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Convexity of Hamiltonian Manifolds

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Abstract. We study point set topological properties of the moment map. In particular, we introduce the notion of a convex Hamiltonian manifold. This notion combines convexity of the momentum image and connectedness of moment map fibers with a certain openness requirement for the moment map. We show that convexity rules out many pathologies for moment maps. Then we show that the most important classes of Hamiltonian manifolds (e.g., unitary vector spaces, compact manifolds, or cotangent bundles) are in fact convex. Moreover, we prove that every Hamiltonian manifold is locally convex.

1. Introduction

Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and let M be a Hamiltonian K-manifold, i.e., a symplectic K-manifold equipped with a moment map $\mu : M \to \mathfrak{k}^*$. The purpose of this note is to study certain point-set topological properties of μ.

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra and $\mathfrak{t}^+ \subseteq \mathfrak{t}^*$ a Weyl chamber. Since every K-orbit of $\mathfrak{k}^*$ meets $\mathfrak{t}^+$ in exactly one point, the image of μ is determined by $\psi(M) := \mu(M) \cap \mathfrak{t}^+$. A celebrated theorem of Kirwan, [5], states that if M is compact and connected then

i) *$\psi(M)$ is a convex set,* and

ii) *all fibers of μ are connected.*

On the other hand, many other Hamiltonian manifolds have these two properties. A very important example of a non-compact Hamiltonian manifold is a finite-dimensional unitary representation of K. Here also *i)*, *ii)* hold. Therefore, it seems worthwhile to introduce a general concept which encompasses both the compact and the unitary case.

In this paper we introduce a certain completeness property called *convexity.* To define it, let $\psi : M \to \mathfrak{t}^+$ be the map which assigns to $x \in M$ the point of intersection of $K\mu(x)$ with $\mathfrak{t}^+$. This is a K-invariant continuous map.

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For any two points $u, v \in \mathfrak{t}^+$ let $\overline{uv}$ be the line segment joining u with v. Then we call M *convex* if $\psi^{-1}(\overline{uv})$ is connected for all $u, v \in \psi(M)$.

It is pretty immediate from the definition that convexity of M implies *i)* and *ii)*. Less obvious is the fact that convexity entrains also a more subtle property, namely that

iii) the map $\psi : M \to \psi(M)$ *is open.*

In section 2 we show that, conversely, *i)*, *ii)*, and *iii)* are equivalent to convexity. To this end, we use results of Sjamaar, [8], on the local structure of the moment map. It is an open problem whether *iii)* is really needed to imply convexity. In any case, convexity is a useful concept since it rules out much of the pathological behavior a moment map may have. To illustrate this we give various examples of bad moment maps in section 3.

On the other hand, we prove that many Hamiltonian manifolds occurring in applications are in fact convex. For example, we show that a connected Hamiltonian manifold is convex whenever it is compact, a complex algebraic Kähler variety, or a cotangent bundle.

Then, in section 5, we show that every Hamiltonian manifold is locally convex. This makes the concept of convexity useful even for arbitrary Hamiltonian manifolds. This fact conceptualizes some topological considerations in the paper [4] by Karshon-Lerman.

The convexity of M has consequences far beyond the topological properties *i)–iii)*. A few are mentioned in this paper like the fact that for convex M, the momentum image $\psi(M)$ is locally a convex polytope. For a much deeper application see the paper [6] where we were able to identify all collective functions on M provided M is convex.

Remark. This paper is an extended version of the second section of the preprint [6].

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2. Convex Hamiltonian manifolds: the definition and a criterion

Let K be a connected, compact Lie group with Lie algebra $\mathfrak{k}$. A *Hamiltonian K-manifold* is a K-manifold M with a K-invariant symplectic form ω and with a *moment map*, i.e., a K-equivariant map $\mu : M \to \mathfrak{k}^*$ such that

$$\langle \xi, d\mu(\eta) \rangle = \omega(\xi x, \eta) \quad \text{for all } \xi \in \mathfrak{k}, x \in M, \eta \in T_x(M). \tag{2.1}$$

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra corresponding to a maximal torus $T \subseteq K$. Since $\mathfrak{t}$ has a unique T-stable complement in $\mathfrak{k}$, we can regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{k}^*$. Let $\mathfrak{t}^+ \subseteq \mathfrak{t}^*$ be a Weyl chamber. The composition $\mathfrak{t}^+ \hookrightarrow \mathfrak{k}^* \twoheadrightarrow \mathfrak{k}^*/K$ is a homeomorphism. We use it to construct a continuous map ψ which makes the following diagram commutative:

$$\begin{array}{ccc} M & \overset{\mu}{\to} & \mathfrak{k}^* \\ \psi\downarrow & & \downarrow \\ \mathfrak{t}^+ & \to & \mathfrak{k}^*/K \end{array} \tag{2.2}$$

In other words, ψ is the unique map with $K\mu(x) \cap \mathfrak{t}^+ = \{\psi(x)\}$ for all $x \in M$.

For any two (not necessarily distinct) points $u, v \in \mathfrak{t}^*$ let $\overline{uv}$ be the line segment joining them. Observe, that a subset C of $\mathfrak{t}^*$ is convex if $\overline{uv} \cap C$ is connected for all $u, v \in C$. This is just a slight reformulation of the classical definition and motivates:

Definition. A Hamiltonian K-manifold is called *convex* if $\psi^{-1}(\overline{uv})$ is connected for all $u, v \in \psi(M)$.

A first reformulation of the concept is:

Proposition 2.1. *A Hamiltonian K-manifold M is convex if and only if $\psi^{-1}(B)$ is connected for every convex subset B of $\mathfrak{t}^+$.*

Proof. Suppose there is a convex subset $B \subseteq \mathfrak{t}^+$ such that $\psi^{-1}(B)$ is disconnected. Then $\psi^{-1}(B)$ is the disjoint union of two non-empty open subsets, say U and V. Let $u \in \psi(U)$ and $v \in \psi(V)$. Since B is convex, we have $\overline{uv} \subseteq B$, hence $X := \psi^{-1}(\overline{uv}) \subseteq \psi^{-1}(B)$. Thus, $X = (X \cap U) \cup (X \cap V)$ is the disjoint union of two non-empty open subsets which means that M is not convex. This shows one direction, the other is trivial. ■

Putting $u = v$ in the definition, we see that for a convex manifold M the fibers of ψ are connected. Moreover, for every $u, v \in \psi(M)$ the intersection $\overline{uv} \cap \psi(M)$ has to be connected, i.e., $\psi(M)$ is convex. These two properties are not enough to imply convexity, though. More precisely, we have the following characterization:

Theorem 2.2. *A Hamiltonian K-manifold is convex if and only if the following conditions are satisfied:*

i) The image $\psi(M)$ is convex.

ii) The fibers of ψ are connected.

iii) The map $\psi : M \to \psi(M)$ is open.

Proof of "$\Leftarrow$". Assume that *i)* through *iii)* hold. If $X := \psi^{-1}(\overline{uv})$ is disconnected then there are open subsets U_1, U_2 of M such that X is the disjoint union of the non-empty subsets $X_i := X \cap U_i$. Suppose $w \in \psi(X_1) \cap \psi(X_2)$. Then $F := \psi^{-1}(w)$ is the union of the non-empty disjoint sets $F \cap U_1$ and $F \cap U_2$. Since this contradicts *ii)*, we have $\psi(X_1) \cap \psi(X_2) = \emptyset$. Furthermore $\psi(X_1) \cup \psi(X_2) = \psi(X) = \overline{uv} \cap \psi(M) = \overline{uv}$, by *i)*. Finally, *iii)* implies that $\psi(X_i) = \overline{uv} \cap \psi(U_i)$ is open in $\overline{uv}$ which contradicts the connectedness of $\overline{uv}$. The proof of the reverse direction is deferred to after Lemma 2.5. ■

All properties which we considered so far make sense for any continuous map ψ from a topological space M to a convex subset of a real vector space and we have shown that the following implications hold:

$$i) \wedge ii) \wedge iii) \quad \Rightarrow \quad \psi \text{ convex} \quad \Rightarrow \quad i) \wedge ii) \tag{2.3}$$

Neither of these implications is reversible in this generality: For the first arrow, let $\psi: M \to \mathbb{R}^2$ be the (real) blow up of $\mathbb{R}^2$ in the origin, i.e., $M = \{((x_0, x_1), [y_0 : y_1]) \in \mathbb{R}^2 \times \mathbf{P}^1(\mathbb{R}) \mid x_0 y_1 = x_1 y_0\}$. Consider the inverse image X of a line segment $\overline{uv} \subseteq \mathbb{R}^2$. If $\overline{uv}$ does not contain the origin then $X \xrightarrow{\sim} \overline{uv}$. Otherwise, X is the union of the exceptional fiber $\psi^{-1}(0,0) \cong S^1$ and an interval meeting in one point. In either case, X is connected, hence ψ is convex. On the other hand, the image of any sufficiently small neighborhood of a point in the exceptional fiber is contained in a small cone and contains the origin. Such a set can't be open, hence *iii)* does not hold. For the second arrow let M be the same as above but with one point in the exceptional fiber removed. Then ψ is still surjective and all fibers are connected. But there is a line segment $\overline{uv}$ whose preimage is disconnected.

This shows that we have to use some special properties of moment maps. The dual of the Cartan algebra $\mathfrak{t}^*$ contains a canonical lattice Γ namely the differentials of all homomorphisms $T \to \mathbb{R}/\mathbb{Z}$. A *rational homogeneous cone* is a subset of $\mathfrak{t}^*$ which is of the form $\sum_{i=1}^N \mathbb{R}_{\geq 0}\gamma_i$ where $\gamma_1, \ldots, \gamma_N \in \Gamma$. A *rational cone* is a translate $u + C$ of a homogeneous rational cone C by a vector $u \in \mathfrak{t}^*$. In this case we say that u is a vertex of $u + C$. Note that $u + C$ can have many vertices namely all points in $u + (C \cap -C)$. The following theorem of Sjamaar contains all the special properties of the moment map which we are going to need.

Theorem 2.3. ([8] Thm. 6.5) *Let M be a Hamiltonian K-manifold. Then for every orbit $Kx \subseteq M$ there is a unique rational cone $C_x \subseteq \mathfrak{t}^*$ with vertex $\psi(x)$ such that:*

i) There exist an arbitrarily small K-stable neighborhood U of Kx such that $\psi(U)$ is a neighborhood of $\psi(x)$ in C_x.

ii) For $u \in \mathfrak{t}^+$ let x and y be in the same connected component of $\psi^{-1}(u)$. Then $C_x = C_y$.

A first application of this theorem is a general openness property of ψ:

Proposition 2.4. *Let M be a Hamiltonian K-manifold such that all fibers of $\psi: M \to \mathfrak{t}^+$ are connected. Let $U \subseteq M$ be open. Then also $\psi^{-1}\psi(U)$ is open.*

Proof. Let U' be the union of all translates kU, $k \in K$. Then U' is open with $\psi(U') = \psi(U)$. Thus, we may replace U by U' and assume that U is K-stable. Let $y \in \psi^{-1}\psi(U)$ and $u := \psi(y)$. Then there is $x \in U$ such that also $\psi(x) = u$. Since $\psi^{-1}(u)$ is connected, Theorem 2.3*ii)* implies that $C_x = C_y =: C$. By part *i)* of that theorem there are open neighborhoods U_x, U_y of x, y, respectively, such that $\psi(U_x)$ and $\psi(U_y)$ are neighborhoods of u in C. Hence $(\psi|_{U_y})^{-1}(\psi(U_x))$ is a neighborhood of y which is contained in $\psi^{-1}\psi(U)$. This proves the assertion. ■

We can now derive an openness criterion for ψ:

Lemma 2.5. *Let M be a Hamiltonian K-manifold such that all fibers of $\psi: M \to \mathfrak{t}^+$ are connected. Assume moreover that every $u \in \psi(M)$ has an arbitrarily small neighborhood B such that $\psi^{-1}(B)$ is connected. Then $\psi: M \to \psi(M)$ is an open map.*

Proof. Let $x \in M$ and U be an open neighborhood of x. We have to show that $\psi(U)$ is a neighborhood of $u := \psi(x)$ in $\psi(M)$. For this we may assume that $\psi(U)$ is a neighborhood of u in C_x. Let B be a neighborhood of u in $\mathfrak{t}^+$ such that $B \cap C_x \subseteq \psi(U)$ and such that $\psi^{-1}(B)$ is connected. Then $V_1 := \psi^{-1}\psi(U)$ is open in M by Proposition 2.4. Clearly, also $V_2 := \psi^{-1}(\mathfrak{t}^* \setminus C_x)$ is open in M. Moreover, V_1 and V_2 are disjoint and cover $\psi^{-1}(B)$. Connectivity implies $\psi^{-1}(B) \subseteq V_1$, i.e., $B \cap \psi(M) \subseteq \psi(U)$ which proves the assertion. ∎

Now we can complete the proof of Theorem 2.2:

Proof of "⇒". Assume that M is convex. Then *i)* and *ii)* clearly hold. Let $u \in \psi(M)$ and $B \subseteq \mathfrak{t}^+$ a convex neighborhood of x. Since also $\psi(M)$ is convex we have $\overline{uv} \subseteq B \cap \psi(M)$ for every $v \in B \cap \psi(M)$. By assumption, $\psi^{-1}(\overline{uv})$ is connected. This implies that $\psi^{-1}(B)$ is connected. Thus *iii)* holds by Lemma 2.5. ∎

One can express properties *i)–iii)* of Theorem 2.2 purely in terms of the moment map μ:

Theorem 2.6. *A Hamiltonian K-manifold is convex if and only if the following conditions are satisfied:*

i′) The intersection $\mu(M) \cap \mathfrak{t}^+$ is convex.

ii′) The fibers of μ are connected.

iii′) Whenever $U \subseteq M$ is a K-invariant *open subset then $\mu(U)$ is open in $\mu(M)$.*

Proof. By construction we have $\psi(M) = \mu(M) \cap \mathfrak{t}^+$, whence *i)*⇔*i′)*. For $x \in M$ let $y := \mu(x)$ and $u := \psi(x)$. Then there is a K-equivariant map $\psi^{-1}(u) \to Ky$, hence $\psi^{-1}(u) = K \times^L \mu^{-1}(y)$ where $L := K_{\mu(x)}$. The equivalence of *ii)* and *ii′)* now follows from the fact that K and L are connected. Finally, let $U \subseteq M$ be open and U' the union of all translates kU. Since $\psi(U') = \psi(U)$ it suffices for checking *iii)* that $\psi(U)$ is open in $\psi(M)$ for all K-invariant U. But then *iii)*⇔*iii′)* follows from the fact that $\mathfrak{k}^* \to \mathfrak{t}^+$ is the quotient map by K. ∎

Remark. In general, one cannot expect μ to be an open map. Take for example $M = T^*(K/H)$ where $K = SU(2)$ and $H = U(1)$, the maximal torus. Then $M = K \times^H \mathfrak{h}^\perp$ where $\mathfrak{h}^\perp \subseteq \mathfrak{k}^*$ is the annihilator of the Lie algebra of H. Then $\mu(M) = \operatorname{ad} K \cdot \mathfrak{h}^\perp = \mathfrak{k}^*$. On the other hand, let $V_0 \subset K/H$ be a small open subset, V its preimage in K and $U := V \times^H \mathfrak{t}^\perp$. Then U is open in M but its image $\mu(U) = V\mathfrak{h}^\perp$ is a conical *proper* subset of $\mathfrak{k}^*$ containing 0. Thus, it is not open.

We conclude this section with two general properties of convex Hamiltonian manifolds. First we show that the cone C_x can be recovered from $\psi(M)$.

Theorem 2.7. *Let M be a convex Hamiltonian K-manifold and $x \in M$. Then C_x is the smallest cone containing $\psi(M)$ and having $\psi(x)$ as a vertex. Moreover, $\psi(M)$ forms a neighborhood of $\psi(x)$ in C_x.*

Proof. Denote this smallest cone by C. By Theorem 2.3, there is a neighborhood U of x such that $\psi(U)$ is a neighborhood of $u := \psi(x)$ in C_x. This shows that $C_x \subseteq C$. For the converse we just have to show that $\psi(M)$ is contained in C_x.

Let $v \in \psi(M)$. Since $\psi(U)$ is open in $\psi(M)$ and since $\overline{uv} \subseteq \psi(M)$ also $\overline{uv} \cap \psi(U)$ is open in $\overline{uv}$. This implies $\overline{uv} \subseteq C_x$, thus $v \in C_x$. ∎

Corollary 2.8. *Let M be a convex Hamiltonian K-manifold. Then the image $\psi(M)$ is locally a polyhedral cone and, in particular, locally closed and semi-analytic in $\mathfrak{t}^*$.*

Remark. See Example 3.2 of the next section for a connected Hamiltonian manifold with $\psi(M) \not\subseteq C_x$ and Example 3.3 where $\psi(M)$ is not locally closed.

3. Pathological examples of moment maps

One can reformulate Proposition 2.4 as follows: If ψ has connected fibers then one can factorize it as

$$M \xrightarrow{\alpha} S \xrightarrow{\beta} \mathfrak{t}^+ \tag{3.1}$$

where α is open, surjective and β is injective. In fact, set-theoretically we let $S = \psi(M)$, $\alpha = \psi$, and β to be the natural inclusion of S in $\mathfrak{t}^+$. We equip S with the topology that $V \subseteq S$ is open if and only if $\psi^{-1}(V)$ is open in M.

How important is it that ψ has connected fibers? There is a straightforward generalization of the factorization (3.1): let $S := M/\sim$ where $x \sim y$ if x and y are in the same connected component of some fiber of ψ. We give S the quotient topology. Then the proof of Proposition 2.4 shows that α is an open map. Moreover, the map β is continuous with discrete fibers.

Example 3.1. Consider the above construction in the following setting. Let $K = T$ be a torus and M_0 any Hamiltonian T-manifold with moment map μ_0. Let V be a manifold and $\gamma : V \to \mathfrak{t}^*$ a smooth map which is everywhere a local isomorphism. Now consider the fiber product $M = M_0 \times_{\mathfrak{t}^*} V$. Since it is locally isomorphic to M_0, it is also Hamiltonian. Its moment map is simply the projection to V composed with γ. Thus, if μ_0 has connected fibers then $S = M/\sim = \gamma^{-1}\mu_0(M_0)$. This way, one can construct Hamiltonian manifolds with arbitrarily disconnected fibers. A concrete example is the following: take $T = (S^1)^2$, $M_0 := T^*(T) = T \times \mathfrak{t}^*$, and identify $\mathfrak{t}^* \cong \mathbb{R}^2$ with $\mathbb{C}$. Let $S = \mathbb{C} \setminus \{0, \frac{2}{3}\}$ and $\gamma : S \to \mathbb{C} = \mathfrak{t}^* : z \mapsto z^3 - z^2$. Then γ is a surjective unramified map. Since γ is an open map, we have constructed a Hamiltonian manifold $M = T \times S$ such that ψ is open with convex image but such that most fibers are disconnected.

Example 3.2. Using the same technique as in Example 3.1, one can also construct a Hamiltonian manifold such that $\beta : S \to \mathfrak{t}^*$ is injective but not open. For this take again $K = (S^1)^2$, $M_0 = T \times \mathfrak{t}^*$ and identify $\mathfrak{t}^*$ with $\mathbb{C}$. Let for some $0 < \epsilon < \pi$ let S' be the set of all non-zero complex numbers with $-\epsilon < \operatorname{Arg} z < \pi$ (i.e., S' is a bit larger than the upper half-plane). Let $\gamma' : S' \to \mathbb{C} = \mathfrak{t}^* : z \mapsto z^2$

and $M' = T \times S'$. Now we apply Lerman's symplectic cut technique, [7], to the preimage Y of the positive real half line which is isomorphic to $S^1 \times S^1 \times \mathbb{R}$: one can cut away the preimage of the sector $-\epsilon < \operatorname{Arg} z < 0$ and replace Y by $Y/(1 \times S^1) \cong S^1 \times \mathbb{R}$. Then one gets a new Hamiltonian manifold M such that S is the set of all $z \in \mathbb{C}$, $z \neq 0$ with $0 \leq \operatorname{Arg} z < \pi$, i.e., S is the upper half-plane together with the positive real half-line. The map $\gamma : S \to \mathbb{C} : z \mapsto z^2$ is injective with image $\mathbb{C} \setminus \{0\}$ but not open: no neighborhood of the positive half-line maps to a neighborhood of the image point. This is basically the Example 3.10 of [4] and which is avoided by our notion of convexity.

Example 3.3. This example is the same as the preceding one but we remove the preimage of $[1, \infty) \times (0, \infty) \subseteq \mathfrak{t}^*$. Then $\psi(M)$ is not locally closed near the point $(1, 0)$.

Example 3.4. Unfortunately, S will not in general be Hausdorff. To show this, let $K = S^1$ and $M_0 := T^*(S^1) \times V = S^1 \times \mathbb{R} \times V$ where V is a non-zero symplectic vector space. The fibers of the moment map are $S^1 \times V$. Now let $H \subseteq V$ be a hyperplane and let M be M_0 where we removed $S^1 \times H$ from one of the fibers of the moment map. Then one of the fibers of the moment map gets disconnected and $S = M/\sim$ is a real line with one of its points doubled. In particular, it is not Hausdorff.

Example 3.5. In the preceding example one could remedy the situation by replacing "$\sim$" by a coarser equivalence relation. More precisely, put $S := M/\approx$ where $x \approx y$ if $\psi(x) = \psi(y)$ and $C_x = C_y$. Then one can show that α is still open and that β has discrete fibers. But also in this definition, S might not be Hausdorff. To construct an example consider the two Hamiltonian S^1-manifolds $M_1 := T^*(S_1) \times \mathbb{R}^2 = S^1 \times \mathbb{R} \times \mathbb{R}^2$, and $M_2 := \mathbb{C} \times \mathbb{R}^2$. Here $K = S^1$ acts in both cases on the first factor and $\mathbb{R}^2$ has the standard symplectic structure $dx \wedge dy$. The moment maps are $\mu(\alpha, t, x, y) = t$ and $\mu(z, x', y') = |z|^2$, respectively. Consider the open subsets $M_1^0 := \mu_1^{-1}(\mathbb{R}^{>0}) = \{t > 0\}$ and $M_2^0 := \mu_2^{-1}(\mathbb{R}_{>0}) = \{z \neq 0\}$. They are isomorphic as Hamiltonian manifolds, the isomorphism being $z = \sqrt{t}\exp(2\pi i\alpha)$. Now we twist this isomorphism by the automorphism of M_1^0

$$(\alpha, t, x, y) \mapsto (\alpha - t^{-2}y, t, x + t^{-1}, y) \tag{3.2}$$

and glue M_1 and M_2 along $M_1^0 \xrightarrow{\sim} M_2^0$. It is easily seen that the graph of the gluing isomorphism is closed in $M_1 \times M_2$ (on the graph holds $x' = x + t^{-1}$ and $|z|^2 = t$. Thus, on its closure we get $t(x' - x) = 1$ and $|z|^2 = t$ which implies $t > 0$ and $z \neq 0$, i.e., the graph is closed). Therefore, the resulting manifold M is Hausdorff. This way we constructed a Hamiltonian S^1-manifold for which all fibers of the moment map but one are connected ($\cong S^1 \times \mathbb{R}^2$) and the zero fiber is disconnected ($\cong (S^1 \times \mathbb{R}^2) \dot\cup \mathbb{R}^2$). Moreover, for x in the first component we have $C_x = \mathbb{R}$ while in the second holds $C_x = \mathbb{R}_{\geq 0}$. This shows that $S = M/\approx$ equals $\mathbb{R}$ with "doubled" origin. In particular, it is not Hausdorff.

Problem 3.6. We have seen that convexity is equivalent to the combination of *i)*, the convexity of $\psi(M)$, *ii)*, the connectedness of the fibers of ψ, and *iii)*, the openness of ψ onto its image. Properties *i)* and *ii)* are well studied in the literature. Therefore, one may wonder whether *iii)* is really an additional constraint. However, the author was not able to come up with an example of an Hamiltonian manifold where *i)* and *ii)* hold, but *iii)* doesn't. Observe, that Examples 3.2 and 3.3 have a non-convex image while in the other examples some fibers of ψ are disconnected. In case, an example with $i) \wedge ii) \wedge \neg iii)$ exists then the space S would have have quite peculiar properties: it is a manifold with non-empty boundary and possibly corners such that there exists a continuous *bijective* map to a *convex* subset of an Euclidean space.

4. Examples of convex Hamiltonian manifolds

Let V be a unitary representation of K. Then every smooth K-stable complex algebraic subvariety of $\mathbf{P}(V)$ is in a canonical way a Hamiltonian K-manifold. We call a Hamiltonian K-manifold *projective* if it arises this way but possibly with the symplectic form and the moment map rescaled by some non-zero factor.

Proposition 4.1. *Let M be a Hamiltonian K-manifold such that for every projective Hamiltonian K-manifold X holds that $\psi_{M\times X}^{-1}(0)$ is connected. Then $\psi_M : M \to \psi(M)$ is an open map with connected fibers.*

Proof. Let $\overline{X}$ equal X with the symplectic structure multiplied by -1. Then we have $\mu_{M\times\overline{X}}(m,x) = \mu_M(m) - \mu_X(x)$. By choosing for X the coadjoint orbit Ku, $u \in \mathfrak{t}^+$ we obtain that $\psi_M^{-1}(u) = \psi_{M\times\overline{X}}^{-1}(0)$ is connected. Now let X_0 be projective such that $\psi_{X_0}(X_0)$ is a neighborhood of 0, e.g., $X_0 = \mathbf{P}(V)$ where V contains stable points for the $K^{\mathbb{C}}$-action. By rescaling, we can arrange that $\psi(X_0)$ is arbitrarily small. Let $X := X_0 \times Ku$. Then $B := \psi_X(X)$ is an arbitrarily small neighborhood of u in $\mathfrak{t}^+$. Moreover, projection to the first factor induces a surjective map $\psi_{M\times\overline{X}}^{-1}(0) \twoheadrightarrow \psi_M^{-1}(B)$. By assumption, the first, hence the second set, is connected. By Lemma 2.5, we conclude that ψ_M is open onto its image. ∎

Now we can give examples of convex Hamiltonian manifolds:

Theorem 4.2. *Every connected Hamiltonian K-manifold M satisfying any one of the conditions i) through iv) below is convex.*

i) M is compact.

ii) The moment map $\mu : M \to \mathfrak{k}^*$ is proper.

iii) The manifold M is a complex algebraic variety, the action of K is the restriction of an algebraic $K^{\mathbb{C}}$-action, and the symplectic structure is induced by a K-invariant Kähler metric.

iv) The manifold M is a complex Stein space, the action of K is the restriction of a holomorphic $K^{\mathbb{C}}$-action, and the symplectic structure is induced by a K-invariant Kähler metric.

Proof. In all cases, it is known that ψ has connected fibers and convex image (see [3] or [8] for *i)*, *ii)* and [2] for *iii)* and *iv)*). Moreover, the classes *i)-iii)* are preserved by taking the product with a projective Hamiltonian K-manifold. Thus Proposition 4.1 implies that ψ is open in these cases.

For *iv)* one has to argue a bit differently since the class of Stein spaces is not stable for products with projective manifolds. But fortunately, Heinzner-Huckleberry prove in [2] a stronger result: given a holomorphic action of G on a complex manifold M then ψ has convex image and connected fibers whenever the G-action is regular. Here "regular" means "each G-orbit of M has an open dense neighborhood which is G-equivariantly the locally biholomorphic image of an irreducible complex G-space U such that U admits a closed, holomorphic, G-equivariant embedding into an algebraic G-variety." It is noted in [2] that actions on Stein spaces are regular. Clearly, the class of manifolds with regular action is stable for taking products with algebraic G-varieties. Thus, ψ is open also in case *iv)*. ∎

Note, that *i)* includes all projective spaces while all linear actions on unitary vector spaces are covered by *iii)*.

5. Local convexity

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Let M be a Hamiltonian K-manifold. Then every $x \in M$ has an arbitrarily small K-stable open neighborhood U which is convex as Hamiltonian manifold and such that $\psi(U)$ is open in C_x.*

Proof. We start with some reductions. Let $y := \psi(x) \in \mathfrak{t}^+ \subseteq \mathfrak{k}^*$ and $L := K_y$. Assume $L \neq K$. It is convenient to identify $\mathfrak{k}^*$ with $\mathfrak{k}$ using a K-invariant scalar product. Then $y \in \mathfrak{t} \subseteq \mathfrak{l} := \operatorname{Lie} L \subseteq \mathfrak{k}$. Let $V_0 \subseteq \mathfrak{t}^+$ be a small open neighborhood of y. Then $V_L := L \cdot V_0$ is a small open neighborhood of Ly in $\mathfrak{l}$. Similarly, $V_K := K \cdot V_0$ is a small open neighborhood of Ky in $\mathfrak{k}$. Moreover, V_L is an orthogonal slice to Ky. This implies that $K \times^L V_L \xrightarrow{\sim} V_K$. Now let $M_L := \mu^{-1}(V_L)$ and $M_K := \mu^{-1}(V_K)$. Then M_K is an open neighborhood of Kx with $K \times^L M_L \xrightarrow{\sim} M_K$. It can be shown ([1] Thm. 26.7) that M_L is a Hamiltonian L-manifold whose moment map μ_L is just the restriction of μ to M_L. Similarly, there is ψ_L which is the restriction of ψ. Suppose the theorem is true for L instead of K. Then there is an arbitrarily small open neighborhood U_L of Lx in M_L which is convex as a Hamiltonian L-manifold. Then $U := K \cdot U_L = K \times^L U_L$ is an arbitrarily small K-invariant open neighborhood of Kx. Moreover, for the preimage of a line segment $\overline{uv} \subseteq \mathfrak{t}^*$ holds

$$\psi^{-1}(\overline{uv}) \cap U = K \times^L (\psi_L^{-1}(\overline{uv}) \cap U_L). \tag{5.1}$$

This shows that U is a convex Hamiltonian K-manifold.

Thus, it remains to consider the case $L = K$, i.e., $y \in (\mathfrak{k}^*)^K$. Since the translate $\mu' := \mu - y$ is again a moment map we may assume $y = 0$. Let $H := K_x$. Since the orbit Kx is isotropic we have $\mathfrak{k}x \subseteq (\mathfrak{k}x)^\perp \subseteq T_xM$. Let S be an H-invariant complement of $\mathfrak{k}x$ in $(\mathfrak{k}x)^\perp$. Then one can show ([8] Thm. 6.3, [1] Thm. 22.1)

- S is a unitary H-module with moment map $\mu_S : S \to \mathfrak{h}^* : v \mapsto (\xi \mapsto \langle v, \xi v\rangle)$;
- $M_0 := K \times^H (\mathfrak{h}^\perp \oplus S)$ is a Hamiltonian K-manifold with moment map $\mu([k,(\alpha,v)]) = k(\alpha + \mu_S(v))$.
- Let $x_0 := [1,(0,0)] \in M_0$. Then $Kx \subseteq M$ and $Kx_0 \subseteq M_0$ have K-invariant open neighborhoods which are isomorphic as Hamiltonian K-manifolds.

Because of the last point we can (and will) assume $M = M_0$ and $x = x_0$.

Let D_r and $\overline{D}_r$ be the open and the closed ball of radius r in S, respectively. Consider the natural scalar $U(1)$-action on S and let $P = \overline{D}_r/\sim$ be the space where all $U(1)$-orbits in $\overline{D}_r \setminus D_r$ are collapsed to a point. One can show (Lerman's symplectic cut, [7]) that P is a compact Hamiltonian H-manifold (in fact, it is a complex projective space) whose moment map μ_P factors μ_S:

$$\begin{array}{ccc} \overline{D} & \twoheadrightarrow & P \\ \cap\!\mid & & \downarrow \mu_P \\ S & \stackrel{\mu_S}{\to} & \mathfrak{h}^* \end{array} \tag{5.2}$$

Now consider

$$V_r := K \times^H (\mathfrak{h}^\perp \times D_r), \quad \overline{V}_r = K \times^H (\mathfrak{h}^\perp \times \overline{D}_r), \quad Q := K \times^H (\mathfrak{h}^\perp \times P). \tag{5.3}$$

Then V_r is an open subset of M, its closure is $\overline{V}_r$, and Q is an image of $\overline{V}_r$. The point is now that $\mathfrak{h}^\perp \times \overline{D}_r \to \mathfrak{k}^* : (\alpha, v) \mapsto \alpha + \mu_S(v)$ is proper. Therefore, also the moment map of Q is proper. Hence, Q is convex by Theorem 4.2*ii)*.

Now we claim that V_r is also convex. First, since V_r is an open subset of Q, we see that the moment map on V_r is open onto its image. Furthermore, since all fibers of $\overline{V}_r \to Q$ are connected (being points or circles) also $\overline{V}_r \to \mathfrak{t}^+$ has connected fibers. Now we use that V_r is the union of all $\overline{V}_s$ with $s < r$. Thus every fiber of $V_r \to \mathfrak{t}^+$ is an increasing union of connected sets and therefore connected. The same argument shows that $\psi(V_r)$ is an increasing union of convex sets, hence itself convex. This proves the claim.

Let $B_s \subseteq \mathfrak{t}^*$ be the open disk with center 0 and radius s. From Theorem 2.7 we know that $\psi(V_r)$ is a neighborhood of 0 in C_x. Thus, given r, we may choose s so small that $B_s \cap C_x \subseteq \psi(V_r)$. Then $U := \psi^{-1}(B_s) \cap V_r$ is an arbitrarily small open neighborhood of Kx which is convex as Hamiltonian manifold and whose image, $B_s \cap C_x$, is open in C_x. ∎

Remark. This proof is similar to that of [4] Prop. 3.7.

6. Some consequences of convexity

If C is a rational cone and $v \in C$ we define the *tangent cone* T_vC of C in v as the smallest cone containing C and having v as a vertex. If $u \in C$ is any vertex then

$$T_vC = C + \mathbb{R}_{\geq 0}(u - v). \tag{6.1}$$

In fact, to show this we may assume that $u=0$. Then T_vC is a homogeneous cone containing C. The point v being a vertex means $-v\in T_vC$ which confirms (6.1).

Corollary 6.1. *Let M be any Hamiltonian K-manifold and $x\in M$. Then there is a neighborhood U of x such that for all $y\in U$ we have $C_y=T_{\psi(y)}C_x$.*

Proof. By Theorem 5.1 we may assume that M is convex and that $\psi(M)$ is open in C_x. But then for every $u\in\psi(M)$, it holds that the cone with vertex u spanned by $\psi(M)$ is the same as that spanned by C_x. We conclude with Theorem 2.7. ∎

An application is the following well known statement.

Corollary 6.2. *Let M be any connected Hamiltonian K-manifold and let $\mathfrak{a}\subseteq\mathfrak{t}^*$ be the affine subspace spanned by $\psi(M)$. Then $\mathfrak{a}$ is also the affine span of every local cone C_x, $x\in M$. Moreover, the set of interior points of $\psi(M)$ (relative to $\mathfrak{a}$) is dense in $\psi(M)$.*

Proof. Let $\mathfrak{a}_x$ be the affine span of C_x. It follows from Corollary 6.1 that is locally constant in x. Since M is connected, $\mathfrak{a}_x$ is independent of x. From $\psi(x)\in C_x\subseteq\mathfrak{a}_x$ follows $\psi(M)\subseteq\mathfrak{a}_x$. Moreover, for every $x\in M$ there is a subset $U_x\subseteq\psi(M)\cap C_x$ which is open in $\mathfrak{a}_x$ and which contains $\psi(x)$ in its closure (Theorem 2.3). Thus $\mathfrak{a}_x$ is spanned by $\psi(M)$ which implies $\mathfrak{a}_x=\mathfrak{a}$. Moreover, $\psi(x)$ is in the closure of the relative interior points of $\psi(M)$. ∎

The last class of examples for convex Hamiltonian manifolds was suggested to me by Tolman:

Theorem 6.3. *Let X be a connected K-manifold and let $M:=T^*(X)$ with its natural Hamiltonian structure. Then M is convex.*

Proof. Let $\iota:X\hookrightarrow M$ be the zero-section. The map $\psi:M\to\mathfrak{t}^*$ is homogeneous with respect to the natural scalar $\mathbb{R}_{>0}$-action on the fibers. Therefore, $\iota(X)\subseteq\psi^{-1}(0)$. Since X is connected, Theorem 2.3 implies that $C=C_{\iota(x)}$ is independent of $x\in X$. Moreover, there is an open neighborhood U of $\iota(X)$ in M such that $\psi(U)$ is a neighborhood of 0 in C. From $M=\cup_{t>0}tU$ we obtain $\psi(M)=C$. In particular, we see that $\psi(M)$ is convex.

Let $\pi:M\to X$ be the projection. For $m\in M$ let $m_0:=\iota\pi(m)\in M$. Choose a convex open neighborhood V of Km_0 such that $\psi(V)$ is open in C. Using the $\mathbb{R}_{>0}$-action we may assume that $m\in V$. Now let V_0 be a small neighborhood of m which is contained in V. Using the convexity of V, we conclude that $\psi(V_0)$ is open in $\psi(V)$, hence open in C. Thus, $\psi:M\to C$ is open.

Finally, suppose that the fiber of ψ over $u\in C$ is disconnected. Thus, $\psi^{-1}(u)=F_1\cup F_2$ where F_1 and F_2 are disjoint, non-empty, and closed. Let $W\subseteq X$ be open, non-empty and K-stable. Then, with X replaced by W, we obtain as above $\psi(\pi^{-1}(W))=\psi(T^*(W))=C$. This means in particular that $\pi^{-1}(W)\cap\psi^{-1}(u)\neq\emptyset$ for every W. Hence, $\pi(\psi^{-1}(u))$ is dense in X. Since X is connected there is $x\in\overline{\pi(F_1)}\cap\overline{\pi(F_2)}$. Let V be a convex neighborhood of $K\iota(x)$ in M. By homogeneity, we may assume that V meets both F_1 and F_2.

But this contradicts the connectedness of fibers of $\psi|_V$. Thus, ψ has connected fibers as well. ■

Remark. The fact that $\psi(M) = C_x$ for any $x \in X \subseteq T^*(X)$ is due to Sjamaar, [8] Thm. 7.6.

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