Journal of Lie Theory
Volume **10** (2000) 359–373

Infinite dimensional manifold structures on principal bundles

Maurice J. Dupré and James F. Glazebrook

Communicated by K. H. Hofmann

Abstract. Infinite dimensional fiber spaces arise naturally in the theory of representations of C*–algebras. Often there are cases where one has to deal with more general notions of differentiability. In order to create a unified framework, we introduce the notion of a $\mathcal{D}$–space and a $\mathcal{D}$–group action within a given category $\mathcal{D}$. Then we present a self–contained account of the type of manifold structure of $\mathcal{D}$–orbit spaces and principal bundles which may be applicable in infinite dimensions.

1. Introduction

The last half of the twentieth century has seen an exponential rise in the number of theories of calculus on vector spaces at ever increasing levels of generality and being applied to even more general theories of manifolds and Lie groups. Thus we witness the evolution of several new branches of mathematics treating such subjects, often referred to collectively as synthetic differential geometry (for the relevant background literature see e.g. [10] [14] [15] [17] [18] [21] [27], and references therein). In this paper we consider some aspects of the general problem of when an orbit space actually becomes a manifold. Here we do not propose an exhaustive treatment of this problem, but instead choose to isolate some of the essential ingredients of the current theory. In particular, we aim to provide a self–contained account of certain fibrations arising from quotient spaces as applicable to the general geometric theory of Banach spaces and Banach algebras. Thus in approaching the subject from a somewhat different perspective, we remark that there are several compelling reasons which motivate such attention, in part originating from the first named author's work on the subject and that of other authors. More specifically, in the study of representations of C*–algebras there are certain fiber spaces that arise very naturally (see e.g. [2] [3] [4] [5] [6] [8] [9] [12] [25]), but these fiber spaces are *not* locally trivial; nevertheless, they can be classified and thus play an important role in the subsequent analysis. The main point is that in the topological classification (see [4] [8] [22]), the resulting cocycles often have differentiable representatives as soon as the notion of differentiability is extended beyond that for maps on manifolds in the usual sense. So it is of

ISSN 0949–5932 / \$2.50

interest to extend some of the standard constructions which work for differentiable manifolds, into a more general framework involving differentiable maps.

In keeping with a categorical framework, we introduce the notion of a $\mathcal{D}$–space from which we can formulate a theorem (Theorem 5.1) concerning orbit spaces under a $\mathcal{D}$–group action, applicable to any manifold theory (in possibly infinite dimensions) for which the differentiability of a map is of a local nature such that differentiable maps satisfy the obvious sheaf–theoretic properties. For completeness, we apply this result to establish several interesting corollaries, some of which may be implicit in the existing literature on the subject in some form or another, but whose proofs are perhaps less elementary than those presented here. In particular, Corollary 5.4 provides a shortened proof of the known fact that the quotient of a Banach Lie group by a Banach Lie subgroup is a Banach analytic manifold (see [29]). Corollary 5.6 asserts that Banach homogeneous spaces (by any reasonable definition) are manifolds, and extends certain results of [26]. Such results are essential when studying the manifold structure of spaces of projections in Banach and C*–algebras (see e.g. [1] [19] and references therein). The techniques of our more general framework enable us to show that certain natural orbit spaces do indeed have manifold structures and the same train of ideas can be extended to discuss the corresponding principal bundles which subsequently arise. For the background on calculus on Banach spaces we refer to [15] [16] [21], and to [29] for the functional analytic viewpoint (for closely related topics, see also e.g. [7] [11] [23] [24]).

2. $\mathcal{D}$–maps and $\mathcal{D}$–spaces

To begin, if $\mathcal{D}$ denotes a category, we take $X \in \mathcal{D}$ to mean X is an object of $\mathcal{D}$ and say that f is a $\mathcal{D}$–map to mean that f is a morphism of $\mathcal{D}$. A (covariant) functor $F : \mathcal{D} \to \mathcal{C}$, of categories $\mathcal{D}$ and $\mathcal{C}$, is *forgetful* if F is injective on $\mathcal{D}$–maps (in effect, if $f, g : X \to Y$ are $\mathcal{D}$–maps and $F(f) = F(g)$, then $f = g$). Further, suppose F satisfies the following condition: if $f : X \to Y$ is an isomorphism of $\mathcal{D}$, if $F(X) = F(Y)$ and $F(f) = \mathrm{Id}$, then $X = Y$. Of course, the isomorphism f appearing in this preceding condition will then have to be Id_X because of the injectivity assumption on F . We can then regard the 'Hom' sets of $\mathcal{D}$ as subsets of the corresponding 'Hom' sets of $\mathcal{C}$, so for instance,

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y)) \ .$$

Consequently, we can think of $X \in \mathcal{D}$ as $F(X) \in \mathcal{C}$ with some kind of extra structure which makes it a member of $\mathcal{D}$, and then $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ simply means that $f : F(X) \to F(Y)$ is a $\mathcal{C}$–map which preserves this extra structure appropriately. Then for $Z \in \mathcal{C}$, we write $Z \in \mathcal{D}$ to understand that there is a given $X \in \mathcal{D}$ with $F(X) = Z$. So if $X, Y \in \mathcal{C}$, we say that $f \in \mathrm{Hom}_{C}(X, Y)$ is a $\mathcal{D}$–*map* when it is understood that we are given $X_{\mathcal{D}}, Y_{\mathcal{D}} \in \mathcal{D}$ such that $F(X_{\mathcal{D}}) = X$ and $F(Y_{\mathcal{D}}) = Y$, and $f \in \mathrm{Hom}_{\mathcal{D}}(X_{\mathcal{D}}, Y_{\mathcal{D}}) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Likewise, if $X, Y \in \mathcal{D}$, we say that $f : X \to Y$ is a $\mathcal{C}$–map to mean that $f : F(X) \to F(Y)$ is a $\mathcal{C}$–map.

Let TOP denote the category of (topological) spaces and continuous maps. Assume now that $F : \mathcal{D} \to \mathrm{TOP}$ is a given forgetful functor that satisfies the following conditions:

(F0) There is a $P \in \mathcal{D}$ with $F(P)$ a point and every $X \in \mathcal{D}$ admits some $\mathcal{D}$–map $X \to P$.

(F1) If $X \in \mathcal{D}$ and $A \subseteq F(X)$, then there is a given $A|^X \in \mathcal{D}$ so that the inclusion map $i : A \hookrightarrow F(X)$ is a $\mathcal{D}$–map $i : (A|^X) \hookrightarrow X$, where with some abuse of notation, we write $A|^A = A$ and understand $A|^X$ to be the element of $\mathcal{D}$ making $A \in \mathcal{D}$. We call $A|^X$ the *$\mathcal{D}$–structure* that A inherits from X, and suppose that

$$F(X)|^X = X \ .$$

(F2) For $f : X \to Y$ a $\mathcal{D}$–map, $A \subseteq F(X)$, $B \subseteq F(Y)$, with $f(A) \subseteq B$, the restriction map

$$f|_A^B : A|^X \longrightarrow B|^Y \ ,$$

defined by f, is also a $\mathcal{D}$–map when A and B have the inherited $\mathcal{D}$–structures.

(F3) Let $X \in \mathcal{D}$ and f a TOP–map of X . If $\mathcal{U}$ is an open cover of X such that $f|U$ is a $\mathcal{D}$–map for each $U \in \mathcal{U}$, then f is a $\mathcal{D}$–map.

(F4) $\mathcal{D}$ has finite products and F is product preserving. Consequently, the ordinary coordinate projections on the $\mathcal{D}$–product are the $\mathcal{D}$–maps determining the actual product structure. Thus a map into a finite product is a $\mathcal{D}$–map if and only if each component is.

(F5) If $X \in$ TOP possesses an atlas of charts with codomains in $\mathcal{D}$ such that the overlap transformations are $\mathcal{D}$–maps, then there is a unique $X_D \in \mathcal{D}$, so that $F(X_D) = X$ and each chart of the atlas is a $\mathcal{D}$–map.

For simplicity, we say that f is a *$\mathcal{D}$–diffeomorphism* to mean f is an isomorphism of $\mathcal{D}$, and say that X is a $\mathcal{D}$–space to mean that X is an object of $\mathcal{D}$. Before proceeding, let us comment on some consequences of these assumptions. Notice that by (F2), if f in (F2) is a $\mathcal{D}$–diffeomorphism with $f(A) = B$, then $f|_A^B$ is a $\mathcal{D}$–diffeomorphism of A onto B . Also, if $X \in \mathcal{D}$ and $Y \in$ TOP and $f : Y \to X$ a homeomorphism, then by (F5) there is a unique $Y_{\mathcal{D}} \in \mathcal{D}$ such that $f : Y_{\mathcal{D}} \to X$, is a $\mathcal{D}$–diffeomorphism. In addition, from (F5) it follows that the disjoint union of $\mathcal{D}$–spaces is again a $\mathcal{D}$–space in a unique way such that each canonical injection is a $\mathcal{D}$–diffeomorphism onto its image. Moreover, P is a final object in $\mathcal{D}$ and for any $X \in \mathcal{D}$, we see that the coordinate projections

$$\pi^X : X \times P \longrightarrow X \ , \ \pi^X : P \times X \longrightarrow X \ ,$$

are $\mathcal{D}$–diffeomorphisms. If $X, Y \in \mathcal{D}$ and $A \subseteq X$, $B \subseteq X$, then using (F1), (F2) and (F4), it follows that

$$(A \times B)|^{X \times Y} = (A|^X) \times (B|^Y) \ ,$$

whereas if $A \subseteq B \subseteq X$, then using (F1) and (F2) and the forgetful property of F, we find that $A|^{B|X} = A|^X$; so we can drop the superscripts when there is no confusion.

Combining (F1), (F2) and (F4), we see that *fiber products* (equivalently, *pullbacks*) exist in $\mathcal{D}$. Specifically, if $f : X \to B$ and $g : Y \to B$, are $\mathcal{D}$–maps, then the fiber product of X and Y over B, denoted by $X \times_B Y$, is the usual subset of $X \times Y$, also bydenoted f^*Y ,

$$X \times_B Y = f^*Y = \{(x, y) : f(x) = g(y)\} \subseteq X \times Y \ ,$$

which following (F1), implies $X \times_B Y \in \mathcal{D}$. We denote the usual coordinate projections by

$$\pi^X : X \times Y \longrightarrow X \ , \ \pi^Y : X \times Y \longrightarrow Y \ .$$

Then in bundle terminology, $P = f^*Y$ is the pullback of Y via f over X, considered as a bundle with projection $\pi^X|_P$, and $\pi^Y|_P : f^*Y \to Y$, is the induced bundle map covering f into the bundle Y over B having projection g (the resulting commutative square is then clearly a pullback diagram in $\mathcal{D}$).

The simplest example of course is just to take $\mathcal{D} =$ TOP and $F =$ Id . Then we have a trivial theory whereby differentiability is in effect just continuity, every space is a $\mathcal{D}$–space and every continuous map is a $\mathcal{D}$–map.

Example 2.1. As a non–trivial example of such a category $\mathcal{D}$ and a forgetful functor F satisfying (F0)–(F5), let us start with a category of topological vector spaces and continuous maps with finite products coinciding with the topological products (for instance: Banach spaces, Fréchet spaces, tame Fréchet spaces and tame continuous linear maps, locally convex topological vector spaces, Lipshitz k–differentiable manifolds modeled on convenient vector spaces [10] [15]) together with any of the various notions of differentiability for maps which satisfy the sheaf property with a chosen level of differentiability, say C^k for $k \in \{0, 1, 2, \dots, \infty, \omega\}$. We then form the category $\mathcal{M}$ of C^k–manifolds and C^k–maps of such manifolds modeled on the open subsets of the linear spaces in the chosen category of topological vector spaces. Next define, C^k–maps on arbitrary subsets of manifolds in $\mathcal{M}$ using local extendability as is easily generalized from [20]. Finally, as our example of $\mathcal{D}$, let us define $C^k\mathcal{M}$-TOP to be the category of spaces having a specified atlas of homeomorphic charts (the set of chart domains being an open cover) to *arbitrary* subsets of manifolds in $\mathcal{M}$ so that the overlap transformations are all C^k–maps, with the obvious definition of C^k–map from one such object to another, by decreeing for local charts, 'pre–' and 'post–' results always in a C^k–map. Clearly then, we can view $\mathcal{M}$ as a subcategory of $C^k\mathcal{M}$-TOP . Taking the obvious forgetful functor F from the category $C^k\mathcal{M}$-TOP to TOP, conditions (F0)–(F5) are clear. Observe here that if $X \in \mathcal{D}$ is locally $\mathcal{D}$–diffeomorphic to a member of $\mathcal{M}$, then $X \in \mathcal{M}$. In view of this, we refer to a member of $C^k\mathcal{M}$-TOP, as a C^k–space.

Remark 2.2. In order to position our definitions within the prevailing literature, we should mention that our above category $\mathcal{D}$ is an example of a *site* (a category with the Grothendieck topology see e.g. [21]). In particular, it is an LSS (*local structure on sets*) category as in [18] with F followed by the forgetful functor of TOP into SET. But here we axiomatize the extension of the domain category and forgetful functor to all possible subsets of manifolds to induce the relevant structures on subsets via the local extension of differentiable maps (see Example 2.1 above). Putting it another way, our main interest here is in extending the category to admit all subsets from subobjects, instead of taking more general limits of atlases that in turn generalize the traditional glueing techniques for manifolds. It appears that the extension via limits as in [18] would also apply in this case. However, we do not pursue the possible consequences here since our interest concerns simplifying the developments needed in showing that certain objects are manifolds. It is possible that such considerations may apply to an LSS category in order to simplify a proof that a given object is an A–space (objects in a model category). But this remains a subject for further investigation.

Remark 2.3. In dealing with Banach spaces, one should be aware of the following. If $E \subset F$ is a closed vector subspace of a Banach space F, and if $X \subset E$ is a submanifold of E, then it does not necessarily follow that X is also a submanifold of F. For instance, E may not admit a complement in F.

In view of this last remark and that in §5 we will have more to say concerning Banach manifolds and Lie groups, we state the following lemma :

Lemma 2.4. *Suppose that M is a C^k–Banach manifold which is modeled on a Banach space E and let $X \subseteq M$. Then X is a C^k–Banach submanifold if and only if X has the property that for each $x \in X$, there is an open set U in M with $x \in U$ and a C^k–retraction $r : U \to U \cap X$. Moreover, $T_x r$ is a continuous linear retraction of $T_x M$ onto $T_x X$.*

Proof. Suppose the local retraction r exists at $x \in X$. After making U smaller and using a local chart on M at x, we may as well assume that M is a Banach space and $x = 0$. Let $p = r'(0)$ and $q = 1 - p$. Thus p and q are continuous linear idempotents in $\mathcal{L}(E)$, the Banach algebra of continuous linear endomorphisms of E. Define $h : U \to E$ by

$$h(u) = p(r(u)) + q(u - r(u)) .$$

Then $h'(0) = 1$, so by the inverse function theorem, there are open sets V and W containing 0 and both contained in U, so that $h|_V^W : V \to W$ is a C^k–diffeomorphism. Set $R = V \cap r^{-1}(V)$, and $T = h(R)$. Let $f = h|_R^T : R \to T$. Then we have $f(R \cap X) = T \cap \mathrm{Im}(p)$. Thus X is a submanifold of M, and the remainder of the proof follows from the chain rule. The converse of course, is an easy consequence of the definition of a submanifold. ∎

3. $\mathcal{D}$–groups and $\mathcal{D}$–bundles

Now any algebraic or topological concept which can be defined in terms of sets or spaces and commutative diagrams of maps, naturally extends to the corresponding $\mathcal{D}$–concept. Thus we can speak of a $\mathcal{D}$–semigroup, by which we mean for instance, a $\mathcal{D}$–space S furnished with a $\mathcal{D}$–map $\psi : S \times S \to S$, which is an associative multiplication. In turn, there is the notion of a $\mathcal{D}$–action of S on a $\mathcal{D}$–space X, but here we need not have to assume that ψ is a $\mathcal{D}$–map. We also have the notion of a $\mathcal{D}$–group whereby multiplication and inversion must both be $\mathcal{D}$–maps. Clearly, any subgroup of a $\mathcal{D}$–group is again a $\mathcal{D}$–group. We can also speak of a $\mathcal{D}$–bundle (following e.g. [13], where a bundle is a triple $\xi = (p, X, B)$, with p a continuous map) as a triple $\xi = (p, X, B)$ where $p : X \to B$ is a $\mathcal{D}$–map, and accordingly a $\mathcal{D}$–bundle map (a commutative square of $\mathcal{D}$–maps), so the category of $\mathcal{D}$–bundles and $\mathcal{D}$–bundle maps then admits finite products and arbitrary disjoint sums. In turn, we can consider a $\mathcal{D}$–group bundle (that is, a given $\mathcal{D}$–group acting on X so that the fibers are orbits and p is taken to be an open, surjective map) and a principal right $\mathcal{D}$–bundle with group H, meaning we have a $\mathcal{D}$–group bundle with a given free $\mathcal{D}$–group action of the $\mathcal{D}$–group H on the right, so the resulting transition map

$$\tau = \tau_\xi : X \times_B X \longrightarrow H ,$$

unique such that $x\tau(x, y) = y$, is in fact a $\mathcal{D}$–map.

Note that local–triviality is not built in here. However, just as in [13], each local $\mathcal{D}$–section defines a $\mathcal{D}$–trivialization, so being locally $\mathcal{D}$–trivial is the same as having local $\mathcal{D}$–sections defined in a neighborhood of each point of the base B . Now the categorically minded reader may observe that in the continuous case, the map p is a 'quotient' map (the base space is topologically the orbit space) when ξ is a continuous group–bundle, but in our situation we do not require p to be a 'quotient map' in the category $\mathcal{D}$. *However, if a $\mathcal{D}$–bundle has local $\mathcal{D}$–sections whose open domains cover the base B, then the map p onto B is clearly a 'quotient' map in the category $\mathcal{D}$ relative to the functor F*, in the sense that the map f from B to the $\mathcal{D}$–space C, is a $\mathcal{D}$–map if and only if $f \circ p$ is. When this last condition holds, we call p a *$\mathcal{D}$–quotient.*

Observe that properties (F0)–(F5) applied to any injection of a factor into a product of $\mathcal{D}$–spaces, make that injection a $\mathcal{D}$–diffeomorphism onto its image. In particular, if we have a $\mathcal{D}$–action $\alpha : G \times X \to X$, then for $g \in G$, the map

$$L_g^\alpha : X \longrightarrow X \ , \ x \mapsto gx \ ,$$

is a $\mathcal{D}$–diffeomorphism of X onto itself. Likewise for R_g^α, if α is a right $\mathcal{D}$–action. Note that there are no assumptions on the operations of G on itself ; it is only required that the action $\alpha : G \times X \to X$ is a $\mathcal{D}$–map.

Let us call $p : X \to B$ a *hereditary $\mathcal{D}$–quotient* if p is a $\mathcal{D}$–map and for each open set $U \subset B$, the restriction $p|^U : p^{-1}(U) \to U$ is a $\mathcal{D}$–quotient as described previously. Note that the disjoint union of hereditary $\mathcal{D}$–quotients is again a $\mathcal{D}$–quotient. This means that when patching together a bunch of $\mathcal{D}$–bundles or hereditary $\mathcal{D}$–quotients, we can simply take their disjoint union and need only consider one. The following proposition is then clear from the definition.

Proposition 3.1. *Suppose that $\xi = (p, X, B)$ is a $\mathcal{D}$–bundle such that every point of B is in the interior of the domain of some $\mathcal{D}$–section of ξ . Then p is a hereditary $\mathcal{D}$–quotient.*

Proposition 3.2. *Suppose that X is a $\mathcal{D}$–space and $p : X \to B$ is a continuous surjective map with $\xi = (p, X, B)$ and suppose that $\zeta = (q, Z, C)$ is a $\mathcal{D}$–bundle where $q : Z \to C$ is a hereditary $\mathcal{D}$–quotient. Let $\mathcal{U}$ be an open cover of B such that for each $U \in \mathcal{U}$, there is a* TOP*–bundle isomorphism $\varphi_U = (h_U; f_U) : \xi|U \to \zeta|V_U$ covering $f_U : U \to V_U$, where V_U is an open set in C and $h_U : p^{-1}(U) \to q^{-1}(U)$ is a $\mathcal{D}$–map. Then B has a $\mathcal{D}$–space structure such that the following hold.*

(1) $f_U : U \to V_U$ is a $\mathcal{D}$–diffeomorphism for each $U \in \mathcal{U}$.

(2) $p : X \to B$ is a hereditary $\mathcal{D}$–quotient.

(3) $\xi = (p, X, B)$ is a $\mathcal{D}$–bundle, locally $\mathcal{D}$–bundle isomorphic to ζ, such that

$$\varphi_U = (h_U; f_U) : \xi|U \to \zeta|V_U \ ,$$

is a $\mathcal{D}$–bundle isomorphism for each $U \in \mathcal{U}$.

Moreover, any one of these facts uniquely determines the $\mathcal{D}$–space structure on B .

Proof. Suppose that $U, W \in \mathcal{U}$ and that $U \cap W = A$. Then $h_W \circ h_U^{-1}$ is a $\mathcal{D}$–diffeomorphism covering

$$f_W \circ f_U^{-1} : f_U(A) \to f_W(A) .$$

Since $q : Z \to C$ is a hereditary $\mathcal{D}$–quotient, it follows that $f_W \circ f_U^{-1}$ is always a $\mathcal{D}$–map and hence is a $\mathcal{D}$–diffeomorphism. Thus (1) and (3) of the conclusion result as immediate consequences of property (F5). If W is any open subset of B and $g : W \to Y$ is any map such that $g \circ (p|^W)$ is a $\mathcal{D}$–map, then by property (F3), we can after restricting, assume that W is contained in some $U \in \mathcal{U}$. Then using $\varphi_U|_W = (h_U|_W; f_U|_W)$, we see that as $q : Z \to C$ is a hereditary $\mathcal{D}$–quotient, $g \circ (f_U|_W)^{-1}$ is a $\mathcal{D}$–map, and hence g is a $\mathcal{D}$–map since $(f_U|_W)$ is a $\mathcal{D}$–diffeomorphism. ∎

4. Local $\mathcal{D}$–models

Suppose $\xi = (p, X, B)$ is now a fixed TOP–bundle and we are given a group H which is a $\mathcal{D}$–space, but we make no assumptions on the relation of the group operations to the topology or to the $\mathcal{D}$–space structure of H . Suppose also that $\alpha_X : X \times H \to X$ is a given continuous map which is algebraically a right action of H on the $\mathcal{D}$–space X . Suppose that p is (topologically) equivalent to the orbit map, that is, p is an open surjection and the orbits are fibers. We point out here that the natural map onto the orbit space is always open, because the continuous action determines a continuous action of the associated discrete topological group. Now let $\alpha_Z : Z \times H \to Z$ be a given $\mathcal{D}$–map which is algebraically a right action of H on the $\mathcal{D}$–space Z . Let $q : Z \to C$ be a continuous map and again, one that is equivalent to the orbit map. Further, let $\zeta = (q, Z, C)$. We would like to make B a $\mathcal{D}$–space so that p is a $\mathcal{D}$–map, and the same for q . This motivates the following definition.

We say that ξ is *$\mathcal{D}$–modeled on ζ over U* to mean that $U \subset B$ and there is an H–equivariant $\mathcal{D}$–diffeomorphism $\varphi : p^{-1}(U) \to q^{-1}(W)$ where $W \subset C$. Of course it follows that $q^{-1}(W)$ is H–invariant in Z and φ induces a unique homeomorphism $\varphi/H : U \to W$, unique such that $(\varphi/H) \circ p|^U = q \circ \varphi$. Thus if U and W are open sets, we call φ a *local $\mathcal{D}$–model for ξ on ζ at any $b \in U$ or at any $c \in W$* .

Let us say that ξ is *weakly $\mathcal{D}$–modeled on ζ at $b \in B$ or at $c \in C$*, to mean that there is an open neighborhood U of b in B, an open subset $W \subseteq C$, and an H–equivariant $\mathcal{D}$–map $\varphi : p^{-1}(U) \to q^{-1}(W)$ which restricted to some open subset $R \subseteq p^{-1}(U)$, is a $\mathcal{D}$–diffeomorphism $\varphi|_R : R \to T$ onto an open subset T of $q^{-1}(W)$, with $b \in p(R)$ or $c \in q(T)$. Here we call φ a *weak local $\mathcal{D}$–model for ξ on ζ at b or at c* .

Proposition 4.1. *Suppose that the action of H on Z is free, that the transition function τ_Z is continuous at the point (z, z) of the diagonal in $Z \times Z$. Suppose $\varphi : p^{-1}(U) \to q^{-1}(W)$ is a weak local $\mathcal{D}$–model for ξ at b, with $R \subseteq p^{-1}(U)$ an open subset and $\varphi|_R : R \to T$ a $\mathcal{D}$–diffeomorphism onto the open subset $T \subseteq q^{-1}(W)$, with $b \in p(R)$ and $z \in T \cap \varphi(p^{-1}(b))$. Then there is an open neighborhood U_0 of b in U and an open subset W_0 of W, so that the restriction* *of $\varphi : p^{-1}(U) \to q^{-1}(W)$ to $\varphi_0 : p^{-1}(U_0) \to q^{-1}(W_0)$, is a H–equivariant homeomorphism. If in addition, the action $\alpha_X : X \times H \to X$ is a $\mathcal{D}$–action, then the map $\varphi_0 : p^{-1}(U_0) \to q^{-1}(W_0)$ is an H–equivariant $\mathcal{D}$–diffeomorphism.*

Proof. Choose $x \in R \cap p^{-1}(b)$ with $z = \varphi(x)$. Since p and q are open maps, after replacing U and V with $p(R)$ and $q(T)$ respectively, we can assume that $p^{-1}(U) = RH$ and $q^{-1}(W) = TH$. As φ is H–equivariant, this means that the resulting restriction of φ is now a surjective map $\varphi : p^{-1}(U) \to q^{-1}(W)$. Since the action of H on X is continuous, there is an open neighborhood R_1 of $x \in R$ and an open neighborhood V_e of the identity $e \in H$, so that $R_1 V_e \subseteq R$. Let $T_1 = \varphi(R_1)$. Then T_1 is an open neighborhood of $z \in T$, and as φ is H–equivariant, it follows that $T_1 V_e \subseteq T$. Now, using the continuity of τ_Z at (z, z), choose an open neighborhood T_0 of $z \in T_1$, such that

$$\tau_Z(T_0 \times_C T_0) \subseteq V_e \ .$$

Again using the fact that p and q are open, on setting $R_0 = R_1 \cap \varphi^{-1}(T_0)$, we now let $U_0 = p(R_0)$ and $W_0 = q(T_0)$. Then U_0 and W_0 are open sets and $b \in U_0$.

Now suppose $u_i \in p^{-1}(U_0)$, for $i = 1, 2$, and $\varphi(u_1) = \varphi(u_2)$. We can then find $x_i \in R_0$ and $h_i \in H$, such that $u_i = x_i h_i$, $i = 1, 2$. Then as φ is H–equivariant, we have

$$\varphi(x_1)h_1 = \varphi(x_1 h_1) = \varphi(u_1) = \varphi(u_2) = \varphi(x_2 h_2) = \varphi(x_2)h_2 \ .$$

Thus $\tau_Z(\varphi(x_1), \varphi(x_2)) = h_1(h_2)^{-1}$. Since $(\varphi(x_1), \varphi(x_2)) \in T_0 \times_C T_0$, it follows that we have $h_1(h_2)^{-1} \in V_e$. Again using the H–equivariance of φ, the above equation gives

$$\varphi(x_1 h_1 (h_2)^{-1}) = \varphi(x_2) \ .$$

Taking into account that $x_1 h_1 (h_2)^{-1}$ and x_2 now both belong to $R_1 V_e \subseteq R$, and $\varphi|_R$ is injective, it follows that the two are equal, and therefore, $u_1 = x_1 h_1 = x_2 h_2 = u_2$. Consider now any $h \in H$ and the right action R_h of h on either X or Z . On X we have a homeomorphism, whereas on Z we find R_h is a $\mathcal{D}$–diffeomorphism. As φ is H–equivariant, we have the usual translation result that $\varphi|_{Rh} : Rh \to Th$, is a homeomorphism and in the case $\alpha_X : X \times H \to X$ is also a $\mathcal{D}$–action, then it follows that $\varphi|_{Rh} : Rh \to Th$ is a $\mathcal{D}$–diffeomorphism. Thus, $\varphi : p^{-1}(U) \to q^{-1}(W)$ is a local homeomorphism or a local $\mathcal{D}$–diffeomorphism as the case may be, and hence with $\varphi_0 = \varphi|p^{-1}(U_0) : p^{-1}(U_0) \to q^{-1}(W_0)$, it follows that φ_0 is an H–equivariant homeomorphism or H–equivariant $\mathcal{D}$–diffeomorphism, again as the case may be. ∎

Suppose we have a group G acting on both X and Y and we consider some subset $A \subseteq X$ together with a map $f : A \to Y$. For $g, h \in G$, let us take gfh to denote the map $gfh : h^{-1}A \to Y$ given by $gfh(x) = g[f(hx)]$, for $x \in h^{-1}A$. Then the following is straightforward.

Proposition 4.2. *Suppose that $p : X \to B$ is a $\mathcal{D}$–map, that G is a group which is a $\mathcal{D}$–space and we are given transitive $\mathcal{D}$–actions of G on the left of X and B such that p is G–equivariant. If $s : U \to X$ is a local $\mathcal{D}$–section of p, then for each $b \in B$ and each $x \in p^{-1}(b)$, there is a local $\mathcal{D}$–section t of p such that $t(b) = x$ and with $t = hsg$, for some $g, h \in G$.*

5. The main theorem and corollaries

Combining the above propositions, we now arrive at the main theorem:

Theorem 5.1. *Let $\xi = (p, X, B)$ and $\zeta = (q, Z, C)$ be as above and suppose that the following conditions hold.*

(1) The action of H on Z is free and the transition map

$$\tau = \tau_Z : Z \times_C Z \to H \ ,$$

is continuous at each point of the diagonal in $Z \times_C Z$.

(2) B is a $\mathcal{D}$–space and p is a $\mathcal{D}$–map which is a hereditary $\mathcal{D}$–quotient and that both actions are $\mathcal{D}$–actions.

(3) There is a weak local $\mathcal{D}$–model φ, as defined above, for ξ on ζ at each $c \in C$.

Then C has a unique $\mathcal{D}$–space structure such that the orbit map φ/H is a $\mathcal{D}$–diffeomorphism whenever φ is a local $\mathcal{D}$–model for ξ on ζ .

Moreover, $\zeta = (q, Z, C)$ is a principal right $\mathcal{D}$–bundle with structure group H, and a continuous map f of C to the $\mathcal{D}$–space Y is a $\mathcal{D}$–map if and only if $f \circ q$ is a $\mathcal{D}$–map. Thus B and C are locally $\mathcal{D}$–diffeomorphic and ξ and ζ are locally $\mathcal{D}$–isomorphic $\mathcal{D}$–bundles.

If in addition, ξ is locally $\mathcal{D}$–trivial, then so too is ζ, H is a $\mathcal{D}$–group and the transition map τ_Z above, is a $\mathcal{D}$–map. Thus both are principal right $\mathcal{D}$–bundles with structure group H . On the other hand, if ξ is just a principal right $\mathcal{D}$–bundle, then so too is ζ, even without local triviality.

Corollary 5.2. *Suppose conditions (1) and (2) of Theorem 5.1 hold and in addition we have a $\mathcal{D}$–action of the group G on the left of Z commuting with the right H–action. Suppose that ξ is weakly modeled on ζ at some point of each G–orbit. Then in addition to all of the conclusions of Theorem 5.1, C admits an induced left G–action, unique such that q is a G–equivariant $\mathcal{D}$–map. If ξ is locally $\mathcal{D}$–trivial, then the induced action of G on the left of C is a $\mathcal{D}$–action of G on its left, unique such that q is a G–equivariant $\mathcal{D}$–map.*

Proof. Since the left action of G commutes with that of H on the right, if $g \in G$, then the left action $L_g : Z \to Z$ of g on Z is H–equivariant. hence the induced map $L_g/H : C \to C$, defines an action of G on the left of C, unique such that q is G–equivariant. Since the right H–actions are $\mathcal{D}$–actions, $L_g : Z \to Z$ is a $\mathcal{D}$–diffeomorphism which is H–equivariant. Thus if φ is a local $\mathcal{D}$–model on ζ at $c \in C$, and if $d = (L_g/H)(c)$, then $L_g\varphi$ is a local $\mathcal{D}$–model on ζ at $d \in C$. Our hypothesis that ξ is weakly modeled on ζ at some point of each G–orbit, now implies (3) of Theorem 5.1, so the conclusions of the latter are now in effect. If ξ is locally $\mathcal{D}$–trivial, which simply means here that ξ has local $\mathcal{D}$–sections whose open domains cover B, then by Theorem 5.1, the same is true of ζ and hence the same is true of $G \times \zeta$. So by Proposition 3.1, the induced left action of G on C is a $\mathcal{D}$–action. ∎

Corollary 5.3. *Suppose that G is a $\mathcal{D}$–group and that H is a subgroup of G . Suppose also that there is a subset K of G such that $e \in K$, $K \cap H = e$, and the multiplication of G restricted to $K \times H$ is a local $\mathcal{D}$–diffeomorphism at (e, e) onto* *a neighborhood of $e \in G$. Then G/H is a $\mathcal{D}$–space in a unique way such that the orbit map $q : G \to G/H$ is a $\mathcal{D}$–map and is the projection of a $\mathcal{D}$–locally trivial right principal $\mathcal{D}$–bundle with structure group H . Moreover, the multiplication of G induces a unique $\mathcal{D}$–action of G on the left of G/H so that q is G–equivariant and G/H is locally $\mathcal{D}$–diffeomorphic to K .*

Proof. Let $X = K \times H$ and $p = \pi^K$ with $B = K$. Define $\varphi : X \to G$ by $\varphi(k, h) = kh$. Then φ is obviously H–equivariant and by hypothesis, it is then a weak local $\mathcal{D}$–model at $e \in G$. Since the left G–action is transitive on G, the conclusion now follows by the previous corollary. ∎

Corollary 5.4. (see also [29]) *Suppose that G is a Banach Lie group and that H is a Banach Lie subgroup of G . Then G/H is a Banach analytic manifold in a unique way such that the orbit map $q : G \to G/H$ is an analytic map and the multiplication of G induces a unique analytic left action of G on G/H, such that q is G–equivariant. Moreover, $\zeta = (q, G, G/H)$ is an analytically locally trivial analytic principal bundle with structure group H . A map f of G/H is analytic if and only if $f \circ q$ is analytic. Likewise, when analytic is replaced by smooth throughout.*

Proof. Since H is an analytic submanifold of G, there exists by Lemma 2.4 a local analytic retraction r of some neighborhood U of the identity $e \in G$ onto $U \cap H$. Then passing to the tangent bundle, the map $T_e r : T_e G \to T_e G$ is seen to be a continuous linear retraction whose image is $T_e H$. Now let $K = \text{Ker } T_e r$, again let $X = K \times H$, and define $\varphi : X \to G$ by $\varphi(k, h) = (\exp k)h$. Further, let us take $\lambda : G \times G \to G$ to be multiplication in G . Then with the natural identification

$$T_{(e,e)}(G \times G) = T_e G \times T_e G \ ,$$

it is easily computed that $T_{(e,e)}\lambda(u, v) = u + v$. Thus as $T \circ \exp = \text{Id}$, it follows that $T_e\varphi$ is the linear homeomorphism of $K \times H$ onto $T_e G$ which is simply the restriction of the addition map. This means that $\varphi : X \to G$ is a local analytic diffeomorphism at $(0, e)$, and as φ is clearly H–equivariant, by Corollary 5.3, G/H has a unique analytic manifold structure so that q is an analytic map which has local analytic sections, and the remaining conclusions are clear. ∎

Suppose that G is a $\mathcal{D}$–group and there is a $\mathcal{D}$–action of G on the left of the $\mathcal{D}$–space X . Let B be an orbit space of this action, so B is automatically a $\mathcal{D}$–space since B is contained in X . Let $x \in B$, such that $B = Gx$, and let G_x denote the isotropy subgroup of G at x . It is natural to ask if when G/G_x is a $\mathcal{D}$–space as above, is G/G_x always $\mathcal{D}$–diffeomorphic to B ? Let $q : G \to G/G_x$ be the orbit map and $p^x : G \to B$ be defined by $p^x(g) = gx$. Let $h : G/G_x \to B$ be the unique bijection with $h \circ q = p^x$. Next, observe that G acts on the left of both itself and B, and both q and p^x are G–equivariant. Thus if s is a section of p^x over $U \subseteq B$, then gsg^{-1} is a section of p^x over gU . In order that G/G_x and B are at least homeomorphic, we need to assume that p^x is an open map onto its image. As G acts transitively on B, if s is a local $\mathcal{D}$–section of p^x, then the sections gsg^{-1} for $g \in G$, have domains covering B and so insure that p^x is open and is a hereditary $\mathcal{D}$–quotient by Proposition 3.1. Thus under the hypothesis of Corollary 5.3, Theorem 5.1 tells us that q is a hereditary $\mathcal{D}$–quotient, and therefore on finding a single local $\mathcal{D}$–section of p^x, it follows that h is a $\mathcal{D}$–diffeomorphism.

Continuing with Banach Lie groups, the above observations result in a further corollary. The reader can take the class of differentiability to be either smooth or analytic throughout. But first, we establish the following proposition.

Proposition 5.5. *Suppose that $p : X \to B$ is a C^k–map with B a C^k–space and X a C^k–Banach manifold. Then the following hold.*

(1) If for $x \in X$ and $b \in B$, s is a local C^k–section of p with $s(b) = x$, then Im (s) *is a C^k–Banach submanifold of X, and s is a C^k–diffeomorphism onto* Im (s) *. Moreover,* $\mathrm{Im}(T_xp) = T_bB$ *and* Ker (T_xp) *topologically split T_xX .*

(2) If p has local C^k–sections whose domains cover B, then B is a C^k–manifold.

(3) If p has local sections whose images cover X, then B is a C^k–Banach manifold, p is a submersion, each fiber of p is a C^k–submanifold of X, and for each fiber F and each point $x \in F$, we have $T_xF =$ Ker (T_xp) *.*

Proof. Commencing with (1), let U be the domain of s, so U is an open subset of B and then let $W = p^{-1}(U) \subseteq X$. Further, set

$$f = p|_W^U : W \longrightarrow U \ , \ r = s \circ f : W \longrightarrow s(U) \subseteq W \ .$$

Then r is a C^k–retraction of X onto $s(U)$ and by Lemma 2.4, $s(U)$ is a C^k–submanifold of X, and for $x \in s(U) = M$, we have $T_xM = \mathrm{Im}\ (T_xr)$. Since $f|M = s^{-1}$, it follows that s is a C^k–diffeomorphism of U onto M . Thus T_bs is a linear homeomorphism of T_bB onto T_xM and the remainder of (1) now follows by applying elementary linear algebra and the chain rule to the equations $r^2 = r$ and $r = s \circ f$. For (2), we note that B is then locally C^k–diffeomorphic to a C^k–Banach manifold, and therefore (3) follows from (1) and (2). ∎

Corollary 5.6. *Suppose that the Banach Lie group G acts smoothly on the subset $X \subseteq M$ where M is a manifold. Let $B \subseteq X$ be an orbit and let $x \in B$. With regards to the above notation, the following are equivalent.*

(1) p^x has a local smooth section.

(2) p^x is open, G_x is a Banach Lie subgroup and for each $g \in G$, Im (T_gp^x) *topologically splits $T_{gx}M$.*

(3) p^x is open, Ker (T_ep^x) *topologically splits T_eG and for each $g \in G$,* Im (T_gp^x) *topologically splits $T_{gx}M$.*

(4) p^x is open, and for each $g \in G$, Ker (T_gp^x) *topologically splits T_gG and* Im (T_gp^x) *topologically splits $T_{gx}M$.*

(5) B is a Banach submanifold of M and p^x is a submersion.

(6) The bundle $\xi = (p^x, G, B)$ is a smoothly locally trivial smooth principal bundle with structure group G_x a Banach Lie subgroup of G, and (Id_G, h) *is a smooth isomorphism of principal bundles.*

(7) h is a smooth G–equivariant diffeomorphism.

Moreover, in the case that $M = X$, conclusions (2), (3) and (4) need only be checked for $g = e$.

Finally, all of the above statements remain true when 'smooth' is replaced by 'analytic' throughout.

Proof. Let $H = G_x$. Suppose that (1) holds. By Propositions 4.2 and 5.5, (2), (3) and (4) hold. If (2) holds, then by Corollary 5.4, $h : G/H \to B$ is a smooth (respectively, analytic) homeomorphism. As $p^x = h \circ q$, then from the chain rule we have Ker $(T_e q) \subseteq$ Ker $(T_e p^x)$, and by Proposition 5.5, we know that Ker $(T_e q) = T_e H$. Thus $T_e H \subseteq \mathrm{Ker}(T_e p^x)$.

As before, let L_g denote left multiplication by g on G and by L_g^α the left action of g on X . Since p^x is G–equivariant, we have $L_g^\alpha p^x = p^x L_g$, so in choosing a local smooth (respectively analytic) extension of L_g^α at $y = p^x(k) = kx$ to a neighborhood in M, we see that $T_k L_g$ maps Ker $(T_k p^x)$ into Ker $(T_{gk} p^x)$, for any $g, k \in G$. Since L is a homomorphism of G into the diffeomorphism group of G, it follows that the map

$$T_k L_g \ : \ \mathrm{Ker}\ (T_k p^x) \xrightarrow{\cong} \mathrm{Ker}\ (T_{gk} p^x) \ ,$$

is an isomorphism for each $g, k \in G$, and so in particular, Ker $(T_k p^x)$ splits $T_k G$ for each $k \in G$. This translation argument also demonstrates that (3) implies (4).

If $v \in$ Ker $(T_e p^x)$, then define the curve $\mathbf{c}$ by $\mathbf{c}(t) = \exp(tv)$, for any $t \in \mathbb{R}$. If $k = \exp(tv)$, then we have $p^x \mathbf{c}(t) = p^x(k)$, and hence

$$(p^x \mathbf{c})'(t) = (T_k p^x)(kv) = (T_k p^x) \circ (T_k L_g)(v) = 0 \ .$$

Thus $p^x \mathbf{c}(t)$ must be independent of t and taking $t = 0$, gives $\exp(tv)x = x$, for all t . Thus $\mathbf{c}(t) = \exp(tv) \in H$, for all t, and therefore $v = \mathbf{c}'(0) \in T_e H$. Hence we conclude that $T_e H =$ Ker $(T_e p^x)$.

It now follows from the same type of translation argument that for any $g \in G$ and F the fiber of $q : G \to G/H$ containing g, we have

$$\mathrm{Ker}\ (T_g q) = T_g H = \mathrm{Ker}\ (T_g p^x) \ .$$

Since $p^x = h \circ q$, differentiating yields

$$T_g p^x = (T_{q(g)} h) \circ (T_g q) \ ,$$

and since $T_g p^x$ and $T_g q$ are both surjective, it now follows that $T_{q(g)} h$ is a linear homeomorphism onto Im $(T_g p^x)$ which topologically splits $T_{gx} M$, by hypothesis. As h is a homeomorphism onto its image, then h is an embedding of manifolds and therefore B is a submanifold of M . Thus (2) implies all the others.

Of course (5) implies all the others, so it merely remains to show that (4) implies (5). The latter in the case $X = M$ is a result of Raeburn [26] (Proposition 1.5, p.372). The argument there proves that x has an open neighborhood in B which is a Banach manifold. However, for $X \neq M$, the translation argument breaks down, since the translations of the group action are not defined on all of M . But in view of the hypothesis of (4), we observe that if y is another point of B, then $p^y = p^x R_g$ for suitably chosen $g \in G$, and therefore the hypothesis of (4) also applies to the point y . Thus y has an open neighborhood in B which is a Banach submanifold of M and thus B itself is a Banach submanifold of M . Then clearly, p^x must be a submersion, as we can apply the above cited result of

[26] directly to the case of G acting on B . Indeed, if E, F and K are Banach spaces with $E \subseteq F \subseteq K$ and E and F both topologically split in K, then there is a continuous linear retraction r^E of K, and its restriction to F defines a continuous linear retraction of F onto E . Finally, (1)–(5) are equivalent to (6) and (7) in view of Theorem 5.1 and the above corollaries. ■

Remark 5.7. We can say that $z = (g_{ij})$ is a $\mathcal{D}$–cocycle with values in the $\mathcal{D}$–group G, relative to the open cover (U_i) of the $\mathcal{D}$–space X, if each g_{ij} is a $\mathcal{D}$–map of $U_{ij} = U_i \cap U_j$ into G . Then from (F5) in §2, we see that the principal bundle defined by the cocycle is a principal $\mathcal{D}$–bundle which is $\mathcal{D}$–locally trivial. Likewise, if $\alpha : G \times F \to F$ is a $\mathcal{D}$–action, then the fiber bundle with fiber F and cocycle z, is a $\mathcal{D}$–bundle. The resulting uniqueness establishes a bijection between the usual sheaf cohomology group $H^1(X, G_{\mathcal{D}})$ and the set of $\mathcal{D}$–isomorphism classes of $\mathcal{D}$–locally trivial principal $\mathcal{D}$–bundles over X with structure group G, where $G_{\mathcal{D}}$ denotes the sheaf of germs of G–valued $\mathcal{D}$–maps. On the other hand, if $\xi = (p, P, B)$ is a $\mathcal{D}$–locally trivial principal right $\mathcal{D}$–bundle with structure group G, and if we let G act on the right of $P \times F$ by $(x, v)g = (xg, g^{-1}v)$, then the action is free and the transition map is $\tau((x_1, v_1), (x_2, v_2)) = \tau_P(x_1, x_2)$ which moreover, is a $\mathcal{D}$–map. If h is a $\mathcal{D}$–local trivialization of ξ, then $h \times \mathrm{Id}_F$ is a local $\mathcal{D}$–model and therefore by Theorem 5.1, $(P \times F)/G$ has a unique $\mathcal{D}$–space structure such that the orbit map is a $\mathcal{D}$–map and a local $\mathcal{D}$–quotient. Consequently, the map $q : (P \times F)/G \to B$ induced by the first factor projection $(P \times F) \to P$, is a $\mathcal{D}$–map, and some routine diagram chasing shows that $\xi[F] = (q, (P \times F)/G, B)$, is a $\mathcal{D}$–locally trivial $\mathcal{D}$–bundle with fiber F. Moreover, if ξ is $\mathcal{D}$–isomorphic to the bundle defined by the $\mathcal{D}$–cocycle z, then $\xi[F]$ is also defined by the same $\mathcal{D}$–cocycle.

In particular, if $K \subset H \subset G$ is an inclusion of subgroups, let us denote by K_H the maximal subgroup of K which is normal in H and observe that if the required transversal subsets exist such that Corollary 5.3 can be applied to make each of the resulting orbit maps $\mathcal{D}$–quotient maps, then the quotient H/K_H is a $\mathcal{D}$–group, and the naturally induced map $G/K \to G/H$ is the projection of the $\mathcal{D}$–bundle with fiber H/K associated to the $\mathcal{D}$–locally trivial principal $\mathcal{D}$–bundle $G/K_H \to G/H$ with structure group H/K_H . The unique map $(G/K_H)/(H/K_H) \to G/H$ is a $\mathcal{D}$–diffeomorphism as the image of the local transversal subset of G for H in G/K_H, gives a local transversal subset for H/K_H in G/K_H . For Banach Lie groups, this can be problematic because one has to determine K_H and check that it is in fact a Lie subgroup. Apart from that, the result holds for Lie groups. However, as shown in [28] , there are some interesting examples where K_H is trivial. On the other hand, if K is normal in H, then $K_H = K$ and so there is no problem.

Acknowledgement : J. Glazebrook wishes to thank IHES, Bures-sur-Yvette (Fr.) for its hospitality during the preparation of this paper. We also thank a referee for comments.

References

[1] Corach, G., H. Porta, and L. Recht, *Differential geometry of systems of projections in Banach algebras*, Pacific J. Math. **143** (2) (1990), 209–228.

[2] Dauns, J., and K. H. Hofmann, *Representation of rings by sections*, Mem. Amer. Math. Soc. **83**, Amer. Math. Soc., Providence RI, 1968.

[3] Dixmier, J., and A. Douady, *Champs continus d'espaces hilbertiens et de C*–algèbres*, Bull. Math. Soc. Fr. **91** (1963), 227–284.

[4] Dupré, M. J., *The Classification and Structure of C*-Algebra bundles*, Mem. Amer. Math. Soc. **21** (222), Amer. Math. Soc., Providence RI, 1979.

[5] Dupré, M. J., and P. A. Fillmore, *Triviality theorems for Hilbert modules*, "Proceedings of the 5^{th} Roumanian Conference on Operator Theory," Birkhäuser Verlag, Boston MA, 1981, 71–79.

[6] Dupré, M. J., and R. M. Gillette, "Banach Modules and Automorphisms of C*-Algebras," Pitman Research Notes in Math. **92**, Pitman Publ., New York, 1983.

[7] Eells, J., *A setting for global analysis*, Bull. Amer. Math. Soc. **72**, (1966) 571–807.

[8] Evans, B. D., *C*–bundles and Compact Transformation Groups*, Mem. Amer. Math. Soc. **39** (269), Amer. Math. Soc., Providence RI, 1982.

[9] Fell, J. M. G., and R. S. Doran, "Representations of *–Algebras, Locally Compact Groups and Banach *–Algebraic Bundles –Vol. 1," Academic Press Inc., New York–London 1988.

[10] Frölicher, A., and A. Kriegl, "Linear Spaces and Differentiation Theory," Pure and Appl. Math. Series, J. Wiley, Chichester, 1988.

[11] Hamilton, R. S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[12] Hofmann, K. H., *Representation of algebras by continuous sections*, Bull. Amer. Math. Soc. **78** (1972), 291–373.

[13] Husemoller, D., "Fibre Bundles," (3rd Ed.) Springer Verlag, New York – Berlin, 1994.

[14] Kock, A., "Synthetic Differential Geometry," London Math. Soc. Lect. Notes **51**, Cambridge Univ. Press, 1981.

[15] Kriegl, A., and P. Michor, "The Convenient Setting of Global Analysis," Math. Surveys and Monographs **53**, Amer. Math. Soc., Providence RI 1997.

[16] Lang, S., "Differentiable Manifolds," Addison Wesley, 1972.

[17] Lawvere, F. W., *Categorical Dynamics*, Lectures at the Univ. of Chicago 1967, in "Topos Theoretical Methods in Geometry" (A. Kock, ed.), Aarhus Math. Inst. Var. Publ. Series **30**, Aarhus Univ. 1979.

[18] Losik, M. V., *Categorical Differential Geometry*, Cah. Topologie Géom. Differ. **35** (1994), 274–290.

[19] Martin, M., and N. Salinas, *Differential geometry of generalized Grassmann manifolds in C*–algebras*, in "Operator Theory: Advances and Applications," **80** Birkhäuser Verlag, Basel, 1995, 206–243.

[20] Milnor, J., "Topology from the Differentiable Viewpoint," Univ. Virginia Press, 1965.

[21] Moerdijk, I., and G. E. Reyes, "Models for Smooth Infinitesimal Analysis," Springer Verlag, New York 1991.

[22] Müller, T., *C*–semigroup bundles and C*–algebras whose irreducible representations are all finite dimensional*, Dissertationes Math. (Rozprawy Mat.) **290** (1990), 5–87.

[23] Omori, H., "Infinite Dimensional Lie Groups, " Translations of Math. Monographs **158**, Amer. Math. Soc., Providence RI, 1997.

[24] Palais, R. S., "Foundations of Global Non–Linear Analysis, " Benjamin, New York, 1968.

[25] Phillips, J., and I. Raeburn, *Automorphisms of C*–algebras and second Čech cohomology*, Indiana J. Math. **29** (1980), 799–822.

[26] Raeburn, I., *The relationship between a commutative Banach algebra and its maximal ideal space*, J. Functional Analysis **25** (1977), 366–390.

[27] Souriau, J.–M., *Groups différentiels et physique matématique*, in "Feuilletages et quantification, Coll. Travaux en Cours," Hermann, Paris, 1984, 75–79.

[28] Steenrod, N., "The Topology of Fibre Bundles," Princeton Univ. Press, 1951.

[29] Upmeier, H., "Symmetric Banach Manifolds and Jordan C*–Algebras," Math. Studies **104**, North Holland, Amsterdam–New York–Oxford, 1985.

Department of Mathematics
Tulane University
New Orleans, LA 70118 USA
mdupre@mailhost.tcs.tulane.edu

Department of Mathematics
Eastern Illinois University
Charleston, IL 61920 USA
and
Department of Mathematics
University of Illinois
Urbana IL 61801 USA
glazebro@math.uiuc.edu

Received May 31, 1999
and in final form March 20, 2000