mathfrak{g}$.

It follows that $r[\alpha]$ is the element of $\mathfrak{n}_+[\alpha] \otimes \mathfrak{n}_-[-\alpha]$ corresponding to the invariant pairing of $\mathfrak{g}$.

Let us now prove by induction on k that if α belongs to $k\Delta_+ - (k-1)\Delta_+$, $P[\alpha]$ belongs to $\hbar^k U_\hbar \mathfrak{n}_+ \otimes U_\hbar \mathfrak{n}_-$ and

$$P[\alpha] = \frac{\hbar^k}{k!} \sum_{\alpha_1,\ldots,\alpha_k \in \Delta_+, \sum_{i=1}^k \alpha_i = \alpha} r[\alpha_1] \cdots r[\alpha_k] + o(\hbar^k).$$

Assume that the statement is proved up to order $k-1$ and let λ belong to $k\Delta_+ - (k-1)\Delta_+$. Then (30) and the induction hypothesis imply that

$$(\tilde{\Delta} \otimes id)(P[\lambda]) = \sum_{\alpha_1,\ldots,\alpha_k \in \Delta_+, \sum_{i=1}^k \alpha_i = \lambda; i_j} \sum_{l,l'>0, l+l'=k} \frac{\hbar^k}{l!l'!} \tag{32}$$
$$e_{\alpha_1,i_1} \cdots e_{\alpha_l,i_l} \otimes e_{\alpha_{l+1},i_{l+1}} \cdots e_{\alpha_k,i_k} \otimes f_{\alpha_1,i_1} \cdots f_{\alpha_k,i_k},$$

where $\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$. Let σ be any permutation of $\{1,\ldots,k\}$.

For any $\alpha_1,\ldots,\alpha_k$ in Δ_+, such that $\sum_{i=1}^k \alpha_i = \lambda$, we have

$$f_{\alpha_1,i_1} \cdots f_{\alpha_k,i_k} = f_{\alpha_{\sigma(1)},i_{\sigma(1)}} \cdots f_{\alpha_{\sigma(k)},i_{\sigma(k)}} + o(\hbar).$$

Indeed, the difference of both sides is a sum of products of the $[f_{\alpha_s,i_s}, f_{\alpha_t,i_t}]$ with elements of $\mathfrak{n}_+$; but $\alpha_s + \alpha_t$ does not belong to Δ_+ by hypothesis on λ, so $[f_{\alpha_s,i_s}, f_{\alpha_t,i_t}] = o(\hbar)$.

The right side of (32) can the be rewritten as

$$\sum_{\alpha_1,\ldots,\alpha_k \in \Delta_+, \sum_{i=1}^k \alpha_i = \lambda; i_j} \sum_{l,l'>0, l+l'=k} \frac{\hbar^k}{l!l'!} \frac{1}{\mathrm{card}\Sigma_{l,l'}}$$
$$\sum_{\sigma \in \Sigma_{l,l'}} e_{\alpha_{\sigma(1)},i_{\sigma(1)}} \cdots e_{\alpha_{\sigma(l)},i_{\sigma(l)}} \otimes e_{\alpha_{\sigma(l+1)},i_{\sigma(l+1)}} \cdots e_{\alpha_{\sigma(k)},i_{\sigma(k)}} \otimes f_{\alpha_1,i_1} \cdots f_{\alpha_k,i_k},$$

where $\Sigma_{l,l'}$ is the set of shuffle transformations of $((1,\ldots,l),(l+1,\ldots,l+l'))$. Therefore the right side of (32) is equal to

$$\frac{\hbar^k}{k!} \tilde{\Delta} \left(\sum_{\alpha_1,\ldots,\alpha_k \in \Delta_+, \sum_{i=1}^k \alpha_i = \lambda; i_j} e_{\alpha_1;i_1} \cdots e_{\alpha_k;i_k} \otimes f_{\alpha_1;i_1} \cdots f_{\alpha_k;i_k} \right) + o(\hbar^k).$$

Let v be the $\hbar$-adic valuation of $P[\lambda]$. Assume that $v < k$. Set $\bar{P}[\alpha] = \hbar^{-v} P[\alpha] \bmod \hbar$. Then if we call Δ_0 the coproduct of $U\mathfrak{n}_+$, and we set $\tilde{\Delta}_0(x) = \Delta_0(x) - x \otimes 1 - 1 \otimes x$, (32) gives $(\tilde{\Delta}_0 \otimes id)(\bar{P}[\alpha]) = 0$, so that $\bar{P}[\alpha]$ belongs to $\mathfrak{n}_+ \otimes U\mathfrak{n}_-$; since $\bar{P}[\alpha]$ also belongs to $U\mathfrak{n}_+[\alpha] \otimes U\mathfrak{n}_-[-\alpha]$ and α does not belong to Δ_+, $\bar{P}[\alpha]$ is zero, contradiction. Therefore $v \geq k$. Let us set $P'[\alpha] = \hbar^{-k} P[\alpha] \bmod \hbar$; we find that

$$(\tilde{\Delta}_0 \otimes id) \left(P'[\alpha] - \frac{1}{k!} \sum_{\alpha_1,\ldots,\alpha_k \in \Delta_+, \sum_{i=1}^k \alpha_i = \lambda; i_j} \bar{e}_{\alpha_1;i_1} \cdots \bar{e}_{\alpha_k;i_k} \otimes \bar{f}_{\alpha_1;i_1} \cdots \bar{f}_{\alpha_k;i_k} \right) = 0,$$

so that $P'[\alpha]$ belongs to

$$\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k\in\Delta_+,\sum_{i=1}^k\alpha_i=\lambda;i_j}\bar e_{\alpha_1;i_1}\cdots\bar e_{\alpha_k;i_k}\otimes\bar f_{\alpha_1;i_1}\cdots\bar f_{\alpha_k;i_k}+\mathfrak n_+\otimes U\mathfrak n_-;$$

as $\mathfrak n_+[\alpha]$ is zero, $P'[\alpha]$ is equal to

$$\frac{1}{k!}\sum_{\alpha_1,\ldots,\alpha_k\in\Delta_+,\sum_{i=1}^k\alpha_i=\lambda;i_j}\bar e_{\alpha_1;i_1}\cdots\bar e_{\alpha_k;i_k}\otimes\bar f_{\alpha_1;i_1}\cdots\bar f_{\alpha_k;i_k},$$

which proves the induction. ■

2.4. The generic case (proof of Corollaries 1.2 and 1.3). We have the equality $U_\hbar\mathfrak n_+\otimes_{\mathbb C[[\hbar]]}\mathbb C((\hbar))=U_{q'}\mathfrak n_+\otimes_{\mathbb C(q')}\mathbb C((\hbar))$, therefore the graded components of $U_{q'}\mathfrak n_+$ have the same dimension as those of $U_\hbar\mathfrak n_+$. Corollaries 1.2 and 1.3) follow.

3. Quantum current algebras of finite type (proofs for Sections 1.)

3.1. PBW theorem and comparison with Feigin-Odesskii algebra (proofs of Theorem 1.3 and Corollary 1.4).

• *Identification of algebras generated by the classical limits of quantum currents relations*

Recall that A is now assumed of finite type. Define $L\mathfrak b_+$ as the Lie subalgebra $(\mathfrak h\otimes\mathbb C[t^{-1}])\oplus(\mathfrak n_+\otimes\mathbb C[t,t^{-1}])$ of $\mathfrak g\otimes\mathbb C[t,t^{-1}]$.

Proposition 3.1. *Define $U_\hbar L\mathfrak b_+$ and $\widetilde U_\hbar L\mathfrak b_+$ as the algebra with generators $h_i[k],i=1,\ldots,n,k\le 0$ and $x_i^+[k],i=1,\ldots,n,k\in\mathbb Z$, and relations*

$$[h_i[k],h_j[l]]=0,\quad [h_i[k],x_j^+[l]]=\frac{q^{kd_ia_{ij}}-q^{-kd_ia_{ij}}}{2\hbar kd_i}x_j^+[k+l],$$

and relations (6) and (7) among the $x_j^+[k]$ (with e_i replaced by x_i^+), resp. (6) and (8). There are algebra isomorphisms from $U_\hbar L\mathfrak b_+/\hbar U_\hbar L\mathfrak b_+$ and $\widetilde U_\hbar L\mathfrak b_+/\hbar\widetilde U_\hbar L\mathfrak b_+$ to $UL\mathfrak b_+$, sending $h_i[k]$ to $\bar h_i\otimes t^k$ and $x_i^+[k]$ to $\bar x_i^+\otimes t^k$.

Proof. $U_\hbar L\mathfrak b_+/\hbar U_\hbar L\mathfrak b_+$ is the algebra with generators $\bar h_i[k],\bar e_i[l],1\le i,j\le n,k\le 0,l\in\mathbb Z$ and relations

$$[\bar h_i[k],\bar e_i[l]]=a_{ij}\bar e_i[k+l],$$

and

$$(z-w)[\bar e_i(z),\bar e_j(w)]=0, \tag{33}$$

$$\mathrm{Sym}_{z_1,\ldots,z_{1-a_{ij}}}\Big(\mathrm{ad}(\bar e_i(z_1))\cdots\mathrm{ad}(\bar e_i(z_{1-a_{ij}}))(\bar e_j(w))\Big)=0,$$

where $\bar e_i(z) = \sum_{k\in\mathbb Z} \bar e_i[k]z^{-k}$. It follows from (33) with $i = j$ that we have $[\bar e_i[n], \bar e_j[m]] = 0$ for all n, m. Therefore, $\mathrm{ad}(\bar e_i(z_1))\cdots \mathrm{ad}(\bar e_i(z_{1-a_{ij}}))(\bar e_j(w))$ is symmetric in the z_i, so that the last equation is equivalent to

$$\mathrm{ad}(\bar e_i(z_1))\cdots \mathrm{ad}(\bar e_i(z_{1-a_{ij}}))(\bar e_j(w)) = 0. \tag{34}$$

On the other hand, $\widetilde U_\hbar L\mathfrak b_+/\hbar \widetilde U_\hbar L\mathfrak b_+$ is the algebra with generators

$$\bar h_i[k]', \bar e_i[l]', 1\le i,j\le n, k\le 0, l\in\mathbb Z$$

and relations

$$[\bar h_i[k]', \bar e_i[l]'] = a_{ij}\bar e_i[k+l]',$$

and

$$(z-w)[\bar e_i(z)', \bar e_j(w)'] = 0, \tag{35}$$

and

$$(\mathrm{ad} e_i[0]')^{1-a_{ij}} e_j[k]' = 0. \tag{36}$$

The algebras presented by the pairs of relations (33) and (34) on one hand, and (35) and (36) on the other, are isomorphic. Indeed, (33) and (35) are equivalent, and (34) implies (36); on the other hand, (33) implies that $[e_i[0]', e_j[k+l]'] = [e_i[k]', e_j[l]']$, so that $[e_i[0]', [e_i[0]', e_j[k+k'+l]']] = [e_i[0]', [e_i[k]', e_j[k'+l]']] = [e_i[k]', [e_i[0]', e_j[k'+l]']]$, because the (33) implies that $[e_i[0]', e_i[k]'] = 0$, therefore $[e_i[0]', [e_i[0]', e_j[k+k'+l]']] = [e_i[k]', [e_i[k']', e_j[l]']]$; one then proves by induction that $(\mathrm{ad} e_i[0]')^p(e_j[k+k_1+\cdots+k_p]) = \mathrm{ad} e_i[k_1]'\cdots \mathrm{ad} e_i[k_p]'(e_j[k])$. With $p = 1-a_{ij}$, this relation shows that the $e_i[k]'$ satisfy (34).

If follows that if $\widetilde F_+$ is the Lie algebra defined in Proposition 1.5, both quotient algebras $U_\hbar L\mathfrak b_+/\hbar U_\hbar L\mathfrak b_+$ and $\widetilde U_\hbar L\mathfrak b_+/\hbar\widetilde U_\hbar L\mathfrak b_+$ are isomorphic to the crossed product of $U\widetilde F_+$ with the derivations $\tilde h_i[k]'$, defined by $\tilde h_i[k]'(\bar e_i[l]) = a_{ij}\bar e_i[k+l]$.

It is clear that there is a unique Lie algebra morphism j_+ from the Lie algebra $\widetilde F_+$ defined in Proposition 1.5 to $\mathfrak n_+\otimes\mathbb C[t,t^{-1}]$, sending $\bar e_i[k]$ to $\bar x_i^+\otimes t^k$. Let us prove that it is an isomorphism.

For this, let us define $\widetilde F$ as the Lie algebra with generators $\tilde x_i^\pm[k], \tilde h_i^\pm[k]$, $1\le i\le n, k\in\mathbb Z$, and relations given by the coefficients of the monomials in

$$(z-w)[\tilde x_i^\pm(z), \tilde x_j^\pm(w)] = 0, \quad \text{if} \quad x, y\in\{\tilde x_i^\pm\},$$

$$[\tilde h_i(z), \tilde h_j(w)] = 0,$$

$$[\tilde h_i(z), \tilde x_j^\pm(w)] = \pm a_{ij}\delta(z/w)\tilde x_j^\pm(w),$$

$$[\tilde x_i^+(z), \tilde x_j^-(w)] = \delta_{ij}\delta(z/w)\tilde h_i(z),$$

$$\mathrm{ad}(\tilde x_i^\pm(z_1))\cdots\mathrm{ad}(\tilde x_i^\pm(z_{1-a_{ij}}))(\tilde x_j^\pm(w)) = 0,$$

where we set $\tilde x(z) = \sum_{k\in\mathbb Z} x[k]z^{-k}$ for x in $\{\tilde x_i^\pm, \tilde h_i\}$.

Lemma 3.1. *(In this Lemma, $\mathfrak{g}$ may be an arbitrary Kac-Moody Lie algebra.) Let W be the Weyl group of $\mathfrak{g}$, and s_i be its elementary reflection associated to the root α_i. Then there is a unique action of W on $\widetilde{F}$ such that*

$$s_i(\widetilde{x}_i^{\pm}[k]) = \widetilde{x}_i^{\mp}[k],$$

$$s_i(\widetilde{x}_j^{\pm}[k]) = \mathrm{ad}(\widetilde{x}_i^{\pm}[0])^{-a_{ij}}(\widetilde{x}_j^{\mp}[k]), \quad \text{if} \quad j \neq i,$$

$$s_i(\widetilde{h}_j^{\pm}[k]) = \widetilde{h}_j^{\pm}[k] - a_{ij}\widetilde{h}_i^{\pm}[k].$$

Proof of Lemma. The proof follows the usual proof for Kac-Moody Lie algebras. For example, if j, k are different from i, we have

$$\begin{aligned}
&(z-w)[s_i(\widetilde{x}_j^{\pm}(z)), s_i(\widetilde{x}_k^{\pm}(w))] \\
&= (z-w)[\mathrm{ad}(\widetilde{x}_i^{\pm}[0])^{-a_{ij}}(\widetilde{x}_j^{\pm}(z)), \mathrm{ad}(\widetilde{x}_i^{\pm}[0])^{-a_{ik}}(\widetilde{x}_k^{\pm}(w))] \\
&= \mathrm{ad}(\widetilde{x}_i^{\pm}[0])^{-a_{ij}-a_{ik}}\left((z-w)[\widetilde{x}_j^{\pm}(z), \widetilde{x}_k^{\pm}(w)]\right)
\end{aligned}$$

because the Serre relations imply that $\mathrm{ad}(\widetilde{x}_i^{\pm}[0])^{1-a_{il}}(\widetilde{x}_l(u)) = 0$ for $j = k, l$ and $u = z, w$; therefore $(z-w)[s_i(\widetilde{x}_j^{\pm}(z)), s_i(\widetilde{x}_k^{\pm}(w))]$ is zero. ∎

Lemma 3.2. *There is a unique Lie algebra isomorphism j from $\widetilde{F}$ to $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, such that $j(\widetilde{x}[k]) = \bar{x} \otimes t^k$, for any x in $\{x_i^{\pm}, h_i\}$ and k in $\mathbb{Z}$.*

Proof. Let $\widetilde{F}_-$ be the Lie algebra with generators $\bar{x}_i^-[k], 1 \leq i \leq n, k \in \mathbb{Z}$ and relations (33) and (34), with $\bar{x}_i^+[k]$ replaced by $\bar{x}_i^-[k]$, and let $\widetilde{H}$ be the abelian Lie algebra with generators $\bar{h}_i[k], 1 \leq i \leq n, k \in \mathbb{Z}$. There are unique Lie algebra morphisms from $\widetilde{F}_{\pm}$ and $\widetilde{H}$ to $\widetilde{F}$, sending the $\bar{x}_i^{\pm}[k]$ to $\widetilde{x}_i^{\pm}[k]$ and the $\bar{h}_i[k]$ to $\widetilde{h}_i[k]$. These morphisms are injections, so that we will indentify $\widetilde{F}_{\pm}$ and $\widetilde{H}$ with their images in $\widetilde{F}$.

Moreover, let $F_{\pm}$ be the free Lie algebras with generators $x_i^{\pm}[k]^F$, $i = 1, \ldots, n$, k integer. Endow $F_{\pm} \oplus \widetilde{H}$ with the Lie algebra structure such that $\widetilde{H}$ is abelian, $F_{\pm}$ is a Lie subalgebra of $F_{\pm} \oplus \widetilde{H}$, and $[\widetilde{h}_i[k], x_i^{\pm}[l]^F] = \pm a_{ij} x_i^{\pm}[k+l]^F$

There are unique derivations $\Phi_{i,k}^{\mp}$ from $F_{\pm}$ to $F_{\pm} \oplus \widetilde{H}$ such that

$$\Phi_{i,k}^{\mp}(x_{i'}^{\pm}[l]) = \delta_{ii'}\widetilde{h}_i[k+l].$$

Let $I_{\pm}^F$ be the ideals of $F_{\pm}$ generated by relations (33) and (34); then computation shows that $I_{\pm}^F$ are preserved by the $\Phi_{i,k}^{\mp}$. It follows that $\widetilde{F}$ is the direct sum of its subspaces $\widetilde{F}_{\pm}$ and $\widetilde{H}$.

The rules $\deg(\widetilde{x}_i^{\pm}[k]) = (\pm\epsilon_i, k)$ and $\deg(\widetilde{h}_i[k]) = (0, k)$ define a Lie algebra grading of $\widetilde{F}$ by $\mathbb{Z}^n \times \mathbb{Z}$, because the relations of $\widetilde{F}$ are homogeneous for this grading. Clearly, $\dim\widetilde{F}_{\pm}[(\pm\epsilon_i, k)] = 1$ for any i and k, so that $\dim\widetilde{F}[(\pm\epsilon_i, k)] = 1$.

Let α be any root on $\mathfrak{g}$. Then there is some simple root $\pm\epsilon_i$ an element w of W such that $\alpha = w(\pm\epsilon_i)$. Then $\widetilde{F}[(\pm\epsilon_i, k)] = \widetilde{F}[(\alpha, k)]$ so that $\dim\widetilde{F}[(\alpha, k)] = 1$.

It is clear that the map j defined in the statement of the Lemma defines a Lie algebra morphism. Define a grading by $\mathbb{Z}^r \times \mathbb{Z}$ on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, by the rules $\deg(x \otimes t^k) = (\deg(x), k)$, for x a homogeneous (for the root grading) element of

$\mathfrak{g}$. Then j is a graded map. Moreover, if α is in $\pm\Delta_+$ and x is a nonzero element of $\mathfrak{g}$ od degree α, then x can be written as a $\sum \lambda_{i_1,\dots,i_p}[x^\pm_{i_1},[\dots,x^\pm_{i_p}]]$; then the image by j of $\sum \lambda_{i_1,\dots,i_p}[x^\pm_{i_1}[0],[\dots,x^\pm_{i_p}[k]]]$ is equal to $x\otimes t^k$; therefore the map induced by j from $\widetilde{F}[(\alpha,k)]$ to $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}][(\alpha,k)]$ is nonzero and therefore an isomorphism.

It follows that Ker$j$ is equal to $\sum_{\alpha\in\mathbb{Z}^n\setminus(\Delta_+\cup\{0\}\cup(-\Delta_+)),k\in\mathbb{Z}}\widetilde{F}[(\alpha,k)]$. Any element of $\widetilde{F}[(\alpha,k)]$ is a linear combination of brackets $[x^\pm_{i_l}[k_l],[\dots,x^\pm_{i_1}[k_1]]]$, with $\sum_{s=1}^l \pm\epsilon_{i_s}=\alpha$. Assume α is not a root of $\mathfrak{g}$ and let l' be the smallest integer such that $\sum_{s=1}^{l'+1}\pm\epsilon_{i_s}$ is not a root of $\mathfrak{g}$. Let us show that each $[x^\pm_{i_{l'+1}}[k_{l'+1}],[\dots,x^\pm_{i_1}[k_1]]]$ vanishes. It follows from the fact that j is an isomorphism when restricted to the parts of degree in $\Delta_+\cup(-\Delta_+)$ that we may write each $[x^\pm_{i_{l'}}[k_{l'}],[\dots,x^\pm_{i_1}[k_1]]]$ as a linear combination $\sum_s\lambda_s[x^\pm_{i_{l'}}[k^{(s)}(i_{l'})]\cdots x^\pm_{i_1}[k^{(s)}(i_1)]]$, where for each s, $i\mapsto k^{(s)}(i)$ is a map from $\{1,\dots,n\}$ to $\mathbb{Z}$, such that $k^{(s)}(i_{l'+1})=k_{l'+1}$. The defining relations for $\mathfrak{n}_\pm$ hold among the $x^\pm_1[k^{(s)}_1],\dots,x^\pm_n[k^{(s)}_n]$, therefore we have Lie algebra maps from $\mathfrak{n}_\pm$ to $\widetilde{F}^\pm$ sending each $x^\pm_i$ to $x^\pm_i[k^{(s)}_i]$. $[x^\pm_{i_{l'+1}}[k^{(s)}(i_{l'+1})]\cdots x^\pm_{i_1}[k^{(s)}(i_1)]]$ is the image of zero by one of these maps, and is therefore zero. It follows that $[x^\pm_{i_{l'}}[k_{l'}],[\dots,x^\pm_{i_1}[k_1]]]$ vanishes, so that Ker$j$ is zero. ∎

End of the proof of the Proposition. The restriction of j to $\widetilde{F}_+$ coincides with the map j_+ define before Lemma 3.1, therefore j_+ induces an isomorphism between $\widetilde{F}_+$ and $\mathfrak{n}_+\otimes\mathbb{C}[t,t^{-1}]$. ∎

• *Crossed product algebras $\mathcal{V}^L$ and $\mathcal{S}^L$*

For k an integer ≤ 0 and $1\leq i\leq n$, define endomorphisms $\widetilde{h_i[k]}$ of FO by

$$(\widetilde{h_i[k]}f)(t^{(i)}_l)=\left[\sum_{j=1}^n\sum_{l=1}^{k_j}\frac{q^{kd_ia_{ij}}-q^{-kd_ia_{ij}}}{2\hbar kd_i}(t^{(j)}_l)^k\right]f(t^{(i)}_l) \tag{37}$$

if $f\in \mathrm{FO}_{\mathbf{k}}$. The $\widetilde{h_i[k]}$ are derivations of FO. These derivations preserve LV, therefore they preserve $\langle LV\rangle$.

Define $\mathcal{V}^L$ and $\mathcal{S}^L$ as the crossed product algebras of $\langle LV\rangle$ and FO with the derivations $\widetilde{x[k]}$: $\mathcal{V}^L$, resp. $\mathcal{S}^L$ is equal to $\langle LV\rangle\otimes\mathbb{C}[h_i[k]^{\mathcal{V}^L},k\leq 0]$, resp. $\mathrm{FO}\otimes\mathbb{C}[h_i[k]^{\mathcal{V}^L},k\leq 0]$; both spaces are endowed with the products given by formula (28), where x now belongs to $\langle LV\rangle$, resp. FO and the X_s are replaced by $h_i[k]$, $k\leq 0$. Define $\hat{\mathcal{V}}^L$ and $\mathcal{S}^L$ as the partial $\hbar$-adic completions
$\mathbb{C}[h_i[k]^{\mathcal{V}^L},k<0]\otimes\langle LV\rangle\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}[h_i[0]^{\mathcal{V}^L}][[\hbar]]$ and
$\mathbb{C}[h_i[k]^{\mathcal{V}^L},k<0]\otimes\mathrm{FO}\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}[h_i[0]^{\mathcal{V}^L}][[\hbar]]$.

Lemma 3.3. *1) The rules* $\deg(h_i[k])=0,\deg(t^k_i)=\epsilon_i$ *define gradings of* $\langle LV\rangle$, FO, $\mathcal{V}^L$, $\mathcal{S}^L$, $\hat{\mathcal{V}}^L$ *and* $\hat{\mathcal{S}}^L$ *by* $\mathbb{N}^n$, *which are compatible with the inclusions. For* X *any of these algebras, we denote by* $X_{\mathbf{k}}$ *its homogeneous component of degree* $\mathbf{k}$. X *is therefore the direct sum of the* $X_{\mathbf{k}}$.

2) For any $\mathbf{k}$, $\langle LV\rangle_{\mathbf{k}}$ *is a free* $\mathbb{C}[[\hbar]]$*-modules with a countable basis.*

3) For any $\mathbf{k}$, $\mathcal{V}^L_{\mathbf{k}}$ *and* $\mathcal{S}^L$ *are free* $\mathbb{C}[[\hbar]][h_i[k]^{\mathcal{V}^L}]$*-modules; and* $\hat{\mathcal{V}}^L_{\mathbf{k}}$ *and* $\hat{\mathcal{S}}^L$ *are free* $\mathbb{C}[[\hbar]][h_i[k]^{\mathcal{V}^L},k<0]\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}[h_i[0]^{\mathcal{V}^L}][[\hbar]]$*-modules.*

Proof. 1) is clear. $\langle LV\rangle_{\mathbf{k}}$ is a $\mathbb{C}[[\hbar]]$-submodule of $\mathrm{FO}_{\mathbf{k}}$, and by Lemma A2, it is a free $\mathbb{C}[[\hbar]]$-module with a countable basis. This shows 2). 3) is a direct consequence of 2). ■

• *Ideals and completions*

Define for N positive integer, I_N as the left ideal of $\langle LV\rangle$ generated by the elements (t_i^k) of FO_{ϵ_i}, $k \geq N$, $i = 1, \ldots, n$. Define $\mathcal{I}_N$ and $\hat{\mathcal{I}}_N$ as the left ideals of $\mathcal{V}^L$ and $\hat{\mathcal{V}}^L$ generated by the same family. For $s \geq 0$, set $I_N^{(s)} = \hbar^{-s}(I_N \cap \hbar^s\langle LV\rangle)$, and $I_N^{(\infty)} = \cup_{s\geq 0} I_N^{(s)}$; define $\mathcal{I}_N^{(s)}$, $\mathcal{I}_N^{(\infty)}$ and $\hat{\mathcal{I}}_N^{(s)}$, $\hat{\mathcal{I}}_N^{(\infty)}$ in the same way.

For any integer a, define $LV^{\geq a}$ as the subspace of FO equal to the direct sum $\oplus_{i=1}^n t_i^a \mathbb{C}[[\hbar]][t_i]$ and let $\langle LV^{\geq a}\rangle$ be the subalgebra of FO generated by $LV^{\geq a}$. Define $I_N^{(0),\geq a}$ as the left ideal of $\langle LV^{\geq a}\rangle$ generated by the $t_i^k, k \geq N$ and $I_N^{\geq a}$ as the ideal of $\langle LV^{\geq a}\rangle$ formed of the elements x such that for some $k \geq 0$, $\hbar^k x$ belongs to $I_N^{(0),\geq a}$.

For any integer a and $\mathbf{k}$ in $\mathbb{N}^r$, define $\mathrm{FO}^{\geq a}$ as the subspace of $\mathrm{FO}_{\mathbf{k}}$ consisting of the rational functions

$$g(t_\alpha^{(i)}) = \frac{1}{\prod_{i=1}^n \prod_{1\leq\alpha\leq k_i, 1\leq\beta\leq k_j (t_\alpha^{(i)} - t_\beta^{(j)})}} f(t_\alpha^{(i)}),$$

where the $f(t_\alpha^{(i)})$ have degree $\geq a$ in each variable $t_\alpha^{(i)}$ and the total degree of g is $\geq (\sum_i k_i)a$. Set $\mathrm{FO}^{\geq a} = \oplus_{\mathbf{k}\in\mathbb{N}^n} \mathrm{FO}_{\mathbf{k}}^{\geq a}$. Then $\mathrm{FO}^{\geq a}$ is a subalgebra of FO.

Define $\mathcal{I}_{(N)}^{\geq a}$ as the set of elements of $\mathrm{FO}^{(\geq a)}$, where $f(t_\alpha^{(i)})$ has total degree N in the variables $t_\alpha^{(i)}$, and let $\mathcal{I}_N^{\geq a}$ be the direct sum $\oplus_{k\geq N} \mathcal{I}_N^{\geq a}$.

Lemma 3.4. *For $(J_N)_{N>0}$ a family of left ideals of some algebra A, say that $(J_N)_{N>0}$ has property $(*)$ if for any integer $N > 0$ and element a in A, there is an integer $k(a, N) > 0$ such that $J_N a \subset J_{k(a,N)}$ for any N large enough, and $k(N, a)$ tends to infinity with N, a being fixed. Then the inverse limit $\lim_{\leftarrow N} A/J_N$ has an algebra structure.*

Say that J_N has property $(**)$ if for any integer $N > 0$ and element a in A, there is are integer $k'(a, N)$ and $k''(a, N) > 0$ such that $J_N a \subset J_{k'(a,N)}$ and $aJ_N \subset J_{k''(a,N)}$ for any N large enough, and $k(N, a)$ tends to infinity with N, a being fixed. In that case also, the inverse limit $\lim_{\leftarrow N} A/J_N$ has an algebra structure.

1) The family $(I_N^{(\infty)})_{N>0}$ of ideals of $\langle LV\rangle$ has property $(*)$;

2) the family $(\mathcal{I}_N^{(\infty)})_{N>0}$ of ideals of $\mathcal{V}^L$ has property $(*)$;

3) the family $(\hat{\mathcal{I}}_N^{(\infty)})_{N>0}$ of ideals of $\hat{\mathcal{V}}^L$ has property $(*)$;

4) the family $(I_N^{\geq a})_{N>0}$ of ideals of $\langle LV^{\geq a}\rangle$ has property $(*)$;

5) the family $(\mathcal{I}_N^{\geq a})_{N>0}$ of ideals of $\mathrm{FO}^{\geq a}$ has property $(**)$.

Proof. Set for any a in A and $N > 0$, $k'(a, N) = \inf\,\{k | J_N a \subset J_k\}$; then $k'(N, a)$ tends to infinity with N, a being fixed and we have $k'(N, a') \geq \inf\,(k(N, a), p)$ if a' belongs to $a + J_p$.

An element of $\lim_{\leftarrow N} A/J_N$ is a family $(a_N)_{N>0}$, $a_N \in A/J_N$, such that $a_{N+1} + J_N = a_N$. For $a = (a_N)_{N>0}$ and $b = (b_N)_{N>0}$ in $\lim_{\leftarrow N} A/J_N$, choose β_N

in b_N and let $N'(N,\beta_N)$ be the smallest integer N' such that $k'(N',\beta_N) \geq N$; $N'(N,\beta_N)$ is independent of the choice of β_N, we denote it $N'(N,b)$. Choose then α_N in $a_{N'(N,b)}$; then $\alpha_N\beta_N + J_N$ is independent of the choice of α_N and β_N; one checks that $\alpha_{N+1}\beta_{N+1} + J_N = \alpha_N\beta_N + J_N$, so that $(\alpha_N\beta_N + J_N)_{N>0}$ defines an element of $\lim_{\leftarrow N} A/J_N$. The product ab is defined to be this element. The construction is similar in the case of property $(**)$.

1) The equality

$$t_i^k * t_j^l = q^{-nd_ia_{ij}} t_i^{k-n} * t_j^{l+n} + q^{-d_ia_{ij}} t_j^l * t_i^k + \sum_{n'=1}^{n-1} (q^{-(n'+1)d_ia_{ij}} - q^{-(n'-1)d_ia_{ij}}) t_j^{l+n'} * t_i^{k-n'} - q^{-(n-1)d_ia_{ij}} t_j^{l+n} * t_i^{k-n},$$

where $n = N - l$ and $k \geq 2N - l$, implies that if $k \geq 2N - l$, $t_i^k * t_j^l$ belongs to I_N. It follows that $I_{2N-l} * t_j^l \subset I_N$. Set then $k(t_j^l, N) = [\frac{1}{2}(N + l/2)] + 2$; for a in $\langle LV \rangle$, and any decomposition dec of a as a sum $\sum_{(j_i),(l_i)} \lambda_{(j_i),(l_i)} t_{j_1}^{l_1} * \cdots * t_{j_p}^{l_p}$, define $k(a, N, dec)$ as the smallest of integers $k(t_{j_p}^{l_p}, \cdots, k(t_{j_1}^{l_1}, N))$; finally, define $k(a, N)$ as the largest of all $k(a, N, dec)$. The family $(I_N)_{N>0}$ has property $(*)$, with this function $k(a, N)$. Then for any a in $\langle LV \rangle$, $(I_N \cap \hbar^s \mathcal{V}^L) a \subset \hbar^s \mathcal{V}^L \cap I_{k(a,N)}$, so that $I_N^{(s)} a \subset I_{k(a,N)}^{(s)}$. It follows that we have also $I_N^{(\infty)} a \subset I_{k(a,N)}^{(\infty)}$, so that the families $(I_N^{(s)})_{N>0}$ and $(I_N^{(\infty)})_{N>0}$ have property $(*)$.

2) follows from the fact that $I_N h_j[l]^{\mathcal{V}^L} \subset I_{N+l}$.

3) follows from the fact that for a any element of $\mathbb{C}[h_i[0]^{\mathcal{V}^L}][[\hbar]]$, we have $\hat{\mathcal{I}}_N^{(\infty)} a \subset \hat{\mathcal{I}}_N^{(\infty)}$.

4) is proved in the same way as 1) ans 2).

5) For $\mathbf{k}$ in $\mathbb{N}^n$, set $|\mathbf{k}| = \sum_{i=1}^n k_i$. Then if f belongs to $\mathrm{FO}_{\mathbf{k}}^{\geq a}$, we have $f * \mathcal{I}_N^{\geq a} \subset \mathcal{I}_{N+a|\mathbf{k}|}^{\geq a}$ and $\mathcal{I}_N^{\geq a} * f \subset \mathcal{I}_{N+a|\mathbf{k}|}^{\geq a}$. Therefore the family $(\mathcal{I}_N^{\geq a})_N$ has property $(**)$. ■

It follows that the inverse limits

$$\lim_{\leftarrow N} \langle LV \rangle / I_N^{(\infty)}, \ \lim_{\leftarrow N} \mathcal{V}^L / \mathcal{I}_N^{(\infty)}, \ \lim_{\leftarrow N} \hat{\mathcal{V}}^L / \hat{\mathcal{I}}_N^{(\infty)},$$

$$\lim_{\leftarrow N} \langle LV^{\geq a} \rangle / I_N^{\geq a} \text{ and } \lim_{\leftarrow N} \mathrm{FO}^{\geq a} / \mathcal{I}_N^{\geq a}$$

have algebra structures. Moreover, as we have $\mathcal{I}_N^{(\infty)} \cap \langle LV \rangle = I_N^{(\infty)}$ and $\hat{\mathcal{I}}_N^{(\infty)} \cap \mathcal{V}^L = \mathcal{I}_N^{(\infty)}$, we have natural algebra inclusions

$$\lim_{\leftarrow N} \langle LV \rangle / I_N^{(\infty)} \subset \lim_{\leftarrow N} \mathcal{V}^L / \mathcal{I}_N^{(\infty)} \subset \lim_{\leftarrow N} \hat{\mathcal{V}}^L / \hat{\mathcal{I}}_N^{(\infty)}.$$

Moreover, there exists a function $\phi(\mathbf{k}, N)$, tending to infinity with N, such that $\mathcal{I}_{N,\mathbf{k}}^{\geq a} \cap \langle LV^{\geq a} \rangle \subset I_{\phi(\mathbf{k},n)}$. Indeed, if the k_i are $\geq a$ and $t_{i_1}^{k_1} * \cdots * t_{i_l}^{k_l}$ belongs to $\mathcal{I}_{N,\mathbf{k}}^{\geq a}$ ($l = |\mathbf{k}|$), then $k_1 + \cdots + k_l \geq N$ so that one of the k_i is $\geq N/l$. The statement then follows from the proof of the above Lemma, 1). It follows that we have an algebra inclusion

$$\lim_{\leftarrow N} \langle LV^{\geq a} \rangle / I_N^{\geq a} \subset \lim_{\leftarrow N} \mathrm{FO}^{\geq a} / \mathcal{I}_N^{\geq a}.$$

If a family $(J_N)_{N>0}$ of left ideals of the algebra A has property $(*)$, and B is any algebra, the family $(J_N\otimes B)_{N>0}$ also satisfies $(*)$; therefore the inverse limit $\lim_{\leftarrow N}(A\otimes B)/(J_N\otimes B)$ has an algebra structure. It follows that we have algebra structures on $\lim_{\leftarrow N}\langle LV\rangle\otimes A/I_N^{(\infty)}\otimes A$, $\lim_{\leftarrow N}\mathcal{V}^L\otimes A/\mathcal{I}_N^{(\infty)}\otimes A$ and $\lim_{\leftarrow N}\hat{\mathcal{V}}^L\otimes A/\hat{\mathcal{I}}_N^{(\infty)}\otimes A$ for any algebra A.

• *Topological Hopf structures on $\mathcal{V}^L$ and $\mathcal{S}^L$*

For $i=1,\ldots,n$ and $l\geq 0$, define $K_i[-l]$ as the element of $\mathcal{V}^L$ (or $\mathcal{S}^L$)

$$K_i[-l]=e^{-\hbar d_ih_i[0]^{\mathcal{V}^L}}S_l(-2\hbar d_ih_i[k]^{\mathcal{V}^L},k<0),$$

where $S_l(z_{-1},z_{-2},\ldots)$ are the Schur polynomials in variables $(z_i)_{i<0}$, which are determined by the relation $\exp(\sum_{i<0}z_it^{-i})=\sum_{l\leq 0}S_l(z_k)t^{-l}$.

Proposition 3.2. *There is unique graded algebra morphism $\Delta_{\mathcal{S}^L}$ from $\mathcal{S}^L$ to $\lim_{\leftarrow N}(\mathcal{S}^L\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L)/(\mathcal{I}_N^{\mathcal{S}}\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L)$, such that*

$$\Delta_{\mathcal{S}^L}(h_i[k])=h_i[k]\otimes 1+1\otimes h_i[k]$$

for $1\leq i\leq n$ and $k\leq 0$, and its restriction to $\mathcal{S}^L_{\mathbf{k}}$ is the direct sum of the maps $\Delta^{\mathbf{k}',\mathbf{k}''}_{\mathcal{S}^L}:\mathrm{FO}^{\geq a}_{\mathbf{k}}\to\lim_{\leftarrow N}(\mathcal{I}^{\geq a}_{\mathbf{k}'}\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L_{\mathbf{k}''})/[\mathrm{FO}^{\geq a}_{\mathbf{k}',N}\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L_{\mathbf{k}''}]$, where $\mathbf{k}=\mathbf{k}'+\mathbf{k}''$, defined by

$$\begin{aligned}&\Delta^{\mathbf{k}',\mathbf{k}''}_{\mathcal{S}^L}(P)\\&=\sum_{p_1,\ldots,p_{N'}\geq 0}\left(\prod_{i=1}^{N'}u_i^{p_i}P'_\alpha(u_1,\ldots,u_{N'})\right)\otimes\left(P''_\alpha(u_{N'+1},\ldots,u_N)\prod_{i=1}^{N'}K_{\epsilon(i)}[-p_i]\right),\end{aligned}$$

where $N'=\sum_{i=1}^n k'_i$, $N''=\sum_{i=1}^n k''_i$, $N=N'+N''$, the arguments of the functions in $\mathrm{FO}_{\mathbf{k}'}$ and $\mathrm{FO}_{\mathbf{k}''}$ are respectively $(t_j^{(i)})_{1\leq i\leq n,1\leq j\leq k'_i}$ and $(t_j'^{(i)})_{1\leq i\leq n,1\leq j\leq k''_i}$; we set

$$(u_1,\ldots,u_{k'_1})=(t_1^{(1)},\ldots,t_{k'_1}^{(1)}),\ldots,(u_{k'_1+\ldots+k'_{n-1}+1},\ldots,u_{N'})=(t_1^{(n)},\ldots,t_{k'_n}^{(n)}),$$

$$(u_{N'+1},\ldots,u_{N'+k''_1})=(t_1'^{(1)},\ldots,t_{k''_1}'^{(1)}),\ldots,$$

$$(u_{N'+k''_1+\ldots+k''_{n-1}+1},\ldots,u_N)=(t_1'^{(n)},\ldots,t_{k''_n}'^{(n)}),$$

$$t^{(i)}_{k'_i+j}=t_j'^{(i)},\quad j=1,\ldots,k''_i,$$

$$(t_1,\ldots,t_{k_1})=(t_1^{(1)},\ldots,t_{k_1}^{(1)}),\ldots,(t_{k_1+\ldots+k_{n-1}+1},\ldots,t_N)=(t_1^{(n)},\ldots,t_{k_n}^{(n)}),$$

and

$$\begin{aligned}&\sum_\alpha P'_\alpha(u_1,\ldots,u_{N'})P''_\alpha(u_{N'+1},\ldots,u_N)\\&=P(t_1,\ldots,t_N)\prod_{1\leq l\leq N',N'+1\leq l'\leq N}\frac{u_{l'}-u_l}{q^{\langle\epsilon(l),\epsilon(l')\rangle}u_{l'}-u_l},\end{aligned}\tag{38}$$

for l in $\{1,\ldots,N'\}$, resp. $\{N'+1,\ldots,N\}$, $\epsilon(l)$ is the element of $\{1,\ldots,n\}$ such that $u_l=t_j^{(\epsilon(l))}$, resp. $u_l=t_j'^{(\epsilon(l))}$ for some j; in (38), the ratios are expanded for $u_l\ll u_{l'}$.

Proof. Define $\mathrm{FO}^{(2)}$ as the $(\mathbb{N}^r)^2$-graded $\mathbb{C}[[\hbar]]$-module

$$\mathrm{FO}^{(2)} = \oplus_{\mathbf{k},\mathbf{k}'\in\mathbb{N}^r} \mathrm{FO}^{(2)}_{\mathbf{k},\mathbf{k}'},$$

where

$$\mathrm{FO}^{(2)}_{\mathbf{k},\mathbf{k}'} = \frac{1}{\prod_{\substack{i<j \\ 1\le\alpha\le k_i \\ 1\le\beta\le k_j}} (t^{(i)}_\alpha - t^{(j)}_\beta) \prod_{\substack{i<j \\ 1\le\alpha\le k'_i \\ 1\le\beta\le k'_j}} (u^{(i)}_\alpha - u^{(j)}_\beta) \prod_{\substack{i,j \\ 1\le\alpha\le k_i \\ 1\le\beta\le k'_j}} (t^{(i)}_\alpha - u^{(j)}_\beta)} \cdot$$
$$\cdot\mathbb{C}[[\hbar]][(t^{(i)}_j)^{\pm 1}, (u^{(i)}_{j'})^{\pm 1}, j = 1,\ldots,k_i, j' = 1,\ldots,k'_i]^{\prod_{i=1}^n \mathfrak{S}_{k_i}\times\mathfrak{S}_{k'_i}},$$

where the groups $\mathfrak{S}_{k_i}$ and $\mathfrak{S}_{k'_i}$ act by permutation of the variables $t^{(i)}_j$ and $u^{(i)}_j$. Define on $\mathrm{FO}^{(2)}$, the graded composition map $*$ as follows: for f in $\mathrm{FO}^{(2)}_{\mathbf{k},\mathbf{k}'}$ and g in $\mathrm{FO}^{(2)}_{\mathbf{l},\mathbf{l}'}$,

$$(f*g)(t_1,\ldots,t_P,u_1,\ldots,u_{P'}) = \mathrm{Sym}_{t^{(1)}_j}\cdots\mathrm{Sym}_{t^{(n)}_j}\mathrm{Sym}_{u^{(1)}_j}\cdots\mathrm{Sym}_{u^{(n)}_j}$$
$$\Bigg(\prod_{1\le i\le N, N+1\le j\le P} \frac{q^{\langle\epsilon_t(i),\epsilon_t(j)\rangle}t_i - t_j}{t_i - t_j} \prod_{1\le i\le N', N+1\le j\le P} \frac{q^{\langle\epsilon_u(i),\epsilon_t(j)\rangle}u_i - t_j}{u_i - t_j}$$
$$\prod_{1\le i\le N, N'+1\le j\le P'} \frac{q^{\langle\epsilon_t(i),\epsilon_u(j)\rangle}t_i - u_j}{t_i - u_j} \prod_{1\le i\le N', N'+1\le j\le P'} \frac{q^{\langle\epsilon_u(i),\epsilon_u(j)\rangle}u_i - u_j}{u_i - u_j}$$
$$f(t_1,\ldots,t_N,u_1,\ldots,u_M)g(t_{N+1},\ldots,t_P,u_{M+1},\ldots,u_P)\Bigg),$$

where $N = \sum_i k_i, N' = \sum_i k'_i, M = \sum_i l_i, M' = \sum_i l'_i, P = N+M, P' = N'+M'$, and $(t_1,\ldots,t_{k_1+k'_1}) = (t^{(1)}_1,\ldots,t^{(1)}_{k_1+k'_1})$, ..., $(t_{k_1+\cdots+k'_{n-1}+1},\ldots,t_P) = (t^{(n)}_1,\ldots,t^{(n)}_{k_n+k'_n})$, and $(u_1,\ldots,u_{l_1+l'_1}) = (u^{(1)}_1,\ldots,u^{(1)}_{l_1+l'_1})$, etc.,

$$(u_{l_1+\cdots+l'_{n-1}+1},\ldots,u_{P'}) = (u^{(n)}_1,\ldots,u^{(n)}_{l_n+l'_n}).$$

We set $\epsilon_x(\alpha) = i$ if $x_\alpha = x^{(i)}_j$ for some j (x is t or u). It is easy to check that defines an algebra structure on $\mathrm{FO}^{(2)}$.

Let Δ_{FO} be the linear map from FO to $\mathrm{FO}^{(2)}$, which maps $\mathrm{FO}_{\mathbf{k}}$ to $\oplus_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathrm{FO}^{(2)}_{\mathbf{k}',\mathbf{k}''}$ as follows

$$\Delta_{\mathrm{FO}}(P)(t^{(1)}_1,\ldots,t^{(n)}_{k'_n},u^{(1)}_1,\ldots,u^{(n)}_{k''_n}) = P(t^{(1)}_1,\ldots,t^{(1)}_{k'_1},u^{(1)}_1,\ldots,u^{(1)}_{k''_1},t^{(2)}_1,\ldots,u^{(n)}_{k''_n}).$$

Then it is immediate that Δ_{FO} defines an algebra morphism.

Consider the $(\mathbb{N}^n)^2$-graded map

$$\mu : \mathrm{FO}^{(2)} \to \cup_a \lim_{\leftarrow N}(\mathrm{FO}^{\ge a}\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L)/[\mathrm{FO}^{\ge a}_N\otimes_{\mathbb{C}[[\hbar]]}\hat{\mathcal{S}}^L]$$

defined by

$$\mu(P) = \sum_{p_1,\ldots,p_{N'}\ge 0} (t_1^{p_1}\cdots t_{N'}^{p_{N'}}\bar{P}_\alpha)\otimes(\bar{P}'_\alpha K_{\epsilon_t(1)}[-p_1]\cdots K_{\epsilon_t(N')}[-p_{N'}])$$

if $P(t_1,\dots,t_N,u_1,\dots,u_{N'})$ belongs to $\mathrm{FO}^{(2)}_{\mathbf{k},\mathbf{k}'}$, and we set

$$P(t_1,\dots,u_{N'})\prod_{1\le l\le N,1\le l'\le N'}\frac{t_l-u_{l'}}{q^{\langle\epsilon_t(l),\epsilon_u(l')\rangle}t_l-u_{l'}}=\sum_\alpha \bar P_\alpha(t_1,\dots,t_N)\bar P'_\alpha(u_1,\dots,u_{N'})$$

(the expansion is for $u_{l'}\ll t_l$).

Let P,Q belong to $\mathrm{FO}_{\mathbf{k},\mathbf{l}}$ and $\mathrm{FO}_{\mathbf{k}',\mathbf{l}'}$. $\mu(P)\mu(Q)$ is equal to

$$\sum_{\alpha,\beta}\left(t_1^{p_1}\cdots t_N^{p_N}P_\alpha(t_1,\cdots,t_N)\otimes P'_\alpha(u_1,\dots,u_P)K_{\epsilon_t(1)}[-p_1]\cdots K_{\epsilon_t(N)}[-p_N]\right) \tag{39}$$
$$\cdot\left(t_1'^{p'_1}\cdots t_{N'}'^{p'_{N'}}Q_\alpha(t'_1,\cdots,t'_{N'})\otimes Q'_\alpha(u'_1,\dots,u'_{P'})K_{\epsilon_{t'}(1)}[-p'_1]\cdots K_{\epsilon_{t'}(1)}[-p'_{N'}]\right);$$

since

$$(\sum_{i\ge 0}t^iK_\alpha[-i])Q(u_1,\dots,u_P)=\prod_{i=1}^P\frac{u_i-q^{\langle\epsilon_u(i),\alpha\rangle}t}{q^{\langle\epsilon_u(i),\alpha\rangle}u_i-t}Q(u_1,\dots,u_P)(\sum_{i\ge 0}t^iK_\alpha[-i]),$$

(39) is equal to

$$\sum_\gamma\sum_{\alpha,\beta}(f_\gamma(t_1,\dots,t_N)t_1^{p_1}\cdots t_N^{p_N}P_\alpha(t_1,\cdots,t_N)\otimes P'_\alpha(u_1,\dots,u_P))\cdot \tag{40}$$
$$\cdot\Big(t_1'^{p'_1}\cdots t_{N'}'^{p'_{N'}}Q_\alpha(t'_1,\cdots,t'_{N'})\otimes g_\gamma(u'_1,\dots,u'_{P'})Q'_\alpha(u'_1,\dots,u'_{P'})K_{\epsilon_t(1)}[-p_1]\cdots$$
$$K_{\epsilon_t(N)}[-p_N]K_{\epsilon_{t'}(1)}[-p'_1]\cdots K_{\epsilon_{t'}(1)}[-p'_{N'}]),$$

with

$$\sum_\gamma f_\gamma(t_1,\dots,t_N)g_\gamma(u'_1,\dots,u'_{P'})=\prod_{1\le i\le N,1\le j\le P'}\frac{u_j-q^{\langle\epsilon_u(j),\alpha\rangle}t_i}{q^{\langle\epsilon_u(j),\alpha\rangle}u_j-t_i}.$$

After some computation, one finds that (40) coincides with $\mu(PQ)$. Therefore μ is an algebra morphism.

It follows that the composition $\mu\circ\Delta_{\mathrm{FO}}$ is an algebra morphism. This composition coincides with $\Delta_{\mathcal{S}^L}$, which is therefore an algebra morphism. ∎

Remark 3.5. $\Delta^{\mathbf{k}',\mathbf{k}''}_{\mathcal{S}^L}$ may also be expressed by the formula

$$\Delta^{\mathbf{k}',\mathbf{k}''}_{\mathcal{S}^L}(P)$$
$$=\sum_{p_1,\dots,p_{N'}\ge 0}\left(\prod_{i=1}^{N'}u_i^{p_i}P'''_\alpha(u_1,\dots,u_{N'})\right)\otimes\left(\prod_{i=1}^{N'}K_{\epsilon(i)}[-p_i]P''''_\alpha(u_{N'+1},\dots,u_N)\right),$$

where

$$\sum_\alpha P'''_\alpha(u_1,\dots,u_{N'})P''''_\alpha(u_{N'+1},\dots,u_N)$$
$$=P(t_1,\dots,t_N)\prod_{1\le l\le N',N'+1\le l'\le N}\frac{u_{l'}-u_l}{u_{l'}-q^{\langle\epsilon(l),\epsilon(l')\rangle}u_l}.$$

∎

Corollary 3.1. *There is a unique algebra morphism $\Delta_{\mathcal{V}^L}$ from $\mathcal{V}^L$ to*

$$\lim_{\leftarrow N}(\mathcal{V}^L \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{V}}^L)/(\mathcal{I}_N^{(\infty)} \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{V}}^L),$$

such that

$$\Delta_{\mathcal{V}^L}(h_i[k]^{\mathcal{V}^L}) = h_i[k]^{\mathcal{V}^L} \otimes 1 + 1 \otimes h_i[k]^{\mathcal{V}^L}, \quad 1 \le i \le n, \quad k \le 0,$$

$$\Delta_{\mathcal{V}^L}(t_i^k) = \sum_{l \ge 0} t_i^{k+l} \otimes K_i[-l] + 1 \otimes t_i^k \quad 1 \le i \le n, \quad k \in \mathbb{Z}.$$

Proof. For each i, $\Delta_{\mathcal{S}^L}$ maps FO_{ϵ_i} to $\lim_{\leftarrow N}(\mathcal{V}^L \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{V}}^L)/(\mathcal{I}_N^{(\infty)} \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{V}}^L)$, because $\mathcal{I}_{N,\mathbf{k}}^{\ge a} \cap \langle LV^{\ge a}\rangle \subset I_{\phi(\mathbf{k},n)}$. It follows that $\Delta_{\mathcal{S}^L}$ maps $\mathcal{V}^L$ to the same space. Call $\Delta_{\mathcal{V}^L}$ the restriction of $\Delta_{\mathcal{S}^L}$ to $\mathcal{V}^L$. This restriction is clearly characterized by its values on $h_i[k]$ and t_i^k. ∎

• *Construction of a Hopf algebra structure*

Recall that we showed in Lemma 3.3 that $\mathcal{V}^L$ is a free $\mathbb{C}[[\hbar]]$-module. Let us set $\mathcal{V}_0^L = \mathcal{V}^L/\hbar\mathcal{V}^L$.

Let $\overline{\mathcal{I}}_N$ be the image of the ideal $\mathcal{I}_N^{(\infty)}$ of $\mathcal{V}^L$ by the projection from $\mathcal{V}^L$ to $\mathcal{V}_0^L$. By Lemma 3.4, 2), and since the map from $\mathcal{V}^L$ to $\mathcal{V}_0^L$ is surjective, the ideals $\overline{\mathcal{I}}_N$ have property $(*)$, so that $\cap_{N>0}\overline{\mathcal{I}}_N$ is a two-sided ideal of $\mathcal{V}_0^L$. Define $\mathcal{W}_0$ as the quotient algebra $\mathcal{W}_0 = \mathcal{V}_0^L / \cap_{N>0} \overline{\mathcal{I}}_N$. We are going to define a Hopf algebra structure on $\mathcal{W}_0$.

Since the $\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N$ have property $(*)$, $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$ has an algebra structure. Moreover, the projection

$$\begin{aligned}&[\lim_{\leftarrow N}(\mathcal{V}^L \otimes_{\mathbb{C}[[\hbar]]} \mathcal{V}^L)/(\mathcal{V}^L \otimes_{\mathbb{C}[[\hbar]]} \mathcal{I}_N^{(\infty)})] \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \\ &\to \lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N),\end{aligned}$$

is an algebra isomorphism. $\Delta_{\mathcal{V}^L}$ induces therefore an algebra morphism $\Delta_{\mathcal{V}_0^L}$ from $\mathcal{V}_0^L$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$.

On the other hand, the $\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N$ have property $(*)$, so that $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$ has an algebra structure. The composition of $\Delta_{\mathcal{V}_0^L}$ with the projection

$$\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N) \to \lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$$

then yields an algebra morphism $p' \circ \Delta_{\mathcal{V}_L^0}$ from $\mathcal{V}_L^{(0)}$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$.

We have for any $k \ge N$, $\Delta_{\mathcal{V}}(t_i^k) \in \mathcal{I}_N \otimes \hat{\mathcal{V}} + \mathcal{V} \otimes \hat{\mathcal{I}}_N$, therefore $\Delta_{\mathcal{V}^L}(\mathcal{I}_N) \subset \mathcal{I}_N \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{V}} + \mathcal{V} \otimes_{\mathbb{C}[[\hbar]]} \hat{\mathcal{I}}_N$, therefore $\Delta_{\mathcal{V}_0^L}(\overline{\mathcal{I}}_N) \subset \overline{\mathcal{I}}_N \otimes \mathcal{V}_0 + \mathcal{V}_0 \otimes \overline{\mathcal{I}}_N$. It follows that $p' \circ \Delta_{\mathcal{V}_L^0}$ maps the intersection $\cap_{N>0}\overline{\mathcal{I}}_N$ to the kernel of the projection $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N) \to \lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$.

We have then an algebra morphism $\tilde{\Delta}_{\mathcal{W}_0}$ from $\mathcal{W}_0$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$.

In Proposition 1.3, we defined a surjective algebra morphism $i_\hbar$ from $U_\hbar L\mathfrak{b}_+$ to $\mathcal{V}^L$. It induces an algebra morphism i from $UL\mathfrak{b}_+$ to $\mathcal{V}_0^L$, which is also surjective.

Let T be the free algebra generated by the $h_i[k]^{(T)}, k \leq 0, i = 1, \ldots, n$ and $e_i[k]^{(T)}, i = 1, \ldots, n, k \in \mathbb{Z}$. We have a natural projection of T on $UL\mathfrak{b}_+$, sending each $x[k]^{(T)}$ to $x \otimes t^k$; composing it with ι, we get a surjective algebra morphism π from T to $\mathcal{V}_0^L$.

We have a unique algebra morphism $\Delta_T : T \to T \otimes T$, such that

$$\Delta_T(x_i[k]^{(T)}) = x_i[k]^{(T)} \otimes 1 + 1 \otimes x_i[k]^{(T)}.$$

Lemma 3.6. *Let $\pi_{\mathcal{V}_0 \to \mathcal{W}_0}$ be the natural projection from $\mathcal{V}_0$ to $\mathcal{W}_0$. We have the identity*

$$\tilde{\Delta}_{\mathcal{W}_0} \circ \pi_{\mathcal{V}_0 \to \mathcal{W}_0} = (\nu \circ (\pi_{\mathcal{V}_0 \to \mathcal{W}_0} \otimes \pi_{\mathcal{V}_0 \to \mathcal{W}_0})) \circ \Delta_T, \tag{41}$$

where ν is the natural projection from $(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$.

Proof. The two sides are algebra morphisms from $\mathcal{W}_0$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$. The identity is satisfied on generators of $\mathcal{W}_0$, therefore it is true. ∎

Let J be the kernel of the projection $\pi_{\mathcal{V}_0 \to \mathcal{W}_0} \circ \pi$ from T to $\mathcal{W}_0$. It follows from (41) that $\Delta_T(J)$ in contained in the kernel of $\nu \circ (\pi \otimes \pi)$, which is the preimage by $\pi \otimes \pi$ of Kerν; Kerν is equal to $\cap_{N>0}(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N)$, which is $(\cap_{N>0}\overline{\mathcal{I}}_N) \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes (\cap_{N>0}\overline{\mathcal{I}}_N)$. Therefore $(\pi \otimes \pi)^{-1}(\text{Ker}\nu)$ is $J \otimes T + T \otimes J$. We have shown that $\Delta_T(J) \subset J \otimes T + T \otimes J$. We have shown:

Proposition 3.3. *Δ_T induces a cocommutative Hopf algebra structure on $T/J = \mathcal{W}_0$.*

We will denote by $\Delta_{\mathcal{W}_0}$ the coproduct induced by Δ_T on $\mathcal{W}_0$.

• *Compatibility of $\Delta_{\mathcal{W}_0}$ with $\Delta_{\mathcal{V}_0^L}$*

Recall that $\Delta_{\mathcal{V}_0^L}$ is an algebra morphism from $\mathcal{V}_0^L$ to $\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L)$. Let us denote by $\Delta_{\mathcal{V}_0^L;N}$ the induced map from $\mathcal{V}_0^L$ to $(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L)$. We have seen that for any integer $p > 0$, $\Delta_{\mathcal{V}_0^L;N}(\overline{\mathcal{I}}_p)$ is contained in the image of $\overline{\mathcal{I}}_p \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_p$ by the projection map $\mathcal{V}_0^L \otimes \mathcal{V}_0^L \to (\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L)$. This image is

$$[(\overline{\mathcal{I}}_p + \overline{\mathcal{I}}_N) \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_p]/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L).$$

Therefore, $\Delta_{\mathcal{V}_0^L;N}(\cap_{p>0}\overline{\mathcal{I}}_p)$ is contained in the intersection of these spaces, which is

$$[\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes (\cap_{p>0}\overline{\mathcal{I}}_p)]/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L).$$

It follows that $\Delta_{\mathcal{V}_0^L;N}$ induces a linear map $\mathcal{W}_0 \to (\mathcal{W}_0 \otimes \mathcal{W}_0)/(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0)$, where $\overline{\mathcal{J}}_N$ is the image of $\overline{\mathcal{I}}_N$ by the projection map $\mathcal{V}_0 \to \mathcal{W}_0$.

It also induces an algebra morphism $\mathcal{W}_0 \to \lim_{\leftarrow N}(\mathcal{W}_0 \otimes \mathcal{W}_0)/(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0)$.

Then this algebra morphism factors through the coproduct map $\Delta_{\mathcal{W}_0}$ defined above. To check this, it is enough to check it on generators $x[k]$ of $\mathcal{W}_0$.

It follows that

Lemma 3.7. *1) $i_\hbar$ induces a map $i : UL\mathfrak{b}_+ \to \mathcal{W}_0$, which is a surjective Hopf algebra morphism.*

2) Let $\mathfrak{a}_L$ be the Lie algebra of primitive elements of $\mathcal{W}_0$. The restriction $\iota_{|L\mathfrak{b}_+}$ of ι to $L\mathfrak{b}_+$ to $\mathfrak{a}_L$ induces a surjective Lie algebra morphism.

Proof. $\Delta_{\mathcal{W}_0} \circ i$ and $(i \otimes i) \circ \Delta_{U\mathfrak{b}_+}$ are both algebra morphisms from $U\mathfrak{b}_+$ to $\mathcal{W}_0 \otimes \mathcal{W}_0$. Their values on the $x \otimes t^k$ coincide, therefore they are equal. This shows 1).

2) follows directly from 1) and from Proposition 2.1. ■

• *Construction of* $\delta_{\mathcal{W}_0}$

Define $\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L$ as the tensor product

$$\mathbb{C}[X_s[0]^{\mathcal{V}^L(1)}, X_s[0]^{\mathcal{V}^L(2)}][[\hbar]] \otimes \mathbb{C}[X_s[k]^{\mathcal{V}^L(1)}, X_s[k]^{\mathcal{V}^L(2)}, k > 0] \otimes \langle LV \rangle \otimes_{\mathbb{C}[[\hbar]]} \langle LV \rangle;$$

endow $\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L$ with the unique $\hbar$-adically continuous algebra structure such that $\mathcal{V}^L \otimes_{\mathbb{C}[[\hbar]]} \mathcal{V}^L \to \mathcal{V}^L \hat{\otimes} \mathcal{V}^L$, $X_s[k]^{\mathcal{V}^L(1)} \mapsto X_s[k]^{\mathcal{V}^L} \otimes 1$, $X_s[k]^{\mathcal{V}^L(2)} \mapsto 1 \otimes X_s[k]^{\mathcal{V}^L}$, $1 \otimes x \otimes 1 \mapsto x \otimes 1$, $1 \otimes 1 \otimes x \mapsto 1 \otimes x$ (where x is in $\langle LV \rangle$) is an algebra morphism.

Define in the same way $\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}$ as the tensor product

$$\mathbb{C}[X_s[k]^{\mathcal{V}^L(1)}, X_s[k]^{\mathcal{V}^L(2)}][[\hbar]] \otimes \langle LV \rangle \otimes I_N^{(\infty)},$$

where the tensor products are over $\mathbb{C}[[\hbar]]$. Each $\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}$ is then a left ideal of $\hat{\mathcal{V}}^L \hat{\otimes} \mathcal{V}^L$.

Clearly, we have $(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/\hbar(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L) = \mathcal{V}_0^L \otimes \mathcal{V}_0^L$. Moreover,

$$[(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)})]/\hbar[(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)})] = (\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N).$$

Then $\Delta_{\mathcal{V}^L}$ is an algebra morphism from $\mathcal{V}^L$ to $\lim_{\leftarrow N}(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L)$. We again denote by $\Delta_{\mathcal{V}^L}$ the composition of this map with the projection on $\lim_{\leftarrow N}(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)})$. Define $\Delta'_{\mathcal{V}^L}$ as $\Delta^L_{\mathcal{V}}$ composed with the exchange of factors.

We have then

$$(\Delta_{\mathcal{V}^L} - \Delta'_{\mathcal{V}^L})(\mathcal{V}^L) \subset \lim_{\leftarrow N}(\hbar \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/[(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}) \cap \hbar \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L].$$

Since $\mathcal{I}_N^{(\infty)}$ is divisible in $\mathcal{V}^L$, we have

$$(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}) \cap \hbar(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L) = \hbar(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}),$$

so $\frac{\Delta_{\mathcal{V}^L} - \Delta'_{\mathcal{V}^L}}{\hbar}$ is a linear map from $\mathcal{V}^L$ to

$$\lim_{\leftarrow N}(\hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{V}}^L)/(\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}).$$

Define $\mathcal{I}_N^{(1)}$ as the image of $\hat{\mathcal{I}}_N^{(1)}$ in $\mathcal{V}_0^L$ by the projection $\mathcal{V}^L \to \mathcal{V}_0^L$.

Let us set $\delta_{\mathcal{V}_0^L} = \frac{\Delta_{\mathcal{V}^L} - \Delta'_{\mathcal{V}^L}}{\hbar}$ mod $\hbar$. Then $\delta_{\mathcal{V}_0^L}$ is a linear map from $\mathcal{V}_0^L$ to

$$\lim_{\leftarrow N}(\mathcal{V}_0^L \otimes \mathcal{V}_0^L)/(\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N).$$

Moreover, $\frac{\Delta_{\mathcal{V}^L} - \Delta'_{\mathcal{V}^L}}{\hbar}$ maps $\mathcal{I}_N^{(\infty)}$ to the inverse limit

$$\lim_{\leftarrow M} (\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}) / (\hat{\mathcal{I}}_M^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_M^{(\infty)}).$$

Therefore, $\delta_{\mathcal{V}_0^L}$ maps $\overline{\mathcal{I}}_N$ to $\lim_{\leftarrow M} (\overline{\mathcal{I}}_N \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_N) / (\overline{\mathcal{I}}_M \otimes \mathcal{V}_0^L + \mathcal{V}_0^L \otimes \overline{\mathcal{I}}_M)$. Therefore, $\delta_{\mathcal{V}_0^L}(\cap_N \overline{\mathcal{I}}_N)$ is zero. It follows that $\delta_{\mathcal{V}_0^L}$ induces a map $\delta_{\mathcal{W}_0}$ from $\mathcal{W}_0$ to $\lim_{\leftarrow N} (\mathcal{W}_0 \otimes \mathcal{W}_0) / (\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N)$.

• *Identities satisfied by* $\delta_{\mathcal{W}_0}$

Lemma 3.8. $\delta_{\mathcal{W}_0}$ *satisfies*

$$(\Delta_{\mathcal{W}_0} \otimes id) \circ \delta_{\mathcal{W}_0} = (\delta_{\mathcal{W}_0}^{2\to 23} + \delta_{\mathcal{W}_0}^{2\to 13}) \circ \Delta_{\mathcal{W}_0}, \tag{42}$$

$$\mathrm{Alt}(\delta_{\mathcal{W}_0} \otimes id) \circ \delta_{\mathcal{W}_0}) = 0, \tag{43}$$

$$\delta_{\mathcal{W}_0}(xy) = \delta_{\mathcal{W}_0}(x) \Delta_{\mathcal{W}_0}(y) + \Delta_{\mathcal{W}_0}(x) \delta_{\mathcal{W}_0}(y) \text{ for } x, y \text{ in } \mathcal{W}_0, \tag{44}$$

where we use the notation of sect. 2.. The two first equalities are identities of maps from $\mathcal{W}_0$ *to* $\lim_{\leftarrow N} \mathcal{W}_0^{\otimes 3} / (\overline{\mathcal{J}}_N \otimes \mathcal{W}_0^{\otimes 2} + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0^{\otimes 2} \otimes \overline{\mathcal{J}}_N)$.

Proof. $\Delta_{\mathcal{V}^L}$ maps $\mathcal{I}_N^{(\infty)}$ to

$$\lim_{\leftarrow M} (\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}) / (\hat{\mathcal{I}}_M^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_M^{(\infty)}).$$

Therefore, $(\Delta_{\mathcal{V}^L} \otimes id) \otimes \Delta_{\mathcal{V}^L}$ and $(id \otimes \Delta_{\mathcal{V}^L}) \otimes \Delta_{\mathcal{V}^L}$ both define algebra morphisms from $\mathcal{V}^L$ to $\lim_{\leftarrow N} (\hat{\mathcal{V}}^L)^{\hat{\otimes} 3} / [\hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} (\hat{\mathcal{V}}^L)^{\hat{\otimes} 2} + \hat{\mathcal{V}}^L \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)} \hat{\otimes} \hat{\mathcal{V}}^L + (\hat{\mathcal{V}}^L)^{\hat{\otimes} 2} \hat{\otimes} \hat{\mathcal{I}}_N^{(\infty)}]$. These morphisms are the restrictions to $\mathcal{V}^L$ of $(\Delta_{\mathcal{S}^L} \otimes id) \otimes \Delta_{\mathcal{S}^L}$ and $(id \otimes \Delta_{\mathcal{S}^L}) \otimes \Delta_{\mathcal{S}^L}$, which coincide, therefore they coincide.

The intersection $\cap_{N>0} \mathcal{I}_N^{(\infty)}$ is a two-sided ideal of $\mathcal{V}^L$. Define $\mathcal{W}$ as the quotient $\mathcal{V}^L / \cap_{N>0} \mathcal{I}_N^{(\infty)}$. Let $\mathcal{J}_N$ be the image of $\mathcal{I}_N^{(\infty)}$ by the projection of $\mathcal{V}^L$ on $\mathcal{W}$. Define $\hat{\mathcal{W}}$ and $\hat{\mathcal{J}}_N$ in the same way, replacing $\mathcal{V}^L$ and $\mathcal{I}_N^{(\infty)}$ by $\hat{\mathcal{V}}^L$ and $\hat{\mathcal{I}}_N^{(\infty)}$. Then $\Delta_{\mathcal{V}^L}$ induces an algebra morphism

$$\Delta_{\mathcal{W}}: \mathcal{W} \to \lim_{\leftarrow N} (\hat{\mathcal{W}} \hat{\otimes} \hat{\mathcal{W}}) / (\hat{\mathcal{J}}_N \hat{\otimes} \hat{\mathcal{W}} + \hat{\mathcal{W}} \hat{\otimes} \mathcal{J}_N).$$

Moreover, $(\Delta_{\mathcal{W}} \otimes id) \circ \Delta_{\mathcal{W}}$ and $(id \otimes \Delta_{\mathcal{W}}) \circ \Delta_{\mathcal{W}}$ define coinciding algebra morphisms from $\mathcal{W}$ to $\lim_{\leftarrow N} (\hat{\mathcal{W}}^{\hat{\otimes} 3}) / [\hat{\mathcal{J}}_N \hat{\otimes} \hat{\mathcal{W}}^{\hat{\otimes} 2} + \hat{\mathcal{W}} \hat{\otimes} \mathcal{J}_N \hat{\otimes} \hat{\mathcal{W}} + \hat{\mathcal{W}}^{\hat{\otimes} 2} \hat{\otimes} \hat{\mathcal{J}}_N]$. Moreover, $\mathcal{W}$ is a free $\mathbb{C}[[\hbar]]$-module, and we have a topological Hopf algebra isomorphism of $\mathcal{W} / \hbar \mathcal{W}$ with $\mathcal{W}_0$. The usual manipulations then imply the statements of the Lemma. ∎

The identities of Lemma 3.8 are the topological versions of the co-Leibnitz, co-Jacobi and Hopf compatibility rules.

• *Topological Lie bialgebra structure on* $\mathfrak{a}_L$

Define $\mathfrak{a}_L^{(N)}$ as the intersection $\mathfrak{a}_L \cap \overline{\mathcal{J}}_N$.

Lemma 3.9. $\overline{\mathcal{J}}_N$ *is the left ideal* $(U\mathfrak{a}_L)\mathfrak{a}_L^{(N)}$ *of* $\mathcal{W}_0 = U\mathfrak{a}_L$. *Moreover,* $\mathfrak{a}_L^{(N)}$ *is a Lie subalgebra of* $\mathfrak{a}_L$.

Proof. $\mathcal{J}_N$ is a left ideal of $\mathcal{W}$, therefore $\overline{\mathcal{J}}_N$ is a left ideal of $\mathcal{W}_0$. Moreover, $\Delta_{\mathcal{W}}(\mathcal{J}_N)$ is contained in the inverse limit $\lim_{\leftarrow M}(\hat{\mathcal{J}}_N \hat{\otimes} \hat{\mathcal{W}} + \hat{\mathcal{W}} \hat{\otimes} \hat{\mathcal{J}}_N)/(\hat{\mathcal{J}}_M \hat{\otimes} \hat{\mathcal{W}} + \hat{\mathcal{W}} \hat{\otimes} \hat{\mathcal{J}}_M)$. It follows that $\Delta_{\mathcal{W}_0}(\overline{\mathcal{J}}_N)$ is contained in $\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N$.

The first statement of Lemma 3.9 now follows from Lemma 2.4. For x, y in $\mathfrak{a}_L^{(N)}$, $[x, y] = xy - yx$ belongs to $\mathfrak{a}_L$ and also to $(U\mathfrak{a}_L)\mathfrak{a}_L^{(N)}$, so it belongs to $\mathfrak{a}_L^{(N)}$. Therefore $\mathfrak{a}_L^{(N)}$ is a Lie subalgebra of $\mathfrak{a}_L$. ■

Lemma 3.10. *1) The restriction of $\delta_{\mathcal{W}_0}$ to $\mathfrak{a}_L$ defines a map $\delta_{\mathfrak{a}_L} : \mathfrak{a}_L \to \lim_{\leftarrow N}(\mathfrak{a}_L \otimes \mathfrak{a}_L)/(\mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(N)})$.*

2) For any element x of $\mathfrak{a}_L$, $\mathrm{ad}(x)(\mathfrak{a}_L^{(N)})$ in contained in $\mathfrak{a}_L^{(N-k(x))}$. The tensor square of the adjoint action therefore induces a $\mathfrak{a}_L$-module structure on $\lim_{\leftarrow N}(\mathfrak{a}_L \otimes \mathfrak{a}_L)/(\mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(N)})$. $\delta_{\mathfrak{a}_L}$ is a 1-cocycle of $\mathfrak{a}_L$ with values in this module.

3) We have $\delta_{\mathfrak{a}_L}(\mathfrak{a}_L^{(N)}) \subset \lim_{\leftarrow M}(\mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(N)})/(\mathfrak{a}_L^{(M)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(M)})$. $(\delta_{\mathfrak{a}_L} \circ id) \circ \delta_{\mathfrak{a}_L}$ therefore defines a map from $\mathfrak{a}_L$ to $\lim_{\leftarrow N}\mathfrak{a}_L^{\otimes 3}/(\mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L^{\otimes 2} + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L + \mathfrak{a}_L^{\otimes 2} \otimes \mathfrak{a}_L^{(N)})$. It satisfies the rule

$$\mathrm{Alt}(\delta_{\mathfrak{a}_L} \otimes id) \circ \delta_{\mathfrak{a}_L} = 0. \tag{45}$$

Proof. Let us show 1). $\delta_{\mathcal{W}_0}$ induces an map $\delta_{\mathcal{W}_0;N} \colon \mathcal{W}_0 \to \mathcal{W}_0^{\otimes 2}/(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N)$. Let a belong to $\mathfrak{a}_L$. Let us write $\delta_{\mathcal{W}_0;N}(a) = \sum_i a_i \otimes b_i \mod \overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N$, with $(a_i)_i$ and $(b_i)_i$ finite families of $\mathcal{W}_0$ such that $(b_i \mod \overline{\mathcal{J}}_N)_i$ is a free family of $\mathcal{W}_0/\overline{\mathcal{J}}_N$. It follows from (42) that

$$\sum_i (\Delta_{\mathcal{W}_0}(a_i) - a_i \otimes 1 - 1 \otimes a_i) \otimes b_i$$

belongs to $\overline{\mathcal{J}}_N \otimes \mathcal{W}_0^{\otimes 2} + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0^{\otimes 2} \otimes \overline{\mathcal{J}}_N$. Its image by the projection $\mathcal{W}_0^{\otimes 3} \to [\mathcal{W}_0^{\otimes 2}/(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N)] \otimes [\mathcal{W}_0/\overline{\mathcal{J}}_N]$ its therefore zero. It follows that each a_i is such that $\Delta_{\mathcal{W}_0}(a_i) - a_i \otimes 1 - 1 \otimes a_i$ belongs to $\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N$. Reasoning by induction on the degree of a_i (for the enveloping algebra filtration of $\mathcal{W}_0$), we find that a_i belongs to $\mathfrak{a}_L + \overline{\mathcal{J}}_N$. Therefore, $\delta_{\mathcal{W}_0;n}(a)$ belongs to the image of $\mathfrak{a}_L \otimes \mathcal{W}_0$ in $\mathcal{W}_0^{\otimes 2}/(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \overline{\mathcal{J}}_N)$. Since $\delta_{\mathcal{W}_0;n}(a)$ is also antisymmetric, it belongs to the image of $\mathfrak{a}_L \otimes \mathfrak{a}_L$ in this space. This shows 1).

Let us show 2). For x en element of $\mathfrak{a}_L$ and y an element of $\mathfrak{a}_L^{(N)}$, $[x, y] = xy - yx$ belongs to $\overline{\mathcal{J}}_N + \overline{\mathcal{J}}_{N-k(x)} = \overline{\mathcal{J}}_{N-k(x)}$; since it also belongs to $\mathfrak{a}_L$, $[x, y]$ belongs to $\mathfrak{a}_L^{(N)}$. That $\delta_{\mathfrak{a}_L}$ is a 1-cocycle then follows from (44).

Let us show 3). $\Delta_{\mathcal{W};M}(\mathcal{J}_N)$ is contained in $(\mathcal{J}_N \otimes \mathcal{W} + \mathcal{W} \otimes \mathcal{J}_N)/(\mathcal{J}_M \otimes \mathcal{W} + \mathcal{W} \otimes \mathcal{J}_M)$. It follows that $\delta_{\mathcal{W}_0;M}(\overline{\mathcal{J}}_N)$ is contained in $(\overline{\mathcal{J}}_N \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \emptyset\mathcal{J}_N)/(\overline{\mathcal{J}}_M \otimes \mathcal{W}_0 + \mathcal{W}_0 \otimes \emptyset\mathcal{J}_M)$. Therefore, $\delta_{\mathfrak{a}_L}(\mathfrak{a}_L^{(N)})$ is contained in $\lim_{\leftarrow M}(\mathfrak{a}_L^{(N)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(N)})/(\mathfrak{a}_L^{(M)} \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(M)})$. (45) in then a consequence of (43). ■

Define the restricted dual $\mathfrak{a}_L^{\star}$ of $\mathfrak{a}_L$ as the subspace of $\mathfrak{a}_L^{*}$ composed of the forms ϕ on $\mathfrak{a}_L$, such for some N, ϕ vanishes on $\mathfrak{a}_L^{(N)}$.

Lemma 3.11. *The dual map to $\delta_{\mathfrak{a}_L}$ defines a Lie algebra structure on $\mathfrak{a}_L^{\star}$.*

Proof. Let ϕ, ψ belong to $\mathfrak{a}_L^\star$. Let N be an integer such that ϕ, ψ vanish on $\mathfrak{a}_L^{(N)}$. For any integer M let $\bar\delta_{\mathfrak{a}_L;M}$ be a lift to $\mathfrak{a}_L^{\otimes 2}$ of the map $\delta_{\mathfrak{a}_L;M}$ from $\mathfrak{a}_L$ to $\mathfrak{a}_L^{\otimes 2}/(\mathfrak{a}_L^{(M)}) \otimes \mathfrak{a}_L + \mathfrak{a}_L \otimes \mathfrak{a}_L^{(M)}$ induced by $\delta_{\mathfrak{a}_L}$.

Let x belong to $\mathfrak{a}_L$. Then $M \geq N$, the number $\langle \phi \otimes \psi, \bar\delta_{\mathfrak{a}_L;M}(x)\rangle$ is independent of the lift $\bar\delta_{\mathfrak{a}_L;M}$ and of M; it defines a linear form $[\phi, \psi]$ on $\mathfrak{a}_L$. The first statement of Lemma 3.10, 3), implies that $[\phi,\psi]$ actually belongs to $\mathfrak{a}_L^\star$. It is clear that $(\phi,\psi) \mapsto [\phi,\psi]$ is linear and antisymmetric in ϕ and ψ. (45) implies that it satisfies the Jacobi identity. ■

• *Topological Lie bialgebra structure on* $L\mathfrak{b}_+$

Define for any integer N, $(L\mathfrak{b}_+)^{(N)}$ as the Lie subalgebra of $L\mathfrak{b}_+$ generated by the $\bar x_i^+ \otimes t^k$, $k \geq N, i = 1, \ldots, n$.

For any x in $L\mathfrak{b}_+$, there exists an integer $l(x)$ such that $\mathrm{ad}(x)((L\mathfrak{b}_+)^{(N)})$ is contained in $(L\mathfrak{b}_+)^{(N-l(x))}$. It follows that $\lim_{\leftarrow N}(L\mathfrak{b}_+)^{\otimes 2}/[(L\mathfrak{b}_+)^{(N)} \otimes L\mathfrak{b}_+ + L\mathfrak{b}_+ \otimes (L\mathfrak{b}_+)^{(N)}]$ and $\lim_{\leftarrow N}(L\mathfrak{b}_+)^{\otimes 3}/[(L\mathfrak{b}_+)^{(N)} \otimes L\mathfrak{b}_+^{\otimes 2} + L\mathfrak{b}_+ \otimes (L\mathfrak{b}_+)^{(N)} \otimes L\mathfrak{b}_+ + L\mathfrak{b}_+^{\otimes 2} \otimes (L\mathfrak{b}_+)^{(N)}]$ have $L\mathfrak{b}_+$-module structures.

Lemma 3.12. *There is a unique map $\delta_{L\mathfrak{b}_+}$ from $L\mathfrak{b}_+$ to*

$$\lim_{\leftarrow N}(L\mathfrak{b}_+)^{\otimes 2}/[(L\mathfrak{b}_+)^{(N)} \otimes L\mathfrak{b}_+ + L\mathfrak{b}_+ \otimes (L\mathfrak{b}_+)^{(N)}],$$

such that $\delta_{L\mathfrak{b}_+}(\bar h_i \otimes t^k) = 0$ and

$$\delta_{L\mathfrak{b}_+}(\bar x_i^+ \otimes t^k) = d_i \mathrm{Alt}[\frac{1}{2}(\bar h_i \otimes 1) \otimes (\bar x_i^+ \otimes 1) + \sum_{l>0} (\bar h_i \otimes t^{-l}) \otimes (\bar x_i^+ \otimes t^{k+l})]$$

and $\delta_{L\mathfrak{b}_+}$ is a 1-cocycle. Moreover, $\delta_{L\mathfrak{b}_+}$ maps $L\mathfrak{b}_+^{(N)}$ to $\lim_{\leftarrow M}(L\mathfrak{b}_+^{(N)} \otimes L\mathfrak{b}_+ + L\mathfrak{b}_+ \otimes L\mathfrak{b}_+^{(N)})/(L\mathfrak{b}_+^{(M)} \otimes L\mathfrak{b}_+ + L\mathfrak{b}_+ \otimes L\mathfrak{b}_+^{(M)})$, and it satisfies the co-Jacobi identity $\mathrm{Alt}(\delta_{L\mathfrak{b}_+} \otimes id) \circ \delta_{L\mathfrak{b}_+} = 0$.

Define the restricted dual $(L\mathfrak{b}_+)^\star$ to $L\mathfrak{b}_+$ as the subspace of $(L\mathfrak{b}_+)^*$ consisting of the forms on $L\mathfrak{b}_+$, which vanish on some $(L\mathfrak{b}_+)^{(N)}$. The argument of Lemma 3.11 implies that $\delta_{L\mathfrak{b}_+}$ induces a Lie algebra structure on $(L\mathfrak{b}_+)^\star$.

Lemma 3.13. *Define on $\mathfrak{g} \otimes \mathbb{C}((t))$, the pairing $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathbb{C}((t))}$ as the tensor product of the invariant pairing on $\mathfrak{g}$ and $\langle f, g\rangle = \mathrm{res}_0(fg\frac{dt}{t})$. $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathbb{C}((t))}$ an isomorphism of $(L\mathfrak{b}_+)^\star$ with the subalgebra $L\mathfrak{b}_-$ of $L\mathfrak{g}$ defined as $\mathfrak{h} \otimes \mathbb{C}[[t]] \oplus \mathfrak{n}_- \otimes \mathbb{C}((t))$. This isomorphism is a Lie algebra antiisomorphism (that is, it is an isomorphism after we change the bracket of $L\mathfrak{b}_-$ into its opposite).*

The map $\iota_{|L\mathfrak{b}_+}$ defined in Lemma 3.7, 2), maps the generators of $L\mathfrak{b}_+^{(N)}$ to $\mathfrak{a}_L^{(N)}$; since $\mathfrak{a}_L^{(N)}$ is a Lie subalgebra of $\mathfrak{a}_L$ (Lemma 3.9), we have $\iota_{|L\mathfrak{b}_+}(L\mathfrak{b}_+^{(N)}) \subset \mathfrak{a}_L^{(N)}$.

It follows that $\iota_{|L\mathfrak{b}_+}$ induces a linear map $\iota^\star$ from $\mathfrak{a}_L^\star$ to $(L\mathfrak{b}_+)^\star = L\mathfrak{b}_-$.

Moreover, we have

$$\delta_{\mathfrak{a}_L} \circ \iota_{|L\mathfrak{b}_+} = (\iota_{L\mathfrak{b}_+}^{\otimes 2}) \circ \delta_{L\mathfrak{b}_+}, \tag{46}$$

because both maps are 1-cocycles of $L\mathfrak{b}_+$ with values in $\lim_{\leftarrow N}\mathfrak{a}_L^{\otimes 2}/(\mathfrak{a}_L^{(N)}\otimes\mathfrak{a}_L+\mathfrak{a}_L\otimes\mathfrak{a}_L^{(N)})$, and coincide on the generators of $L\mathfrak{b}_+$. Equation (46) then implies that $\iota^\star:\mathfrak{a}_L^\star\to L\mathfrak{b}_-$ is a Lie algebra morphism.

Let us set $(L\mathfrak{b}_-)_{pol}=\mathfrak{h}\otimes\mathbb{C}[t^{-1}]\oplus\mathfrak{n}_-\otimes\mathbb{C}[t,t^{-1}]$; $(L\mathfrak{b}_-)_{pol}$ is the polynomial part of $L\mathfrak{b}_-$.

Lemma 3.14. *The image of $\iota^\star$ contains $(L\mathfrak{b}_-)_{pol}$.*

Proof. As we have seen, $\mathcal{V}^L$ is graded by $\mathbb{N}^n$. Each ideal $\mathcal{I}_N^{(\infty)}$is a graded ideal, so that $\mathcal{W}=\mathcal{V}/\cap_N\mathcal{I}_N^{(\infty)}$ is also graded by $\mathbb{N}^n$. Moreover, the degree 0 and ϵ_i components of $\mathcal{I}_N^{(\infty)}$ are respectively 0 and $\oplus_{k\geq N}\mathbb{C}[[\hbar]][h_i[k]^{\mathcal{V}^L},k\leq 0]t_i^k$. Therefore, the components of $\cap_N\mathcal{I}_N^{(\infty)}$ of degree 0 and ϵ_i are zero. The components of $\mathcal{W}$ of degrees 0 and ϵ_i are therefore respectively $\mathbb{C}[[\hbar]][h_i[k],k\leq 0]$ and $\oplus_{l\in\mathbb{Z}}\mathbb{C}[[\hbar]][h_i[k]^{\mathcal{V}^L},k\leq 0]t_i^l$.

$\mathcal{W}_0$ is also graded by $\mathbb{N}^n$, and its components of degrees 0 and ϵ_i are $\mathbb{C}[h_i[k],k\leq 0]$ and $\oplus_{l\in\mathbb{Z}}\mathbb{C}[h_i[k]^{\mathcal{V}^L},k\leq 0]t_i^l$.

The primitive part $\mathfrak{a}_L$ of $\mathcal{W}_0$ is therefore also graded by $\mathbb{N}^n$, and the computation of $\Delta_{\mathcal{W}_0}$ on $\mathcal{W}_0[0]$ and $\mathcal{W}_0[\epsilon_i]$ shows that $\mathfrak{a}_L[0]=\oplus_{1\leq i\leq n,k\leq 0}\mathbb{C}h_i[k]^{\mathcal{V}^L}$ and $\mathfrak{a}_L[\epsilon_i]=\oplus_{k\in\mathbb{Z}}\mathbb{C}t_i^k$.

Define linear forms $h_{i,k}^*$ and $e_{i,k}^*$ on $\mathfrak{a}_L$ by the rules that $h_{i,k}^*$ vanishes on $\oplus_{\alpha\neq 0}\mathfrak{a}_L[\alpha]$, and the restriction of $h_{i,k}^*$ to $\mathfrak{a}_L[0]$ maps $h_j[l]^{\mathcal{V}^L}$ to $\delta_{ij}\delta_{kl}$; and $e_{i,k}^*$ vanishes on $\oplus_{\alpha\neq\epsilon_i}\mathfrak{a}_L[\alpha]$, and the restriction of $e_{i,k}^*$ to $\mathfrak{a}_L[\epsilon_i]$ maps t_i^l to δ_{kl}.

It follows from the computation of $\mathcal{I}_N^{(\infty)}[0]$ and $\mathcal{I}_N^{(\infty)}[\epsilon_i]$ that the $h_{i,k}^*$ and $e_{i,k}^*$ vanish on all the $\overline{\mathcal{J}}_N$, resp. on the $\overline{\mathcal{J}}_N,N\geq k$, adn therefore on all the $\mathfrak{a}_L^{(N)}$, resp. on the $\mathfrak{a}_L^{(N)},N\geq k$. It follows that the $h_{i,k}^*$ and $e_{i,k}^*$ actully belong to $\mathfrak{a}_L^\star$.

Since the images of $\bar{h}_i\otimes t^k$ and $\bar{x}_i^+\otimes t^k$ by $\iota_{|L\mathfrak{b}_+}$ are $h_i[k]^{\mathcal{V}^L}$ and t_i^k, the images of $h_{i,k}^*$ and $e_{i,k}^*$ by $\iota^\star$ are the generators $\bar{h}_i\otimes t^k,1\leq i\leq n,k\geq 0$ and $\bar{x}_i^-\otimes t^k,1\leq i\leq n,k\in\mathbb{Z}$, of $(L\mathfrak{b}_-)_{pol}$.

The statement follows because $\iota^\star$ is a Lie algebra morphism. ∎

Lemma 3.14 implies that the kernel of $\iota_{|L\mathfrak{b}_+}$ is contained in contains the annihilator of $(L\mathfrak{b}_-)_{pol}$ in $L\mathfrak{b}_+$. Since this annihilator is zero, $\iota_{|L\mathfrak{b}_+}$ is injective. It follows that $\iota_{|L\mathfrak{b}_+}$ is an isomorphism.

Therefore, $\iota:UL\mathfrak{b}_+\to\mathcal{W}_0$ is also an isomorphism. Recall that ι was obtained from the surjective $\mathbb{C}[[\hbar]]$-modules morphism $\iota_\hbar=p\circ i_\hbar$, where p is the projection of $\mathcal{V}^L$ on $\mathcal{W}$.

We now use:

Lemma 3.15. *Let E and F be $\mathbb{C}[[\hbar]]$-modules, such that F is torsion-free and E is separated (i.e. $\cap_{N>0}\hbar^N E=0$). Let $\pi:E\to F$ be a surjective morphism of $\mathbb{C}[[\hbar]]$-modules, such that the induced morphism $\pi_0:E/\hbar E\to F/\hbar F$ is an isomorphism of vector spaces. Then π is an isomorphism.*

Proof. Let x belong to Ker$p$. $\pi_0(x \bmod \hbar)$ is zero, therefore x belongs to $\hbar E$. Set $x=\hbar x_1$. $\hbar\pi(x_1)$ is zero; since F is torsion-free, x_1 belongs to Ker$p$. Therefore, Ker$p\subset\hbar$Ker$p$. It follows that Ker$p\subset\cap_{N>0}\hbar^N E$, so that Ker$p=0$. It follows that π is an isomorphism. ∎

Recall that $U_\hbar L\mathfrak{n}_+$ was defined as the quotient $\mathcal{A}/(\cap_{N>0}\hbar^N\mathcal{A})$. It follows that $U_\hbar L\mathfrak{n}_+$ is separated. The above Lemma therefore shows that $p\circ i_\hbar$ is an isomorphism. Since p and $i_\hbar$ are both surjective, they are both isomorphisms. Corollary 1.4 follows, together with $\cap_{N>0}\mathcal{I}_N^{(\infty)}=0$ (from where also follows that $\cap_{N>0}\mathcal{I}_N=0$), and also, by Lemma 3.3, Theorem 1.3, 1).

It is then clear that the map $U_\hbar L\mathfrak{n}_+\to U_\hbar L\mathfrak{n}_+^{top}$ is injective and that $U_\hbar L\mathfrak{n}_+^{top}$ is the $\hbar$-adic completion of $U_\hbar L\mathfrak{n}_+$. This proves Theorem 1.3, 2).

There is a unique algebra morphism ς from $\widetilde{U}_\hbar L\mathfrak{n}_+$ to $U_\hbar L\mathfrak{n}_+$, which sends each $e_i[k]^{\widetilde{\mathcal{A}}}$ to $e_i[k]$. As we have seen in Proposition 3.1, ς induces an isomorphism between $\widetilde{U}_\hbar L\mathfrak{n}_+/\hbar\widetilde{U}_\hbar L\mathfrak{n}_+$ and $U_\hbar L\mathfrak{n}_+/\hbar U_\hbar L\mathfrak{n}_+$. Moreover, $\widetilde{U}_\hbar L\mathfrak{n}_+$ is separated and by Theorem 1.3, 1), $U_\hbar L\mathfrak{n}_+$ is free. Lemma 3.15 then implies that ς is an isomorphism. ∎

Remark 3.16. Let $\mathrm{FO}^{(0)}$ be the subspace of FO formed of the functions satisfying $f(\binom{(i)}{\alpha})=0$ when $t_1^{(i)}=q_{d_i}^2t_2^{(i)}=\cdots=q_{d_i}^{-2a_{ij}}t_{a_{ij}}^{(i)}=q_{d_i}^{-a_{ij}}t_1^{(j)}$ for any i,j. We showed in [11] that the image of $U_\hbar L\mathfrak{n}_+$ in FO is contained in $\mathrm{FO}^{(0)}[\hbar^{-1}]$. It is natural to expect that this image is actually the subspace of $\mathrm{FO}^{(0)}$ consisting in the functions such that $f(t_1,\cdots,t_N)=O(\hbar^k)$ whenever k out of the N variables t_i coincide.

3.2. Nondegeneracy of the pairing $\langle\ ,\ \rangle_{U_\hbar L\mathfrak{n}_\pm}$ (proof of Theorem 1.4).

Let us define $T(LV)$ as the tensor algebra $\oplus_{k\geq 0}(LV)^{\otimes_{\mathbb{C}[[\hbar]]}k}$, where $LV=\oplus_{i=1}^n\mathbb{C}[[\hbar]][t_i,t_i^{-1}]$. Denote in this algebra, the element t_i^l of LV as $f_i[l]^{(T)}$.

Define a pairing

$$\langle\ ,\ \rangle_{\mathrm{FO}\times T(LV)}:\mathrm{FO}\times T(LV)\to\mathbb{C}((\hbar))$$

as follows: if P belongs to $\mathrm{FO}_{\mathbf{k}}$,

$$\begin{aligned}&\langle P,f_{i_1}[l_1]^{(T)}\cdots f_{i_{N'}}[l_N]^{(T)}\rangle_{\mathrm{FO}\times T(LV)}\\&=\delta_{\mathbf{k},\sum_{j=1}^N\epsilon_{i_j}}\mathrm{res}_{u_N=0}\cdots\mathrm{res}_{u_1=0}\\&\left(P(t_1,\cdots,t_N)\prod_{l<l'}\frac{u_{l'}-u_l}{q^{\langle\epsilon_{i_{l'}},\epsilon_{i_l}\rangle}u_{l'}-u_l}u_1^{l_1}\cdots u_N^{l_N}\frac{du_1}{u_1}\cdots\frac{du_N}{u_N}\right),\end{aligned}\tag{47}$$

where we set as usual $(t_1,\ldots,t_{k_1})=(t_1^{(1)},\ldots,t_{k_1}^{(1)})$, etc.,

$$(t_{k_1+\cdots+k_{n-1}+1},\ldots,t_{k_1+\cdots+k_n})=(t_1^{(n)},\ldots,t_{k_n}^{(n)}),$$

and $u_1=t_1^{(i_1)}$, $u_2=t_1^{(i_2)}$ if $i_2\neq i_1$ and in general $u_s=t_{\nu_s+1}^{(i_s)}$, where ν_s is the number of indices t such that $t<s$ and $i_t=i_s$.

Lemma 3.17. *The pairing $\langle\ ,\ \rangle_{\mathrm{FO}\times T(LV)}$ verifies $(T(LV))^\perp=0$.*

Proof. Assume that the polynomial P of $\mathrm{FO}_{\mathbf{k}}$ is such that (47) vanishes for any families of indices (i_k) and (l_k). Fix a family of indices (i_k) such that $\mathbf{k}=\sum_{j=1}^N\epsilon_{i_j}$. Since (47) vanishes for any family (l_k), the rational function $P(t_1,\cdots,t_N)\prod_{l<l'}\frac{u_{l'}-u_l}{q^{\langle\epsilon_{i_{l'}},\epsilon_{i_l}\rangle}u_{l'}-u_l}$ vanishes, therefore P is zero. ∎

Let $\langle\, , \,\rangle_{\langle LV\rangle\times T(LV)}$ be the restriction of $\langle\, , \,\rangle_{\mathrm{FO}\times T(LV)}$ to $\langle LV\rangle \times T(LV)$. Lemma 3.17 implies that $T(LV)^{\perp} = 0$ for this pairing. Using the isomorphism of Theorem 1.3 between $\langle LV\rangle$ and $U_\hbar L\mathfrak{n}_+$, we may view $\langle\, , \,\rangle_{\langle LV\rangle\times T(LV)}$ as a pairing $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times T(LV)}$ between $U_\hbar L\mathfrak{n}_+$ and $T(LV)$. So again $T(LV)^{\perp} = 0$ for $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times T(LV)}$.

Let p be the quotient map from $T(LV)$ to $U_\hbar L\mathfrak{n}_+$. Composing

$$\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times T(LV)}$$

with $p\otimes id$, we get a pairing $\langle\, , \,\rangle_{T(LV)\times T(LV)}$ between $T(LV)$ and itself. It follows from (47) and (10) that $\langle\, , \,\rangle_{T(LV)\times T(LV)}$ is given by formula (11). Moreover, it follows from [11], Proposition 4.1 (relying on an identity of [18]) that $\langle\, , \,\rangle_{T(LV)\times T(LV)}$ induces a pairing $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times U_\hbar L\mathfrak{n}_-}$ between $U_\hbar L\mathfrak{n}_+$ and $U_\hbar L\mathfrak{n}_-$.

Since $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times U_\hbar L\mathfrak{n}_-}$ is induced by the pairing $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times T(LV)}$, and the annihilator of $T(LV)^{\perp}$ for this pairing is zero, we get that $(U_\hbar L\mathfrak{n}_+)^{\perp} = 0$ for $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_+\times U_\hbar L\mathfrak{n}_-}$. Exchanging the roles of $U_\hbar L\mathfrak{n}_+$ and $U_\hbar L\mathfrak{n}_-$, we find that $(U_\hbar L\mathfrak{n}_-)^{\perp} = 0$. Theorem 1.4 follows. ∎

Remark 3.18. This argument is completely similar to the proof of Theorem 1.2, the pairing between $U_\hbar L\mathfrak{n}_+$ and FO playing the role of the pairing beween $U_\hbar\mathfrak{n}_+$ and $\mathrm{Sh}(V)$.

3.3. The form of the R-matrix (proof of Proposition 1.4).

Let us define $A_+^{a,b}$ as the subalgebra of $U_\hbar L\mathfrak{n}_+$ generated by the $e_i[k]$, $i = 1, \dots, n$, $a \le k \le b$.

Lemma 3.19. *$A_+^{a,b}$ is a graded subalgebra of $U_\hbar L\mathfrak{n}_+$. We have $A_+^{a,b} + I^+_{\le a} + I^+_{\ge b} = U_\hbar L\mathfrak{n}_+$. Moreover, the graded components of $A_+^{a,b}$ are finite $\mathbb{C}[[\hbar]]$-modules.*

Proof. Let us define $A_+^{\le a}$ and $A_+^{\ge b}$ as the subalgebras of A_+ generated by the $e_i[k]$, $k \le a$ (resp. $k \ge b$). It follows from Theorem 1.3 that the product defines a surjective morphism from $A_+^{\le a} \otimes A_+^{a,b} \otimes A_+^{\ge b}$ to A_+. The Lemma follows. ∎

Since $I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha] \subset (I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp\perp}$, it follows from Lemma 3.19 that $(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp\perp}$ is a submodule of A_+ with a complement of finite type. Moreover, this module is also divisible, so that $A_+[\alpha]/(I^+_{\ge a}[\alpha]+I^+_{\le b}[\alpha])^{\perp\perp}$ is torsion-free. Since it is finitely generated, it follows that $A_+[\alpha]/(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp\perp}$ is a free, finite-dimensional $\mathbb{C}[[\hbar]]$-module.

On the other hand, $(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp}$ is a submodule of

$$\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(A_+[\alpha]/(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha]), \mathbb{C}[[\hbar]]),$$

and is therefore a $\mathbb{C}[[\hbar]]$-module of finite type. It is a submodule of $A_-[-\alpha]$, so it is torsion-free. It follows that $(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp}$ is also a free, finite-dimensional $\mathbb{C}[[\hbar]]$-module.

By construction, the pairing induced by $\langle\, , \,\rangle_{U_\hbar L\mathfrak{n}_\pm}$ between $(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp}$ and $A_+[\alpha]/(I^+_{\ge a}[\alpha] + I^+_{\le b}[\alpha])^{\perp\perp}$ is nondegenerate.

The fact that $P_{a,b}[\alpha]$ defines an element of $\lim_{\leftarrow a,b} A_+/(I^+_{\leq a} + I^+_{\geq b})^{\perp\perp} \otimes_{\mathbb{C}[[\hbar]]} A_-[\hbar^{-1}]$ follows from the following fact: if $F \subset G$ is an inclusion of finite dimensional vector spaces, and id_F and id_G are the identity elements of $F \otimes F^*$ and $G \otimes G^*$, then their images in $G \otimes F^*$ by the natural maps coincide.

Let x be any product of the $f_i[k]$, with $c \leq k \leq d$. Then x is orthogonal to $I^+_{\leq -c} + I^+_{\geq -d}$. It follows that $\cup_{a,b}(I^+_{\leq a} + I^+_{\geq b})^{\perp} = 0$, therefore $\cap_{a,b}(I^+_{\leq a} + I^+_{\geq b})^{\perp\perp} = 0$.

The proof of Proposition 1.4 follows then the proof of Proposition 1.1. ■

Remark 3.20. We have in the $\mathfrak{sl}_2$ case

$$P = \sum_r \sum_{i_1 < \cdots < i_r, n_r \geq 0} \frac{\hbar^{n_1 + \cdots + n_r}}{[n_1]^!_q \cdots [n_r]^!_q} e^{n_1}_{i_1} \cdots e^{n_r}_{i_r} \otimes f^{n_1}_{-i_1} \cdots f^{n_r}_{-i_r};$$

for $r \leq 2$, this formula is shown in [7], App. B. It would be interesting to obtain analogous explicit formulas in Yangian or elliptic cases.

4. Toroidal algebras (proofs of Propsositions 1.5, 1.6)

4.1. Proof Proposition 1.5.

1) follows from the argument of the beginning of the proof of Proposition 3.1. The first statements of 2) are obvious.

The proof of Proposition 3.1 then implies that j_+ induces a surjective Lie algebra morphism from $\widetilde{F}$ to $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, which restricts to an isomorphim between $\oplus_{\alpha \in \pm\Delta_+, \alpha \text{ real}; k \in \mathbb{Z}} \widetilde{F}[(\alpha, k)]$ and $(\oplus_{\alpha \in \pm\Delta_+, \alpha \text{ real}} \mathfrak{g}[\alpha]) \otimes \mathbb{C}[t, t^{-1}]$, which are the real roots part of both Lie algebras, and that $\widetilde{F}_+[(\alpha, k)] = 0$ if α does not belong to Δ_+.

It follows that $\mathrm{Ker} j_+$ is a graded subalgebra of $\widetilde{F}_+$, contained in

$$\oplus_{\alpha \in \Delta_+, \alpha \text{ imaginary}, k \in \mathbb{Z}} \widetilde{F}_+[(\alpha, k)].$$

■

4.2. Proof of Proposition 1.6.

Let us prove 1). Let us denote by $Z(\widetilde{F}_+)$ the center of $\widetilde{F}_+$. Let us first prove that $\mathrm{Ker} j_+ \subset Z(\widetilde{F}_+)$. Let x belong to $\mathrm{Ker} j_+$. We may assume that x is homogeneous of degree $n\delta$. Then any $n\delta + \alpha_i$, which is a real root. $[\widetilde{e}_i[k], x]$ is homogeneous of degree $n\delta + \alpha_i$, which is a real root. Since the restriction of j_+ on the subspace of $\widetilde{F}$ of degree $n\delta + \alpha_i$ is injective, $j_+([\widetilde{e}_i[k], x])$ is nonzero unless $[\widetilde{e}_i[k], x]$ is itself zero. But $j_+([\widetilde{e}_i[k], x])$ is equal to $[j_+(\widetilde{e}_i[k]), j_+(x)]$, which is zero because $j_+(x) = 0$. Therefore, $[\widetilde{e}_i[k], x]$ is equal to zero.

On the other hand, since j_+ is surjective and the center of $L\mathfrak{n}_+$ is zero, $\mathrm{Ker} j_+ = Z(\widetilde{F}_+)$. This proves 1).

Let us prove 2). The argument used in the proof of 1) implies that $\mathrm{Ker} j_+$ is contained in the center $Z(\widetilde{F})$ of $\widetilde{F}$. In the same way, one proves that $\mathrm{Ker} j_-$ is contained in $Z(\widetilde{F})$, therefore $\mathrm{Ker} j \subset Z(\widetilde{F})$. On the other hand, $\widetilde{F}$ is perfect.

It follows that we have a surjective Lie algebra morphism $j' : \mathfrak{t} \to \widetilde{F}$, such that the composition $\mathfrak{t} \to \widetilde{F} \to L\mathfrak{g}$ is the natural projection of $\mathfrak{t}$ on $L\mathfrak{g}$. Let j'_+ be the restriction of j' to $\mathfrak{t}_+$. For any i, k, we have $j'_+(e_i[k]^{\mathfrak{t}}) = \widetilde{e}_i[k] + k_{i,k}$, with $k_{i,k}$ in $\mathrm{Ker}(j)$. Let λ be any linear map from $\mathfrak{t}_+$ to $\mathrm{Ker}(j)$, such that $\lambda(e_i[k]^{\mathfrak{t}}) = k_{i,k}$. Set $\widetilde{j}'_+ = j'_+ - \lambda$. Then $\widetilde{j}'_+$ is a Lie algebra map from $\mathfrak{t}_+$ to $\widetilde{F}$. Since $\widetilde{j}'_+(e_i[k]^{\mathfrak{t}}) = \widetilde{e}_i[k]$ and the $e_i[k]^{\mathfrak{t}}$ generate $\mathfrak{t}_+$, the image of $\widetilde{j}'_+$ is contained in $\widetilde{F}_+$.

Moreover, $\widetilde{j}'_+$ is graded, and it coincides with j'_+ on the nonsimple roots subspace $[\mathfrak{t}_+, \mathfrak{t}_+] = \oplus_{\alpha \in \Delta_+ \setminus \{\epsilon_i\}} \mathfrak{t}_+[\alpha]$ of $\mathfrak{t}_+$. It follows that the restrictions of j' on $[\mathfrak{t}_+, \mathfrak{t}_+]$ and $[\mathfrak{t}_-, \mathfrak{t}_-]$ are graded.

Let us show that $\widetilde{j}'_+$ is surjective. Since the composition of $\widetilde{j}'_+$ with the projection $j : \widetilde{F}_+ \to L\mathfrak{n}_+$ is the natural projection, it suffices to show that any element x of $\mathrm{Ker} j_+$ is contained in $\widetilde{j}'_+(\mathfrak{t}_+)$. x belongs to the image of j', so let us set $x = j'(y)$, with $y = y_+ + y_- + y_0$, $y_\pm$ in $[\mathfrak{t}_\pm, \mathfrak{t}_\pm]$ and y_0 in $\mathfrak{h}_{\mathfrak{t}} \oplus \oplus_{i=1}^n \mathfrak{t}_+[\epsilon_i] \oplus \oplus_{i=1}^n \mathfrak{t}_-[-\epsilon_i]$, where $\mathfrak{h}_{\mathfrak{t}}$ is the Cartan subalgebra of $\mathfrak{t}$ (defined as $\mathfrak{h}[\lambda^{\pm 1}] \oplus Z_0$, see Remark 4.3). Then $j'(y_\pm)$ belong to $[\widetilde{F}_\pm, \widetilde{F}_\pm]$ and $j'(y_0)$ belongs to $\widetilde{H} \oplus \oplus_{i=1}^n \widetilde{F}_+[\epsilon_i] \oplus \oplus_{i=1}^n \widetilde{F}_-[-\epsilon_i] \oplus \mathrm{Ker}(j)$. Moreover, the map from $\mathfrak{h}_{\mathfrak{t}} \oplus \oplus_{i=1}^n \mathfrak{t}_+[\epsilon_i] \oplus \oplus_{i=1}^n \mathfrak{t}_-[-\epsilon_i]$ to $\widetilde{H} \oplus \oplus_{i=1}^n \widetilde{F}_+[\epsilon_i] \oplus \oplus_{i=1}^n \widetilde{F}_-[-\epsilon_i]$ induced by j' is injective, therefore $y_0 = 0$. It follows that $y_- = 0$ and $x = j'(y_+) = \widetilde{j}'_+(y_+)$, because $\widetilde{j}'_+$ coincides with j' on $[\mathfrak{t}_+, \mathfrak{t}_+]$.

Lemma 4.1. *1) Assume that A is not of type $A_1^{(1)}$. There is a unique Lie algebra map j'' from $\widetilde{F}_+$ to $\mathfrak{t}_+$ such that $j''(\widetilde{e}_i[k]) = e_i[k]^{\mathfrak{t}}$, for any $i = 0, \ldots, n$ and k integer.*

2) Assume that A is the Cartan matrix of type $A_1^{(1)}$. There is a unique Lie algebra map j'' from $\widetilde{F}_+$ to $\mathfrak{t}_+ / \oplus_{l \in \mathbb{Z}} \mathbb{C} K_\delta[l]$ such that $j''(\widetilde{e}_i[k]) = e_i[k]^{\mathfrak{t}}$, for any $i = 0, 1$ and k integer.

Proof. One should just check that the defining relations of $\widetilde{F}_+$ are satisfied by the $e_i[k]^{\mathfrak{t}}$ (in the $A_1^{(1)}$ case, by the images of $e_i[k]^{\mathfrak{t}}$ in $\mathfrak{t}_+ / \oplus_{l \in \mathbb{Z}} \mathbb{C} K_\delta[l]$). This is the case when A is not of type $A_1^{(1)}$, because in that case we set $e_i = \bar{x}_i \otimes \lambda^{\delta_{i0}}$ and we always have $\langle \bar{x}_i, \bar{x}_j \rangle_{\bar{\mathfrak{g}}} = 0$ for $i \neq j$.

If A is of type $A_1^{(1)}$, we have $x_0 = \bar{f} \otimes \lambda$, $x_1 = \bar{e}$, therefore

$$[x_0[l], x_1[m]] = (-\bar{h}\lambda \otimes t^{l+m}, -m K_\delta[l+m]),$$

so that $[x_0[l+1], x_1[m]] = [x_0[l], x_1[m+1]]$ holds in $\mathfrak{t}_+ / \oplus_{l \in \mathbb{Z}} \mathbb{C} K_\delta[l]$. ∎

Let us now prove Proposition 1.6, 2). The composition $\widetilde{j}'_+ \circ j''$ are Lie algebra maps from $\widetilde{F}_+ \to \mathfrak{t}_+ \to \widetilde{F}_+$ ($\widetilde{F}_+ \to \mathfrak{t}_+ / \oplus_{k \in \mathbb{Z}} \mathbb{C} K_\delta[k] \to \widetilde{F}_+$ in the $A_1^{(1)}$ case), which map the generators $\widetilde{e}_i[k]$ to themselves. Therefore, $\widetilde{F}_+$ can be viewed as a subalgebra of $\mathfrak{t}_+$ (of $\mathfrak{t}_+ / \oplus_{k \in \mathbb{Z}} \mathbb{C} K_\delta[k]$ in the $A_1^{(1)}$ case). This subalgebra contains the elements $e_i[k]^{\mathfrak{t}}$ of $\mathfrak{t}_+$ (resp. of $\mathfrak{t}_+ / \oplus_{k \in \mathbb{Z}} \mathbb{C} K_\delta[k]$). Since the Lie subalgebra of $\mathfrak{t}_+$ generated by the $e_i[k]^{\mathfrak{t}}$ is $\mathfrak{t}_+$ itself, the image of $\widetilde{F}_+$ is equal to $\mathfrak{t}_+$ (resp. to $\mathfrak{t}_+ / \oplus_{k \in \mathbb{Z}} \mathbb{C} K_\delta[k]$). This proves Proposition 1.6, 2). ∎

Remark 4.2. Proposition 1.6, 1), can also be obtained using the presentation given in [23] of $\mathfrak{t}$. In this paper, one shows that $\mathfrak{t}$ is isomorphic to the algebra $\dot{F}$ with generators $\dot{e}_i^{\pm}[k], \dot{h}_i[k]$ and $\dot{c}$ and relations $\mathrm{ad}(\dot{e}_i^{\pm}[0])^{1-a_{ij}}(\dot{e}_j^{\pm}[k]) = 0$, $[\dot{e}_i^{\pm}[k], \dot{e}_i^{\pm}[l]] = 0$, $[\dot{h}_i[k], \dot{e}_j^{\pm}[l]] = \pm a_{ij}\dot{e}_j^{\pm}[k+l]$, $[\dot{e}_i^{+}[k], \dot{e}_j^{-}[l]] = \delta_{ij}\dot{h}_i[k+l] + i\delta_{i+j,0}\langle e_i, f_i\rangle_{\bar{\mathfrak{g}}} c$, $[\dot{h}_i[k], \dot{h}_j[k]] = k\delta_{k+l,0}\langle h_i, h_j\rangle_{\bar{\mathfrak{g}}} c$, c central. It is then clear that there is a Lie algebra map from $\mathfrak{t}$ to $\dot{F}$. On the other hand, the system of relations $[\dot{e}_i^{+}[k], \dot{e}_i^{+}[l]] = \mathrm{ad}(\dot{e}_i^{\pm}[0])^{1-a_{ij}}(\dot{e}_j^{\pm}[k]) = 0$ is *not* a presentation of $\mathfrak{t}_+$, because the ideal generated by these relations is not preserved by the analogues of the $\Phi_{i,k}^{\pm}$ of the proof of Lemma 3.2. ■

Remark 4.3. *Toroidal Manin triples.* It is easy to define an extension of the Lie algebra $\mathfrak{t}$ with an invariant scalar product. Recall first ([23, 20]) that if $\mathfrak{g}$ is the central extension of the Lie algebra $\bar{\mathfrak{g}}[\lambda, \lambda^{-1}]$, $\mathfrak{t}$ is the universal central extension of $\bar{\mathfrak{g}}[\lambda^{\pm 1}, \mu^{\pm 1}]$. We have therefore

$$\mathfrak{t} = \bar{\mathfrak{g}}[\lambda^{\pm 1}, \mu^{\pm 1}] \oplus Z(\mathfrak{t}).$$

$Z(\mathfrak{t})$ is isomorphic to $\Omega^1_{\mathbb{A}}/d\mathbb{A}$, where $\mathbb{A} = \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}]$. We have

$$Z(\mathfrak{t}) = \oplus_{k,l\in\mathbb{Z}} K_{k\delta}[l] \oplus \mathbb{C}c,$$

with $K_{k\delta}[l] =$ the class of $\frac{1}{k}\lambda^k\mu^{l-1}d\mu$ if $k \neq 0$, $K_0[l] =$ the class of $\mu^l\frac{d\lambda}{\lambda}$, $c =$ the class of $\frac{d\mu}{\mu}$.

Define for k, l in $\mathbb{Z}$, $\widetilde{D}_{k\delta}[l]$ as the derivations of $\bar{\mathfrak{g}}[\lambda^{\pm 1}, \mu^{\pm 1}]$ equal to $\lambda^k\mu^l(l\lambda\partial_\lambda - k\mu\partial_\mu)$ if $k \neq 0$ and to $\mu^l\lambda\partial_\lambda$ if $k = 0$, and $\tilde{d}$ as the derivation $\mu\partial_\mu$.

Endow $\mathbb{C}^{\times 2}$ with the coordinates (λ, μ) and consider on this space the Poisson structure defined by $\{\lambda, \mu\} = \lambda\mu$. Let $\mathrm{Ham}(\mathbb{C}^{\times 2})$ be the Lie algebra of Hamiltonian vector fields on $\mathbb{C}^{\times 2}$ generated by the functions $\lambda^k\mu^l$, $k, l \in \mathbb{Z}^2$, $\log\lambda$ and $\log\mu$. For any function f on $\mathbb{C}^{\times 2}$, denote by V_f the corresponding Hamiltonian vector field. Then $\mathrm{Ham}(\mathbb{C}^{\times 2})$ is a Lie algebra, and the map $V_{\lambda^k\mu^l} \mapsto \widetilde{D}_{k\delta}[l]$, for $(k,l) \neq (0,0)$, $V_1 \mapsto 0$, $V_{\log\lambda} \mapsto \widetilde{D}_0[0]$, $V_{\log\mu} \mapsto \tilde{d}$, defines a Lie algebra map from $\mathrm{Ham}(\mathbb{C}^{\times 2})$ to $\mathrm{Der}(\bar{\mathfrak{g}}[\lambda^{\pm 1}, \mu^{\pm 1}])$.

The formula $V_f(\sum_i a_i db_i) = \sum_i \{f, a_i\}db_i + a_i d\{f, b_i\}$ defines an action of $\mathrm{Ham}(\mathbb{C}^{\times 2})$ on $\Omega^1_{\mathbb{A}}/d\mathbb{A}$, that is on $Z(\mathfrak{t})$. Define $\bar{D}_{k\delta}[l]$ and $\bar{d}$ as the following endomorphisms of $\mathfrak{t}$: $\bar{D}_{k\delta}[l](x, 0) = (\widetilde{D}_{k\delta}[l](x), 0), \bar{d}(x, 0) = (\tilde{d}(x), 0)$, and $\bar{D}_{k\delta}[l](0, \omega) = (0, V_{\lambda^k\mu^l}(\omega))$ for $(k,l) \neq (0,0)$, $\bar{D}_0[0](0,\omega) = (0, V_{\log\lambda}(\omega))$, $\bar{d}(0, \omega) = (0, V_{\log\mu}(\omega))$. These endomorphisms again define derivations of $\mathfrak{t}$, and we have now a Lie algebra map from $\mathrm{Ham}(\mathbb{C}^{\times 2})$ to $\mathrm{Der}(\mathfrak{t})$. Let $\tilde{\mathfrak{t}}$ be the corresponding crossed product Lie algebra of $\mathfrak{t}$ with $\mathrm{Ham}(\mathbb{C}^{\times 2})$. We denote by $D_{k\delta}[l]$ and d the elements of $\tilde{t}$ implementing the extensions of the derivations $\bar{D}_{k\delta}[l]$ and $\bar{d}$ to $\mathfrak{t}$.

Define for a, b integers, $x[a, b]$ as the element $(x \otimes \lambda^a\mu^b)$ of $\bar{\mathfrak{g}}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Define the bilinear form $\langle\ ,\ \rangle_{\tilde{\mathfrak{t}}}$ by

$$\langle x[a,b], x'[a',b']\rangle_{\tilde{\mathfrak{t}}} = \langle x, x'\rangle_{\bar{\mathfrak{g}}}\delta_{a+a',0}\delta_{b+b',0},$$

$$\langle D_{k\delta}[l], K_{k'\delta}[l']\rangle_{\widetilde{\mathfrak{t}}} = \delta_{k+k',0}\delta_{l+l',0}, \langle d, c\rangle_{\widetilde{\mathfrak{t}}} = 1,$$

and all other pairings of elements $x[a,b], K_{k\delta}[l], D_{k'\delta}[l'], c$ and d are zero.

Then $\langle\ ,\ \rangle_{\widetilde{\mathfrak{t}}}$ is an invariant nondegenerate bilinear form on $\widetilde{\mathfrak{t}}$.

Let us define D as the image of $\mathrm{Ham}(\mathbb{C}^{\times 2})$ in $\widetilde{\mathfrak{t}}$. Let us set $D_>, D_<$ and D_0 as its subspaces $\oplus_{k>0,l\in\mathbb{Z}}\mathbb{C}D_{k\delta}[l]$, $\oplus_{k<0,l\in\mathbb{Z}}\mathbb{C}D_{k\delta}[l]$ and $\oplus_{l\in\mathbb{Z}}\mathbb{C}D_0[l]\oplus\mathbb{C}d$. We have then $D = D_>\oplus D_0\oplus D_<$. In the same way, define $Z_>, Z_<$ and Z_0 as the subspaces $\oplus_{k>0,l\in\mathbb{Z}}\mathbb{C}K_{k\delta}[l]$, $\oplus_{k<0,l\in\mathbb{Z}}\mathbb{C}K_{k\delta}[l]$ and $\oplus_{l\in\mathbb{Z}}\mathbb{C}K_0[l]\oplus\mathbb{C}c$ of $Z(\mathfrak{t})$. We have then $Z(\mathfrak{t}) = Z_>\oplus Z_<\oplus Z_0$.

Recall we defined $\mathfrak{h}_{\mathfrak{t}}$ as the subalgebra $\bar{\mathfrak{h}}[\lambda^{\pm1}]\oplus Z_0$ of $\mathfrak{t}$. $\widetilde{\mathfrak{h}}_{\mathfrak{t}} = \mathfrak{h}_{\mathfrak{t}}\oplus D_0$ is then a Lie subalgebra of $\widetilde{\mathfrak{t}}$. In the spirit of the new realizations, we split $\widetilde{\mathfrak{h}}_{\mathfrak{t}}$ in two parts.

Let us set $\mathfrak{h}_+ = \bar{\mathfrak{h}}[\lambda]\oplus Z_0$, $\mathfrak{h}_- = \bar{\mathfrak{h}}[\lambda^{-1}]\oplus D_0$; then $\mathfrak{h}_+ + \mathfrak{h}_- = \widetilde{\mathfrak{h}}_{\mathfrak{t}}$, and $\mathfrak{h}_+\cap\mathfrak{h}_-$ is $\bar{\mathfrak{h}}$.

Define $L\mathfrak{n}_+$ and $L\mathfrak{n}_-$ as the linear spans of the $x[a,b]$, $a\in\mathbb{Z}, b>0$ ($b\geq 0$ if $x\in\bar{\mathfrak{n}}_+$), resp. of the $x[a,b]$, $a\in\mathbb{Z}, b<0$ ($b\leq 0$ if $x\in\bar{\mathfrak{n}}_-$). $L\mathfrak{n}_\pm$ are Lie subalgebras of $\bar{\mathfrak{g}}[\lambda^{\pm1},\mu^{\pm1}]$. $L\mathfrak{n}_\pm\oplus Z_\pm$ and $\widetilde{L\mathfrak{n}}_\pm = L\mathfrak{n}_\pm\oplus D_\pm\oplus Z_\pm$ are also Lie subalgebras of $\widetilde{\mathfrak{t}}$. Set $\widetilde{\mathfrak{t}}_\pm = \widetilde{L\mathfrak{n}}_\pm\oplus\mathfrak{h}_\pm$.

Endow $\widetilde{\mathfrak{t}}\times\bar{\mathfrak{h}}$ with the scalar product $\langle\ ,\ \rangle_{\widetilde{\mathfrak{t}}\times\bar{\mathfrak{h}}}$ defined by

$$\langle (x,h),(x',h')\rangle_{\widetilde{\mathfrak{t}}\times\bar{\mathfrak{h}}} = \langle x,x'\rangle_{\widetilde{\mathfrak{t}}} - \langle h,h'\rangle_{\bar{\mathfrak{h}}}.$$

Let $p_\pm$ be the natural projection of $\widetilde{\mathfrak{t}}_\pm$ on $\bar{\mathfrak{h}}$. Identify $\widetilde{\mathfrak{t}}_\pm$ as the Lie subalgebras of $\{(x,\pm p_\pm(x)), x\in\widetilde{\mathfrak{t}}_\pm\}$ of $\widetilde{\mathfrak{t}}\times\bar{\mathfrak{h}}[\lambda,\lambda^{-1}]$. $\widetilde{\mathfrak{t}}_\pm$ are supplementary isotropic subspaces of $\widetilde{\mathfrak{t}}\times\bar{\mathfrak{h}}[\lambda,\lambda^{-1}]$ and define therefore a Manin triple. This Manin triple is a central and cocentral extension (by $Z(\mathfrak{t})$ and D) of the Manin triple

$$(\bar{\mathfrak{g}}[\lambda^{\pm1},\mu^{\pm1}]\times\bar{\mathfrak{h}}, L\mathfrak{n}_+\oplus\bar{\mathfrak{h}}[\lambda], L\mathfrak{n}_-\oplus\bar{\mathfrak{h}}[\lambda^{-1}])$$

which is a part of the new realizations Manin triple $(\mathfrak{g}[\mu^{\pm1}]\times\mathfrak{h}, L\mathfrak{b}_+, L\mathfrak{b}_-)$.

One may also consider "intermediate" Manin triples, for example

$$\Big(\{\bar{\mathfrak{g}}[\lambda^{\pm1},\mu^{\pm1}]\oplus Z_>\oplus D_<\}\times\bar{\mathfrak{h}}, L\mathfrak{n}_+\oplus\bar{\mathfrak{h}}[\lambda]\oplus Z_>, L\mathfrak{n}_-\oplus\bar{\mathfrak{h}}[\lambda^{-1}]\oplus D_<\Big).$$

It is a natural problem to quantize the corresponding Lie bialgebra structures on $\widetilde{\mathfrak{t}}_\pm$. For this, one can think of the following program:

1) to compute the centers of $U_\hbar L\mathfrak{n}_+$ and (following [15]) the center of FO. By duality, these central elements should provide derivations of $U_\hbar L\mathfrak{n}_-$ (and FO) of imaginary degree. Compute these derivations and relations between them. One could expect that the algebra generated by the derivations is some difference analogue of the Lie algebra $\mathrm{Ham}(\mathbb{C}^{\times 2})$.

2) it should then be easy, following Theorem 1.3, to prove that the analogue of $i_\hbar$ is an isomorphism, and to derive from there the quantization of the Lie bialgebra $L\mathfrak{b}_+$.

We hope to return to these questions elsewhere.

A Appendix: Lemmas on $\mathbb{C}[[\hbar]]$-modules

Lemma A1. *Let E be a finitely generated $\mathbb{C}[[\hbar]]$-module. Let $E_{tors} = \{x \in E | \hbar^k x = 0$ for some $k > 0\}$ be the torsion part of E. Then E_{tors} is isomorphic to a direct sum $\oplus_{i=1}^p \mathbb{C}[[\hbar]]/(\hbar^{n_i})$, where n_i are positive integers, and E is isomorphic to the direct sum of E_{tors} and a free module $\mathbb{C}[[\hbar]]^{p'}$.*

Proof. As E is finitely generated, we have a surjective $\mathbb{C}[[\hbar]]$-modules morphism $\mathbb{C}[[\hbar]]^N \to E$. Let K be the kernel of this morphism. Then E is isomorphic to $\mathbb{C}[[\hbar]]^N/K$.

Let us determine the form of K. Let us set $\bar{K}_i = K \cap \hbar^i \mathbb{C}[[\hbar]]^N$. Then we have $\hbar \bar{K}_i \subset \bar{K}_{i+1}$. Let us set $E_0 = \mathbb{C}^N$, and $F_i = \hbar^{-i}\bar{K}_i \mod \hbar$. Then we have $F_0 \subset F_1 \subset \cdots \subset E_0$. Let p the integer such that $F_k = F_p$ for $k \geq p$. We can then find a basis $(v_i)_{1\leq i \leq N}$ of E_0 such that $(v_1, \ldots, v_{\mathrm{dim} F_0})$ is a basis of F_0, $(v_1, \ldots, v_{\mathrm{dim} F_1})$ is a basis of F_1, etc., $(v_1, \ldots, v_{\mathrm{dim} F_p})$ is a basis of F_p. Then K is the submodule $\oplus_i (\oplus_{k=\mathrm{dim} F_{i-1}+1}^{\mathrm{dim} F_i} \hbar^i \mathbb{C}[[\hbar]] v_k)$ of $E_0[[\hbar]]$.

It follows that the quotient $E_0[[\hbar]]/K$ is isomorphic to a direct sum

$$\oplus_{i=1}^p \mathbb{C}[[\hbar]]/(\hbar^{n_i}) \oplus \mathbb{C}[[\hbar]]^{p'}.$$

The statement of the Lemma follows. ∎

Corollary A1. *Any $\mathbb{C}[[\hbar]]$-submodule of a finite-dimensional free $\mathbb{C}[[\hbar]]$-module is free.*

Proof. This follows from the fact that such a submodule has no torsion and from the above Lemma. ∎

We have also

Lemma A2. *Let E be a free $\mathbb{C}[[\hbar]]$-module with countable basis $(v_i)_{i\geq 0}$. Any countably generated $\mathbb{C}[[\hbar]]$-submodule of E is free and has a countable basis.*

Proof. We repeat the reasoning of the proof of Lemma A1. Let $(w_i)_{i\geq 0}$ be a countable family of E and let F be the sub-$\mathbb{C}[[\hbar]]$-module of E generated by the w_i.

Set $\bar{F}_i = F \cap \hbar^i E$ and $F_i = \hbar^{-i} F \mod \hbar$. Generating families and bases for the F_i can be constructed inductively as follows.

A generating family for F_0 is $(w_i \mod \hbar)_{i\geq 0}$. We can then construct by induction a partition of $\mathbb{N}$ in subsets (i_k) and (j_k) such that $(w_{i_k} \mod \hbar)_{k\geq 0}$ is a basis of $\mathrm{Span}(w_i \mod \hbar)_{i\geq 0}$.

Let $\lambda_{kk'}$ be the scalars such that $w_{j_k} - \sum_{k'} \lambda_{kk'} w_{i_{k'}}$ belongs to $\hbar E$. Set $w_k^{(1)} = \hbar^{-1}[w_{j_k} - \sum_{k'} \lambda_{kk'} w_{i_{k'}}]$. Then a generating family of F_1 is $(w_{i_k}, w_k^{(1)} \mod \hbar)$. We then construct by induction a partition of $\mathbb{N}$ in subsets $(i_k^{(1)})$ and $(j_k^{(1)})$ such that $(w_{i_k}, w_{i_k^{(1)}}^{(1)} \mod \hbar)$ is a basis of F_1.

It is clear how to continue this procedure. Then $(w_{i_k}, w_{i_k^{(1)}}^{(1)}, w_{i_k^{(2)}}^{(2)} \ldots)$ forms a basis of F. ∎

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B. Enriquez
IRMA and Université Louis Pasteur,
7 rue René Descartes,
67084 Strasbourg, France
enriquez@math.u-strasbg.fr

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