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The closure diagrams for nilpotent orbits of real forms of $\mathbf{F}_4$ and $\mathbf{G}_2$

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Abstract. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two adjoint nilpotent orbits in a real semisimple Lie algebra $\mathfrak{g}_0$. We shall write $\mathcal{O}_1 \geq \mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$. This defines a partial order on the set of such orbits and we refer to this order as the closure ordering. We determine the closure ordering for adjoint nilpotent orbits when $\mathfrak{g}_0$ is of type G I, F I, or F II. The cases G I and F II are rather trivial and are included only for the sake of completeness.

1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, G the adjoint group of $\mathfrak{g}$, and $\mathfrak{g}_0$ a real form of $\mathfrak{g}$. The conjugation σ of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ defines a real structure on $\mathfrak{g}$ and G. Let $G_{\mathbf{R}}$ be the group of real points of G and G_0 its identity component for the Lie group topology. Thus G_0 is the adjoint group of $\mathfrak{g}_0$.

Let $\mathcal{N}$ denote the nilpotent variety of $\mathfrak{g}$, i.e., the set of $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)$ is nilpotent. We set $\mathcal{N}_{\mathbf{R}} = \mathfrak{g}_0 \cap \mathcal{N}$. The variety $\mathcal{N}$ is irreducible and consists of finitely many G-orbits. There is an open dense G-orbit in $\mathcal{N}$ called the principal orbit. Let $\mathcal{O}_1, \mathcal{O}_2$ be two G-orbits in $\mathcal{N}$. If $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$, we shall write $\mathcal{O}_1 \geq \mathcal{O}_2$. If $\mathcal{O}_1 \geq \mathcal{O}_2$ and $\mathcal{O}_1 \neq \mathcal{O}_2$ we write $\mathcal{O}_1 > \mathcal{O}_2$. The relation "$\geq$" is a partial order on the set of G-orbits in $\mathcal{N}$ to which we refer as the *closure ordering*. We mention that $\mathcal{O}_1 > \mathcal{O}_2$ implies $\dim(\mathcal{O}_1) > \dim(\mathcal{O}_2)$. We shall use analogous definitions and terminology for $G_{\mathbf{R}}$-orbits and G_0-orbits in $\mathcal{N}_{\mathbf{R}}$.

The closure ordering for G-orbits in $\mathcal{N}$ is known (see [?] and the references given there). For the convenience of the reader we reproduce in Figure 1 the closure diagrams for the two relevant cases, F_4 and G_2. The numbers on the left hand side of the diagrams are the dimensions of the orbits on that level.

If $\mathfrak{g}$ is of classical type (A, B, C, or D) and G the corresponding classical group, for types B and D we assume that G is the full orthogonal group, the

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closure ordering for $G_{\mathbf{R}}$-orbits in $\mathcal{N}_{\mathbf{R}}$ was determined in our papers [?, ?]. In most cases either $G_{\mathbf{R}} = G_0$ or the closure ordering of G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ can be easily deduced from the closure ordering of the $G_{\mathbf{R}}$-orbits. There is one notable exception, when $\mathfrak{g}_0$ is of type $\mathfrak{so}(p,q)$. This particular case has been dealt with in our joint paper with J. Sekiguchi and N. Lemire [?] but not yet solved.

Our main result is the description of the closure ordering of G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ when $\mathfrak{g}_0$ is of type F I, i.e., the split real form of F_4. For the sake of completeness we also include the cases F II and G I. In all three cases $G_{\mathbf{R}} = G_0$, i.e., $G_{\mathbf{R}}$ is connected.

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ and $\mathfrak{k}, \mathfrak{p}$ the complexifications of $\mathfrak{k}_0, \mathfrak{p}_0$. We denote by θ the corresponding Cartan involution of $\mathfrak{g}_0$. We use the same letter θ to denote the complex linear extension of this involution to $\mathfrak{g}$, as well as the corresponding involutory automorphism of G. We set $K = \{a \in G : \theta(a) = a\}$. The Lie algebra of K is $\mathfrak{k}$. In general K is not connected (e.g. when $\mathfrak{g}_0 = \mathfrak{so}(p,q)$) but it is connected in the three cases that we consider in this paper: F I, F II, and G I. By K^0 we denote the identity component of K.

Let $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{p}$, the nilpotent variety of $\mathfrak{p}$. It is well known that $\mathcal{N}_1$ is a finite union of K^0-orbits. Furthermore for $X \in \mathcal{N}_1$, $K^0 \cdot X$ is a connected component of $\mathfrak{p} \cap G \cdot X$, and

$$\dim(K^0 \cdot X) = \frac{1}{2} \dim(G \cdot X).$$

There is a one-to-one correspondence between the G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ and the K^0-orbits in $\mathcal{N}_1$ known as the Kostant-Sekiguchi correspondence. For the precise description of this correspondence we refer the reader to the book [?]. Barbasch and Sepanski [?] have shown recently that the Kostant-Sekiguchi correspondence preserves the closure ordering. This means that our original problem of describing the closure ordering of G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ is equivalent to the same problem for K^0-orbits in $\mathcal{N}_1$. We shall actually solve the latter problem.

The K^0-orbits in $\mathcal{N}_1$ were enumerated in our two papers [?, ?] for all noncompact real forms $\mathfrak{g}_0$ of simple exceptional complex Lie algebras $\mathfrak{g}$. In order to avoid possible confusion, we shall use the same numbering of the K^0-orbits in $\mathcal{N}_1$ as in those papers. (This numbering is the same as in [?].) The trivial zero orbit will be given number 0. We have only recently computed representatives of the K^0-orbits in $\mathcal{N}_1$ and G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ and obtained a concrete description of the Kostant-Sekiguchi correspondence [?].

I would like to thank Prof. J. Sekiguchi for his interest and several conversations concerning the problem treated in this paper.

2. Statement of the result

The real forms G I, F I, and F II are of inner type. Hence we can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k}$. Let R (resp. R_0) be the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with respect to $\mathfrak{h}$. Let B be a base of R. For $\alpha \in R$ we denote by $\mathfrak{g}^\alpha$ the corresponding root space. We remark that R_0 is the set of $\alpha \in R$ for which $\mathfrak{g}^\alpha \subset \mathfrak{k}$. By W (resp. W_0) we denote the Weyl group of R (resp. R_0).

We define the *extended base* B_e of R to be $B \cup \{\alpha_0\}$ where α_0 is the negative of the highest root of R. There is a unique base B_0 of R_0 such that $B_0 \subset B_e$.

If $\mathfrak{g}_0$ is of type G I we have $B = \{\alpha_1, \alpha_2\}$, α_1 being short, and $B_0 = \{\beta_1, \beta_2\}$ with $\beta_1 = \alpha_1, \beta_2 = \alpha_0$. We also have $K = (\mathrm{SL}_2 \times \mathrm{SL}_2)/Z_2$ and $\dim \mathfrak{p} = 8$. For the other positive roots we use the notations: $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = 2\alpha_1 + \alpha_2$, $\alpha_5 = 3\alpha_1 + \alpha_2$, and $\alpha_6 = 3\alpha_1 + 2\alpha_2$.

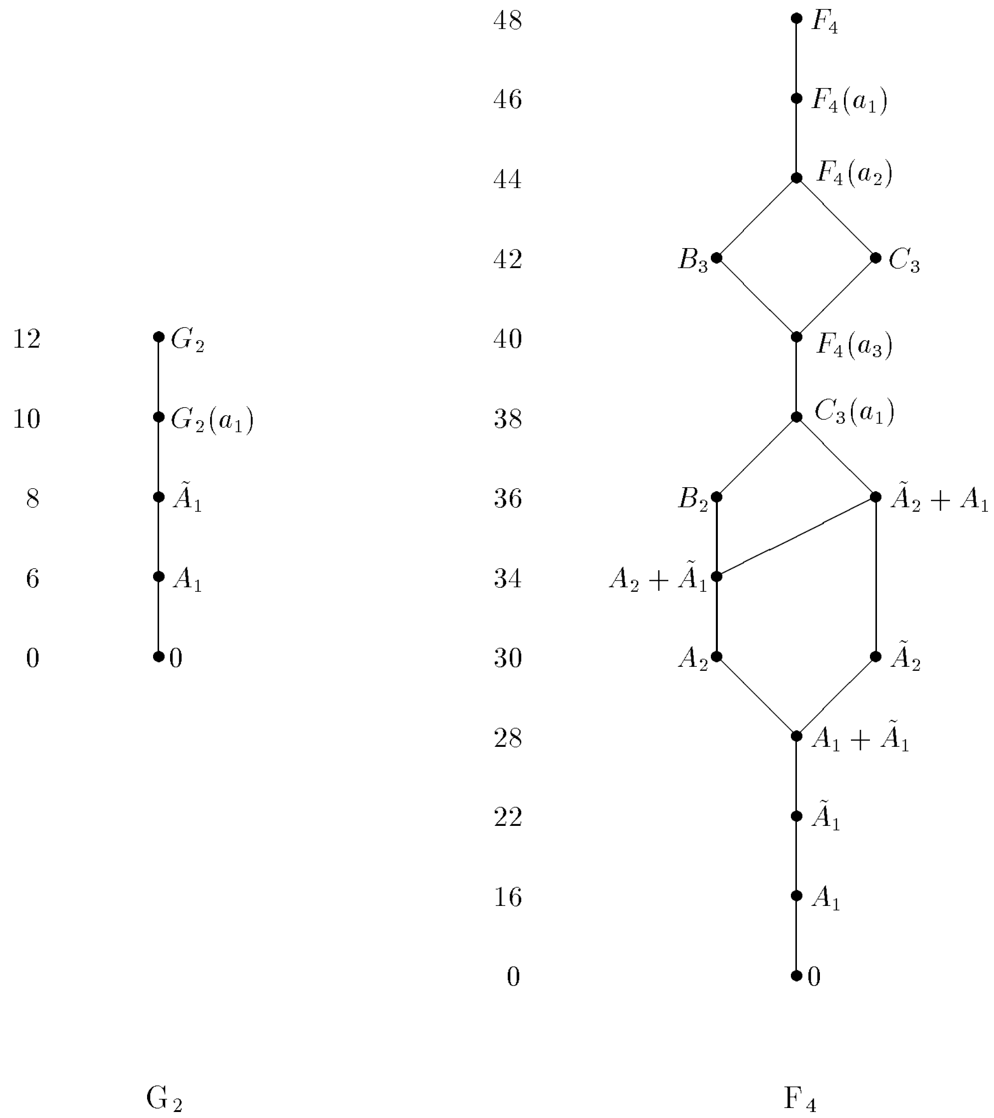

Figure 1: Closure diagrams for G_2 and F_4

If $\mathfrak{g}$ is of type F_4, we write $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ using the same notation as in Bourbaki [?] and in [?]. If $\mathfrak{g}_0$ is of type F I then $K = (\mathrm{Sp}_6 \times \mathrm{SL}_2)/Z_2$, $\dim \mathfrak{p} = 28$, and $B_0 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ with $\beta_1 = \alpha_4$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_2$, and $\beta_4 = \alpha_0$. When $\mathfrak{g}_0$ is of type F II then $K = \mathrm{Spin}_9$, $\dim \mathfrak{p} = 16$, and $B_0 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ with $\beta_1 = \alpha_0$, $\beta_2 = \alpha_1$, $\beta_3 = \alpha_2$, and $\beta_4 = \alpha_3$.

A *normal triple* is an ordered triple (E, H, F) of elements $E, F \in \mathfrak{p}$ and $H \in \mathfrak{k}$ spanning a three dimensional simple subalgebra of $\mathfrak{g}$ and satisfying the relations:

$$[H,E]=2E,\ [H,F]=-2F,\ [F,E]=H.$$

Given a K^0-orbit $\mathcal{O}\subset\mathfrak{p}$, one can choose a normal triple (E,H,F) such that $E\in\mathcal{O}$, $H\in\mathfrak{h}$, and $\beta_i(H)$ are nonnegative integers. Furthermore the integers $\beta_i(H)$ determine $\mathcal{O}$ uniquely. We can choose $w\in W$ such that $\alpha_i(w(H))$ are nonnegative integers. These integers determine uniquely the nilpotent G-orbit containing $\mathcal{O}$. We shall write $H'=w(H)$.

The positive roots of F_4 and its structure constants with respect to a Chevalley basis are listed in the Appendix. The bases that we use are the same as in [?], but our table of the structure constants for F_4 is given in the Appendix in a more "user friendly way" than in the mentioned paper. The positive roots are enumerated by integers from 1 to 24. The negative root $-\alpha_i$ is also written as α_{-i}. The root vector in the Chevalley basis corresponding to the root α_i is denoted by X_i.

In Tables 1-3 below we shall list the non-zero nilpotent K-orbits $\mathcal{O}$ in $\mathfrak{p}$ for the three cases that we consider. For each of these orbits we list the integers $\beta_i(H)$, the integers $\alpha_i(H')$, a representative E of the orbit $\mathcal{O}$, and the complex dimension of $\mathcal{O}$. In some cases two or more representatives (of different types) are given. For instance the orbit 4 in Table 1 has a representative $X_{-2}+X_{-4}$ of type $A_1+\tilde{A}_1$ meaning that $\{\alpha_{-2},\alpha_{-4}\}$ is a base of a closed root subsystem of type $\{\alpha_{-2},\alpha_{-4}\}$ with α_{-2} a long root and α_{-4} a short root. The same orbit also has the representative $X_{-2}+X_{-5}$ of type A_2. In the last column we give the label from Figure 1 for the G-orbit in $\mathcal{N}$ containing the given K-orbit in $\mathcal{N}_1$.

Table 1: G I **table**

$\mathcal{O}$	$\beta_i(H)$	$\alpha_i(H')$	$E\in\mathcal{O}$	$\dim\mathcal{O}$	Type of E	$G\cdot\mathcal{O}$
1	11	01	X_{-2}	3	A_1	A_1
2	13	10	X_{-3}	4	$\tilde{A}_1$	$\tilde{A}_1$
3	22	02	X_5+X_{-3}	5	$A_1+\tilde{A}_1$	$G_2(a_1)$
4	04	02	$X_{-2}+X_{-4}$	5	$A_1+\tilde{A}_1$	$G_2(a_1)$
			$X_{-2}+X_{-5}$		A_2	
5	48	22	$X_{-4}+X_5$	6	G_2	G_2

We can now state our main result.

Theorem 2.1. *Let $\mathfrak{g}_0$ be of type* G I, F I, *or* F II. *Then the closure ordering of the nilpotent K-orbits in $\mathfrak{p}$ is as given in* Figure 2 *on the following page.*

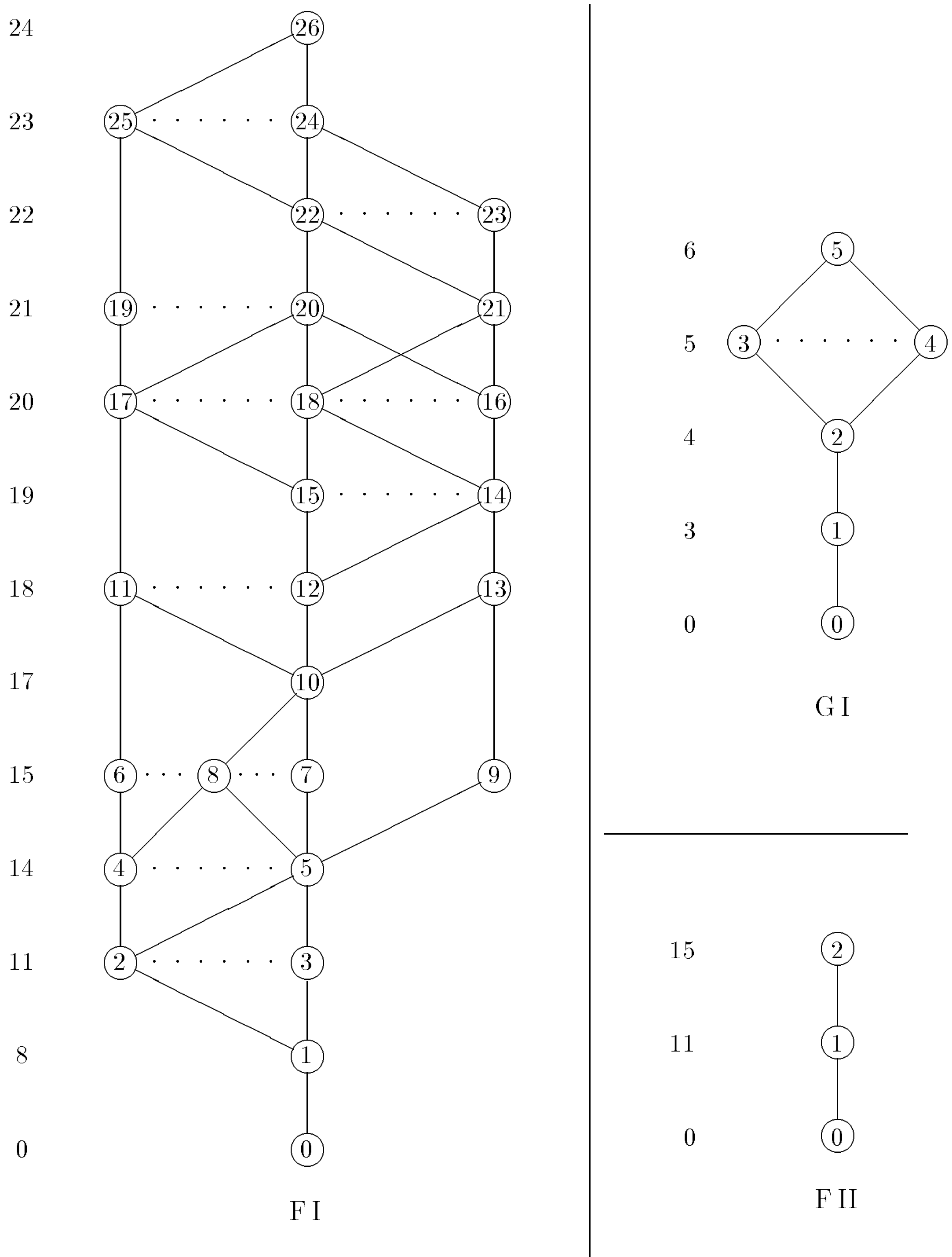

Figure 2: Closure diagrams for $F\,I$, $F\,II$ and $G\,I$.

The horizontal dotted lines in Figure 2 indicate that the K^0-orbits joined by these lines are contained in the same G-orbit. Again the numbers on the left hand side of the diagrams are the dimensions of the orbits on that level.

Table 2: F I **table**

$\mathcal{O}$	$\beta_i(H)$	$\alpha_i(H')$	$E \in \mathcal{O}$	dim	Type	$G \cdot \mathcal{O}$	
1	001 1	1000	X_{-1}	8	A_1	A_1	
2	100 2	0001	X_{-8}	11	$\tilde{A}_1$	$\tilde{A}_1$	
			$X_{-1} + X_{-14}$		$2A_1$		
3	010 0	0001	$X_{-1} + X_{22}$	11	$2A_1$	$\tilde{A}_1$	
4	001 3	0100	$X_{-5} + X_{-15}$	14	$A_1 + \tilde{A}_1$	$A_1 + \tilde{A}_1$	
			$X_{-5} + X_{-11} + X_{-18}$		$3A_1$		
5	101 1	0100	$X_{23} + X_{-8}$	14	$A_1 + \tilde{A}_1$	$A_1 + \tilde{A}_1$	
			$X_{23} + X_{-5} + X_{-11}$		$3A_1$		
6	000 4	2000	$X_{-1} + X_{-23}$	15	A_2	A_2	
			$X_{-1} + X_{-14} + X_{-21}$		$2A_1 + \tilde{A}_1$		
			$X_{-1} + X_{-14} + X_{-20} + X_{-22}$		$4A_1$		
7	200 0	2000	$X_{-1} + X_{18}$	15	A_2	A_2	
			$X_{18} + X_{23} + X_{-7}$		$2A_1 + \tilde{A}_1$		
			$X_{18} + X_{23} + X_{-5} + X_{-11}$		$4A_1$		
8	002 0	2000	$X_{-18} + X_{23} + X_{-8}$	15	$2A_1 + \tilde{A}_1$	A_2	
			$X_{23} + X_{-5} + X_{-11} + X_{-18}$		$4A_1$		
9	020 0	0002	$X_{-8} + X_{19}$	15	$\tilde{A}_2$	$\tilde{A}_2$	
10	110 2	0010	$(X_{22} + X_{-11}) + X_{-12}$	17	$A_2 + \tilde{A}_1$	$A_2 + \tilde{A}_1$	
11	102 4	2001	$X_{-18} + X_{-14} + X_{23}$	18	A_3	B_2	
12	012 2	2001	$X_{22} + X_{-15}$	18	B_2	B_2	
			$X_{-11} + X_{22} + X_{-18}$		A_3		
13	111 1	0101	$(X_{21} + X_{-12}) + X_{-11}$	18	$\tilde{A}_2 + A_1$	$\tilde{A}_2 + A_1$	
14	103 1	1010	$(X_{20} + X_{-12}) + X_{22}$	19	$B_2 + A_1$	$C_3(a_1)$	
15	111 3	1010	$(X_{22} + X_{-15}) + X_{-14}$	19	$B_2 + A_1$	$C_3(a_1)$	
16	004 0	0200	$(X_{-5} + X_{17}) + X_{-11} + X_{-18}$	20	$B_2 + 2A_1$	$F_4(a_3)$	
17	020 4	0200	$(X_{22} + X_{-15}) + X_{-14} + X_{-20}$	20	$B_2 + 2A_1$	$F_4(a_3)$	Ex.3.3
18	202 2	0200	$(X_{20} + X_{-12}) + X_{-14} + X_{22}$	20	$B_2 + 2A_1$	$F_4(a_3)$	
19	004 8	2200	$X_{-14} + X_{23} + X_{-21}$	21	B_3	B_3	
			$X_{-14} + X_{23} + X_{-20} + X_{-22}$		D_4		
20	204 4	2200	$X_{-14} + X_{-18} + X_{21}$	21	B_3	B_3	
			$X_{20} + X_{-18} + X_{22} + X_{-14}$		D_4		
21	131 3	1012	$X_{-15} + X_{19} + X_{-14}$	21	C_3	C_3	
22	040 4	0202	$(X_{-15} + X_{21} + X_{-20}) + X_{-14}$	22	$C_3 + A_1$	$F_4(a_2)$	
23	222 2	0202	$(X_{-15} + X_{19} + X_{-14}) + X_{20}$	22	$C_3 + A_1$	$F_4(a_2)$	
24	224 4	2202	$X_{20} + X_{-18} + X_{-14} + X_{19}$	23	B_4	$F_4(a_1)$	
25	404 8	2202	$X_{20} + X_{-18} + X_{22} + X_{-19}$	23	B_4	$F_4(a_1)$	
26	444 8	2222	$X_{-18} + X_{20} + X_{-17} + X_{19}$	24	F_4	F_4	

Table 3: F II **table**

$\mathcal{O}$	$\beta_i(H)$	$\alpha_i(H')$	$E \in \mathcal{O}$	$\dim \mathcal{O}$	Type of E	$G \cdot \mathcal{O}$
1	0001	0001	X_{-4}	11	$\tilde{A}_1$	$\tilde{A}_1$
2	4000	0002	$X_{-4} + X_{-19}$	15	$\tilde{A}_2$	$\tilde{A}_2$

3. The proof: Part I

Let $\mathcal{O}^i$ denote the i-th orbit as listed in Table 2. In this section we shall prove that if $\mathcal{O}^i$ and $\mathcal{O}^j$ are joined by a solid line in Figure 2 (diagram F I), with i above j, then $\mathcal{O}^i > \mathcal{O}^j$. The proofs are based on the theorem below, which is valid for any simple complex Lie algebra $\mathfrak{g}$ and its non-compact real form $\mathfrak{g}_0$.

In order to state this theorem we need some additional notations. Let (E, H, F) be a normal triple and $\mathcal{O} = K^0 \cdot E \subset \mathfrak{p}$. Then for any integer j we define

$$\begin{aligned} \mathfrak{g}_H(0,j) &= \{X \in \mathfrak{k} : [H,X] = jX\}, \\ \mathfrak{g}_H(1,j) &= \{X \in \mathfrak{p} : [H,X] = jX\}, \end{aligned}$$

and

$$\mathfrak{p}_i(H) = \sum_{j \geq i} \mathfrak{g}_H(1,j).$$

Finally let Q_H be the parabolic subgroup of K^0 having

$$\mathfrak{q}_H := \sum_{j \geq 0} \mathfrak{g}_H(0,j)$$

as its Lie algebra.

Theorem 3.1. *Let (E, H, F) be a normal triple and $\mathcal{O} = K^0 \cdot E \subset \mathfrak{p}$. Denote by $Z_K(H)$ the centralizer of H in K and by $Z_K(H)^0$ its identity component. Then*

(a) *The pair $(Z_K(H)^0, \mathfrak{g}_H(0,2))$ is a prehomogeneous vector space and $Z_K(H)^0 \cdot E$ its dense open orbit.*

(b) *The pair $(Q_H, \mathfrak{p}_2(H))$ is a prehomogeneous vector space and $Q_H \cdot E$ its dense open orbit.*

(c) *If $\mathcal{O}' \subset \mathcal{N}_1$ is also a K^0-orbit, then $\mathcal{O} \geq \mathcal{O}'$ if and only if $\mathcal{O}' \cap \mathfrak{p}_2(H) \neq \emptyset$.*

The assertions (a) and (b) constitute a symmetric space analog of a result of Kostant [**?**, Theorem 4.3] (see also [**?**, Lemma 4.1.4]). The proof of these two assertions given in [**?**, Proposition 1] for the special case $\mathfrak{g}_0 = \mathfrak{so}(p,q)$ can be easily extended to the general case. The assertion (c) follows from [**?**, Satz 2, pp. 182-184].

We shall now apply Theorem 3.1 to the case when $\mathfrak{g}_0$ is of type F I. Let (E^i, H^i, F^i) denote the normal triple with $E^i \in \mathcal{O}^i$ and H^i as given in Table 2.

Each space $\mathfrak{g}_{H^i}(1,j)$ with $j \neq 0$ is a sum of certain root spaces, and the same is true for the space $\mathfrak{p}_2(H^i)$. In Table 4 we list the indices k of all roots $\alpha = \alpha_k$ such that $\mathfrak{g}^\alpha \subset \mathfrak{p}_2(H^i)$. We list first the indices of roots α for which $\mathfrak{g}^\alpha \subset \mathfrak{g}_{H^i}(1,2)$ and separate them by a semi-colon from the indices of the roots β (if any) for which $\mathfrak{g}^\beta \subset \mathfrak{p}_3(H^i)$.

Table 4: Root spaces in $\mathfrak{g}_{H^i}(1,2)$ and $\mathfrak{p}_3(H^i)$

i	Indices of roots	
1	-1;	
2	-1, -5, -8, -11, -14;	
3	-1, -5, 22, 23;	
4	-5, -8, -11, -12, -15, -18; -1	
5	-5, -8, -11, 23; -1	
6	-1, -5, -8, -11, -12, -14, -15, -17, -18, -19, -20, -21, -22, -23;	
7	-1, -5, -8, -11, -14, 18, 20, 21, 22, 23;	
8	-5, -8, -11, -12, -15, -18, 23; -1	
9	-8, -12, 19, 21; -1, -5, 22, 23	
10	-11, -12, -14, 22, 23; -1, -5, -8	
11	-14, -18, 23; -1, -5, -8, -11, -12, -15	
12	-11, -15, -18, 22; -1, -5, -8, -12, 23	
13	-11, -12, 21; -1, -5, -8, 22, 23	
14	-12, -15, 20, 21, 22; -1, -5, -8, -11, 23	
15	-14, -15, 22; -1, -5, -11, -12, 23	
16	-5, -8, -11, -12, 14, -15, 17, -18, 19, 20, 21, 22; -1, 23	
17	-11, -14, -15, -17, -18, -20, 22, 23; -1, -5, -8, -12	
18	-12, -14, -15, 20, 21, 22; -1, -5, -8, -11, 23	
19	-14, -17, -19, -20, -21, -22, 23; -1, -5, -8, -11, -12, -15, -18	
20	-14, -18, 20, 21, 22; -1, -5, -8, -11, -12, -15, 23	
21	-14, -15, 19; -1, -5, -8, -11, -12, 21, 22, 23	
22	-11, -14, -15, -17, -18, 19, -20, 21; -1, -5, -8, -12, 22, 23	
23	-14, -15, 19, 20; -1, -5, -8, -11, -12, 21, 22, 23	
24	-14, -18, 19, 20; -1, -5, -8, -11, -12, -15, 21, 22, 23	
25	-17, -18, -19, 20, 21, 22; -1, -5, -8, -11, -12, -14, -15, 23	Ex.3.2
26	-17, -18, 19, 20; -1, -5, -8, -11, -12, -14, -15, 21, 22, 23	

Next we examine all pairs (i,j) such that i is joined to j by a solid line in Figure 2 and i is above j. For each such pair we have constructed a normal triple (E,H,F) such that $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}^j$. By Theorem 3.1, (c), it follows that $\mathcal{O}^i > \mathcal{O}^j$.

In Table 5, for each such pair (i,j), we exhibit the element E as a sum of root vectors. If the roots involved in E form a basis of a closed root subsystem of R, we indicate its type in the third column. The last column identifies the element H by listing the integers $\alpha_1(H),\ldots,\alpha_4(H)$ in that order. If these integers are not listed, then $H = H^j$ and one can use Table 2 and the integers $\beta_1(H),\ldots,\beta_4(H)$ listed there to compute $\alpha_1(H),\ldots,\alpha_4(H)$.

We give more details in two cases. The coroot corresponding to the root α_k is denoted by $H_k (\in \mathfrak{h})$. Note that $H_{-k} = -H_k$.

Example 3.2. $i = 25$, $j = 22$. Then $E = (X_{-19} + X_{21} + X_{-18}) + (X_{-5})$ and the type is given as $C_3 + A_1$. The meaning of the parantheses is that the roots $\alpha_{-19}, \alpha_{21}, \alpha_{-18}$ form a base for the C_3-component and α_{-5} a base for the A_1-component. In this case

$$\alpha_1(H) = -10,\ \alpha_2(H) = 8,\ \alpha_3(H) = -4,\ \alpha_4(H) = 4.$$

Since

$$\begin{aligned} \alpha_{19} &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \\ \alpha_{21} &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \\ \alpha_{18} &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \\ \alpha_5 &= \alpha_1 + \alpha_2, \end{aligned}$$

we indeed have $E \in \mathfrak{g}_H(1,2)$. The element F is not given in the table, but it can be easily computed. In this case it is a linear combination of X_{19}, X_{-21}, X_{18}, and X_5.

It is a simple matter to compute

$$\beta_1(H) = 4,\ \beta_2(H) = -4,\ \beta_3(H) = 8,\ \beta_4(H) = -4$$

and show that $w(H) = H^{22}$ for a suitable $w \in W_0$. This proves that $E \in \mathcal{O}^{22}$. Since the indices $-19, 21, -18$, and -5 appear in row 25 of Table 4, we also have $E \in \mathfrak{p}_2(H^{25})$.

Example 3.3. $i = 19$, $j = 17$. In this case $E = X_{22} + X_{-18} + X_{-14} + X_{23}$ and (since a type is not given) the roots $\alpha_{22}, \alpha_{-18}, \alpha_{-14}$, and α_{23} do not form a basis for a closed root subsystem. Since the integers $\alpha_1(H), \ldots, \alpha_4(H)$ are not listed, this means that $H = H^{17}$. From Table 2 we have

$$\beta_1(H) = 0,\ \beta_2(H) = 2,\ \beta_3(H) = 0,\ \beta_4(H) = 4.$$

By using $-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, we find that

$$\alpha_1(H) = -6,\ \alpha_2(H) = 0,\ \alpha_3(H) = 2,\ \alpha_4(H) = 0.$$

Since

$$\begin{aligned} \alpha_{22} &= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ \alpha_{18} &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \\ \alpha_{14} &= \alpha_1 + 2\alpha_2 + 2\alpha_3, \\ \alpha_{23} &= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \end{aligned}$$

we indeed have $E \in \mathfrak{g}_H(1,2)$.

The root spaces for the roots α_{20}, α_{-22}, α_{11}, α_{-23}, α_{18}, and α_{14} are contained in $\mathfrak{g}_H(1,-2)$. By using the structure constants for F_4 given in the Appendix, we obtain that

$$[X_{20} + X_{-22}, E] = H_{22}, \quad [X_{11} + X_{-23}, E] = H_{23},$$

$$[X_{18}, E] = -H_{18}, \quad [X_{14}, E] = -H_{14}.$$

The roots α_{22}, α_{23}, α_{18}, α_{14} are linearly independent and so $\mathfrak{h} = \mathrm{ad}(E)\mathfrak{g}_H(1,-2)$. Hence there exists $F \in \mathfrak{g}_H(1,-2)$ such that $[F, E] = H$. As in the previous example, we can check that $E \in \mathfrak{p}_2(H^{19}) \cap \mathcal{O}^{17}$.

Table 5: Normal triples (E,H,F) with $E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}^j$

i	j	Type	E	$\alpha_1(H),\ldots,\alpha_4(H)$	
26	25	B_4	$X_{22}+X_{-18}+X_{20}+X_{-17}$		
26	24	B_4	$X_{20}+X_{-18}+X_{-14}+X_{19}$		
25	19	B_3	$X_{-18}+X_{20}+X_{-17}$	$-6,4,-4,4$	
25	22	C_3+A_1	$(X_{-19}+X_{21}+X_{-18})+(X_{-5})$	$-10,8,-4,4$	Ex.3.2
24	22		$X_{19}+X_{21}+X_{-14}+X_{-18}$	$-10,0,4,0$	
24	23	C_3+A_1	$(X_{-15}+X_{19}+X_{-14})+(X_{20})$	$-10,2,2,2$	
22	20	B_3	$X_{-14}+X_{-18}+X_{21}$		
22,23	21	C_3	$X_{-15}+X_{19}+X_{-14}$		
19	17		$X_{22}+X_{-18}+X_{-14}+X_{23}$		Ex.3.3
20	17	$A_3+\tilde{A}_1$	$(X_{20}+X_{-18}+X_{22})+(X_8)$	$-2,4,0,-2$	
20	18	B_2+2A_1	$(X_{22}+X_{-15})+(X_{-14})+(X_{20})$		
20	16		$X_{20}+X_{22}+X_{-5}+X_{-18}$	$-6,4,0,0$	
21	18		$X_{21}+X_{22}+X_{-14}+X_{-15}$	$-6,2,0,2$	
21	16		$X_{19}+X_{21}+X_{22}+X_{-15}$	$-6,4,0,0$	
17	11	A_3	$X_{-18}+X_{-14}+X_{23}$		
17,18	15	B_2+A_1	$(X_{22}+X_{-15})+(X_{-14})$		
16,18	14	B_2+A_1	$(X_{20}+X_{-12})+(X_{22})$		
14,15	12	B_2	$X_{22}+X_{-15}$		
14	13	$\tilde{A}_2+A_1$	$(X_{-12}+X_{21})+(X_{-11})$		
11	6	A_2	$X_{-14}+X_{-18}$		
11	10	$A_2+\tilde{A}_1$	$(X_{-14}+X_{23})+(X_{-12})$		
12,13	10	$A_2+\tilde{A}_1$	$(X_{-11}+X_{22})+(X_{-12})$		
13	9	$\tilde{A}_2$	$X_{-12}+X_{21}$		
10	8	$2A_1+\tilde{A}_1$	$(X_{-11})+(X_{23})+(X_{-12})$		
10	7	A_2	$X_{-14}+X_{23}$		
6,8	4	$A_1+\tilde{A}_1$	$(X_{-18})+(X_{-8})$		
7,8,9	5	$A_1+\tilde{A}_1$	$(X_{23})+(X_{-8})$		
4,5	2	$\tilde{A}_1$	X_{-8}		
5	3	$2A_1$	$(X_{-5})+(X_{23})$		
2,3	1	A_1	X_{-1}		

4. The proof: Part II

In order to complete the proof of Theorem 2.1, we must show that $\mathcal{O}^r \not> \mathcal{O}^s$ whenever in Figure 2 the pair (r,s) is not connected by an edge. We need not check *all* possible pairs because of transitivity. For example: $\mathcal{O}^4 \not> \mathcal{O}^3$ follows once we have proved $\mathcal{O}^6 \not> \mathcal{O}^3$ because $\mathcal{O}^6 > \mathcal{O}^4$. Therefore it suffices to show that $\mathcal{O}^r \not> \mathcal{O}^s$ when (r,s) is one of the following eight pairs:

$$\begin{array}{ccccc} (23,6), & (25,23), & (9,4), & (19,9), & (24,19), \\ (16,15), & (7,4), & (6,3). & & \end{array} \qquad (\dagger)$$

In this section we shall prove that $\mathcal{O}^r \not> \mathcal{O}^s$ for the pairs (r,s) listed in

Table 7 below, covering the last three pairs (r, s) listed above. This will be derived from a general theorem which we prove next.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a simple Lie algebra, as in the Introduction, and let V be a Z_2-graded finite-dimensional $\mathfrak{g}$-module. This means that V is equipped with a direct decomposition $V = V_0 \oplus V_1$ such that

$$\mathfrak{k} \cdot V_0 \subset V_0, \quad \mathfrak{k} \cdot V_1 \subset V_1, \quad \mathfrak{p} \cdot V_0 \subset V_1, \quad \mathfrak{p} \cdot V_1 \subset V_0.$$

We refer to V_0 (resp. V_1) as the even (resp. odd) subspace of V. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ denote the representation of $\mathfrak{g}$ on V.

Now let (E, H, F) be a normal triple in $\mathfrak{g}$. Since $E \in \mathfrak{p}$, the linear operator $\rho(E)$ maps V_0 to V_1 and V_1 to V_0. We introduce the integers

$$d_0(j, E) = \dim V_0 \cap \ker \rho(E)^j, \quad d_1(j, E) = \dim V_1 \cap \ker \rho(E)^j,$$

for $j \geq 1$. They depend only on the orbit $\mathcal{O} = K^0 \cdot E$ (and the module V), and so we can set

$$d_0(j, \mathcal{O}) = d_0(j, E), \quad d_1(j, \mathcal{O}) = d_1(j, E).$$

Theorem 4.1. *Let $\mathcal{O}, \mathcal{O}' \subset \mathcal{N}_1$ be K^0-orbits with $\dim \mathcal{O} > \dim \mathcal{O}'$, and let V be a finite-dimensional Z_2-graded $\mathfrak{g}$-module. If $d_0(j, \mathcal{O}) > d_0(j, \mathcal{O}')$ or $d_1(j, \mathcal{O}) > d_1(j, \mathcal{O}')$ for some $j \geq 1$, then $\mathcal{O} \not> \mathcal{O}'$.*

Proof. Assume that, say, $d_0(j, \mathcal{O}) > d_0(j, \mathcal{O}')$ for some $j \geq 1$. Let L be the space of linear operators $u : V \to V$ such that $u(V_0) \subset V_1$ and $u(V_1) \subset V_0$. Let $L_0(k, m)$ be the subset of L consisting of all $u \in L$ such that

$$\dim V_0 \cap \ker u^k \geq m.$$

Clearly $L_0(k, m)$ is an affine subvariety of L.

If $m = d_0(j, \mathcal{O})$, then $\rho(\mathcal{O}) \subset L_0(j, m)$ and since $L_0(j, m)$ is closed in L, we deduce that

$$\rho(\overline{\mathcal{O}}) \subset L_0(j, m).$$

Since $m = d_0(j, \mathcal{O}) > d_0(j, \mathcal{O}')$, $\rho(\mathcal{O}')$ is not contained in $L_0(j, m)$. It follows that $\mathcal{O}' \not\subset \overline{\mathcal{O}}$, i.e., $\mathcal{O} \not> \mathcal{O}'$. ∎

In order to apply this theorem, we have to compute the integers $d_i(j, E)$. Fortunately there is a simple method for doing these computations by using the $\mathfrak{sl}_2$-theory. By restriction, V is a Z_2-graded module for the Z_2-graded subalgebra $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ of $\mathfrak{g}$ spanned by the normal triple (E, H, F).

Note that $H \in \mathfrak{s}_0$ while $E, F \in \mathfrak{s}_1$. We can decompose V into direct sum of simple Z_2-graded $\mathfrak{s}$-modules X_{ij}, see [?, Lemma 1]. We recall that $i = 0$ or 1 and j is a nonnegative integer. X_{ij} is also simple as ungraded $\mathfrak{s}$-module, it has dimension $j + 1$, and the highest weight vector is even if $i = 0$ and odd if $i = 1$. Let m_{ij} be the multiplicity of X_{ij} in V, i.e., symbolically we have

$$V = \sum_{i,j} m_{ij} X_{ij}.$$

Table 6: The integers $d_i(j,k)$ for the module $V(\omega_4)$

k	$d_0(j,k)$	$d_1(j,k)$	
1	11,14	9,12	
2	9,14	7,11,12	
3	9,13,14	7,12	Case (6,3)
4	8,14	6,9,12	Case (7,4)
5	8,12,14	6,11,12	
6	8,14	6,6,12	Case (6,3)
7	8,10,14	6,10,12	Case (7,4)
8	8,11,14	6,9,12	
9	5,8,13,13,14	3,8,11,12	
10	6,10,13,14	4,9,11,12	
11	6,9,11,14	4,6,9,11,12	
12	6,8,11,13,14	4,7,9,12	
13	5,8,12,13,14	3,8,10,12	
14	5,8,11,12,14	3,7,9,12	
15	5,8,11,13,14	3,7,9,11,12	Case (16,15)
16	5,8,11,11,14	3,6,9,12	Case (16,15)
17	5,8,11,13,14	3,6,9,10,12	
18	5,7,11,12,14	3,7,9,11,12	
19	5,8,8,11,11,14	3,3,6,6,9,9,12	
20	5,6,8,9,11,12,14	3,5,6,8,9,11,12	
21	3,4,7,8,11,12,13,13,14	1,4,5,8,9,11,11,12	
22	3,4,7,8,11,12,13,13,14	1,4,5,8,9,10,11,12	
23	3,4,7,8,11,11,13,13,14	1,4,5,8,9,11,11,12	
24	3,4,6,7,9,9,11,11,13,13,14	1,3,4,6,7,9,9,11,11,12	
25	3,4,6,7,9,10,11,12,13,14	1,3,4,6,7,8,9,10,11,11,12	
26	2,2,4,4,6,6,8,8,10,10,11, 11,12,12,13,13,14	0,2,2,4,4,6,6,8,8,9,9,10, 10,11,11,12	

The $\mathfrak{sl}_2$-theory gives the formulas:

$$\begin{aligned} d_i(1,E) &= \sum_{j\geq 0} m_{ij}, \\ d_i(2,E)-d_i(1,E) &= \sum_{j\geq 1} m_{1-i,j}, \\ d_i(3,E)-d_i(2,E) &= \sum_{j\geq 2} m_{ij}, \\ &\vdots \end{aligned}$$

for $i=0,1$. Hence the problem reduces to the computation of the multiplicities m_{ij}.

The linear operator $\rho(H)$ is semisimple and leaves the subspaces V_0 and V_1 invariant. Define integers

$$N(i,j,H) = \dim\{X \in V_i : \rho(H)X = jX\}$$

for $i=0,1$ and $j \in \mathbf{Z}$. Then, by $\mathfrak{sl}_2$-theory,

$$m_{ij} = N(i,j,H) - N(1-i,j+2,H)$$

for $i = 0, 1$ and arbitrary $j \geq 0$. The integers $N(i, j, H)$ can be computed by using the Weyl character formula.

Let us now return to our special case where $\mathfrak{g}_0$ is of type F I. We take V to be the fundamental $\mathfrak{g}$-module $V(\omega_4)$ (by $\omega_1, \dots, \omega_4$ we denote the fundamental weights of F_4 as in [?]). It has dimension 26 and its weights are the short roots of R (all simple weights) and 0 which has multiplicity 2. We postulate that the highest weight vector of V is even. This gives rise to a Z_2-grading $V = V_0 \oplus V_1$ with $\dim V_0 = 14$ and $\dim V_1 = 12$.

In Table 6 we list the integers $d_i(j, k) = d_i(j, \mathcal{O}^k)$ for each of the 26 nonzero K-orbits in $\mathcal{N}_1$. By using Theorem 4.1, it follows from Table 6 that $\mathcal{O}^r \not\geq \mathcal{O}^s$ for the pairs (r, s), with $\dim \mathcal{O}^r > \dim \mathcal{O}^s$, listed in Table 7. In each case we exhibit $i \in \{0, 1\}$ and j such that $d_i(j, r) > d_i(j, s)$.

We did compute tables similar to Table 6 for several other simple $\mathfrak{g}$-modules (including the adjoint module) but no new information was gained.

Table 7: Some pairs (r, s) with $\mathcal{O}^r \not\geq \mathcal{O}^s$

r	s	$d_i(j, r)$	$d_i(j, s)$
23	19	$d_1(2, 23) = 4$	$d_1(2, 19) = 3$
22	19	$d_1(2, 22) = 4$	$d_1(2, 19) = 3$
19	18	$d_0(2, 9) = 8$	$d_0(2, 18) = 7$
17	14	$d_0(4, 17) = 13$	$d_0(4, 14) = 12$
16	15	$d_1(4, 16) = 12$	$d_1(4, 15) = 11$
12	9	$d_1(1, 12) = 4$	$d_1(1, 9) = 3$
11	9	$d_1(1, 11) = 4$	$d_1(1, 9) = 3$
7	4	$d_1(2, 7) = 10$	$d_1(2, 4) = 9$
6	3	$d_0(2, 6) = 14$	$d_0(2, 3) = 13$

5. The proof: Part III

In this section we prove that $\mathcal{O}^r \not\geq \mathcal{O}^s$ for the first five of the eight pairs (r, s) listed in (†) in the previous section. This will complete the proof.

For that purpose we examine the structure of $\mathfrak{p}$ as a K-module in more detail. First recall that $K = (\mathrm{Sp}_6 \times \mathrm{SL}_2)/Z_2$ and that $\mathfrak{p} \cong V_0 \otimes V_1$ where V_0 is the third fundamental module of Sp_6 and V_1 is the defining 2-dimensional module of SL_2. We remark that the second and third fundamental modules of Sp_6 both have dimension 14 but the former has 0 as a weight of multiplicity 2 while 0 is not a weight of the latter.

Let $R_1 = R \setminus R_0$, i.e., R_1 is the set of roots $\alpha \in R$ such that $\mathfrak{g}^\alpha \subset \mathfrak{p}$. Denote by R_1^+ (resp. R_1^-) the set of positive (resp. negative) roots in R_1. Each of the sets R_1^+, R_1^- consists of 14 roots. The subspaces

$$\mathfrak{p}^+ = \sum_{\alpha \in R_1^+} \mathfrak{g}^\alpha, \quad \mathfrak{p}^- = \sum_{\alpha \in R_1^-} \mathfrak{g}^\alpha$$

are simple Sp_6-modules isomorphic to V_0, and $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

An isomorphism $\varphi : \mathfrak{p}^+ \to \mathfrak{p}^-$ of Sp_6-modules is given by $\varphi(X) = [X_{-24}, X]$. In Table 8 we give the φ-images of the basis of $\mathfrak{p}^+$ consisting of root vectors.

When viewed as a K-module, $\mathfrak{p}$ has only simple weights. The weight diagram of this module is exhibited in Figure 3 where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are clearly visible as a left and right half of the diagram. We have also indicated by arrows how the root vectors of $\beta_1, \beta_2, \beta_3$ act on the weight spaces of $\mathfrak{p}^+$. For β_4 only two arrows are shown. The highest weight vector is X_{-1} and the lowest X_1.

Table 8: The isomorphism φ

X	$\varphi(X)$	X	$\varphi(X)$
X_1	X_{-23}	X_{17}	X_{-15}
X_5	X_{-22}	X_{18}	X_{-14}
X_8	$-X_{-21}$	X_{19}	$-X_{-12}$
X_{11}	X_{-20}	X_{20}	$-X_{-11}$
X_{12}	X_{-19}	X_{21}	X_{-8}
X_{14}	$-X_{-18}$	X_{22}	$-X_{-5}$
X_{15}	$-X_{-17}$	X_{23}	$-X_{-1}$

If (E, H, F) is a normal triple, we set

$$\mathfrak{p}_2^{\pm}(H) = \mathfrak{p}^{\pm} \cap \mathfrak{p}_2(H).$$

Let $\pi : \mathfrak{p} \to \mathfrak{p}^+$ be the projector with kernel $\mathfrak{p}^-$ and note that it commutes with the action of Sp_6. Any $Z \in \mathfrak{p}$ can be written uniquely as $Z = X + \varphi(Y)$ with $X, Y \in \mathfrak{p}^+$. The orbit $\mathrm{SL}_2 \cdot Z$ consists of all vectors

$$(aX + bY) + \varphi(cX + dY)$$

with $ad - bc = 1$. Hence

$$\pi(\mathrm{SL}_2 \cdot Z) \subset \langle X, Y \rangle$$

where $\langle X, Y \rangle$ denotes the subspace of $\mathfrak{p}^+$ spanned by X and Y.

From the previous two sections we know the closures of the orbits $\mathcal{O}^6$, $\mathcal{O}^4$, $\mathcal{O}^2$, and $\mathcal{O}^1$:

$$\begin{aligned}
\overline{\mathcal{O}_6} &= \mathcal{O}_6 \cup \mathcal{O}_4 \cup \mathcal{O}_2 \cup \mathcal{O}_1 \cup \{0\}, \\
\overline{\mathcal{O}_4} &= \mathcal{O}_4 \cup \mathcal{O}_2 \cup \mathcal{O}_1 \cup \{0\}, \\
\overline{\mathcal{O}_2} &= \mathcal{O}_2 \cup \mathcal{O}_1 \cup \{0\}, \\
\overline{\mathcal{O}_1} &= \mathcal{O}_1 \cup \{0\}.
\end{aligned}$$

The pair $(\mathrm{Sp}_6 \times \mathrm{GL}_1, \mathfrak{p}^+)$, where GL_1 is the maximal torus of the SL_2 factor of K leaving $\mathfrak{p}^+$ and $\mathfrak{p}^-$ invariant, is a regular prehomogeneous vector space (see [?, p.145]). The orbits of this space were classified by Igusa [?]. There are four non-zero orbits:

$$\mathfrak{p}^+ \cap \mathcal{O}^6, \quad \mathfrak{p}^+ \cap \mathcal{O}^4, \quad \mathfrak{p}^+ \cap \mathcal{O}^2, \quad \mathfrak{p}^+ \cap \mathcal{O}^1$$

with representatives $X_1 + X_{23}$, $X_8 + X_{21}$, X_8, X_1 and dimensions $14, 13, 10, 7$, respectively.

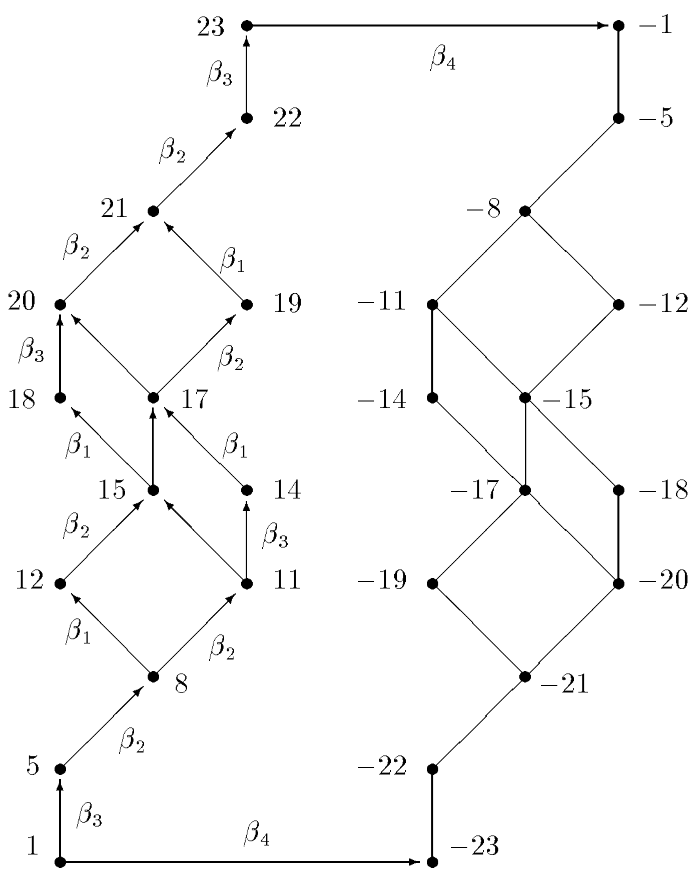

Figure 3: The weight diagram of $\mathfrak{p}$

By definition [?, Definition 1, p. 35], the singular set S of this prehomogeneous vector space is the complement of the dense open orbit, i.e., $S = \mathfrak{p}^+ \setminus \mathcal{O}^6$. Since this prehomogeneous space is regular, S is the zero set of a relative invariant. By using this invariant (which is given explicitly in [?]) it is easy to verify that the 7-dimensional subspace spanned by $X_k, 17 \leq k \leq 23$ (the "upper half" of $\mathfrak{p}^+$ in Figure 3) is contained in S.

We start with the pairs $(23, 6)$ and $(25, 23)$. Let E be the reprensentative of $\mathcal{O}^{23}$ from Table 2:

$$E = X_{-15} + X_{19} + X_{-14} + X_{20} = (X_{19} + X_{20}) + \varphi(X_{17} + X_{18}).$$

We have

$$\pi(\mathrm{SL}_2 \cdot E) \subset \langle X_{19} + X_{20}, X_{17} + X_{18} \rangle \subset S,$$

and so $\mathcal{O}^{23} \subset S + \mathfrak{p}^-$. Since $S + \mathfrak{p}^-$ is closed and $\mathfrak{p}^+ \cap \mathcal{O}^6 = \mathfrak{p}^+ \setminus S$, we have $\mathcal{O}^{23} \not> \mathcal{O}^6$.

Since $X_{19}+X_{20}$ and $X_{17}+X_{18}$ belong to $\mathfrak{p}^+\cap\mathcal{O}^4$ while their sum is in $\mathfrak{p}^+\cap\mathcal{O}^6$, it is easy to deduce (use the action of the maximal torus of K) that in fact

$$\pi(\mathrm{SL}_2\cdot E)\subset\mathfrak{p}^+\cap(\mathcal{O}^6\cup\mathcal{O}^4).$$

On the other hand

$$\mathfrak{p}_2^+(H^{25})=\langle X_{20},X_{21},X_{22},X_{23}\rangle\subset\overline{\mathcal{O}^2}.$$

We conclude that $\mathcal{O}^{23}\cap\mathfrak{p}_2(H^{25})=\emptyset$. By Theorem 3.1, $\mathcal{O}^{25}\not>\mathcal{O}^{23}$.

We now turn to the pair $(9,4)$. Let E be the representative of $\mathcal{O}^9$ from Table 2:

$$E=X_{-8}+X_{19}=X_{19}+\varphi(X_{21}).$$

Then

$$\pi(\mathrm{SL}_2\cdot E)=\langle X_{19},X_{21}\rangle\setminus\{0\}\subset\mathfrak{p}^+\cap\mathcal{O}^2,$$

and by acting with Sp_6, we infer that

$$\overline{\mathcal{O}^9}\subset\mathfrak{p}^+\cap\overline{\mathcal{O}^2}+\mathfrak{p}^-.$$

Hence $\mathfrak{p}^+\cap\overline{\mathcal{O}^9}\subset\mathfrak{p}^+\cap\overline{\mathcal{O}^2}$, and so $\mathcal{O}^9\not>\mathcal{O}^4$.

Next we consider the pair $(19,9)$. Since

$$\pi(\mathcal{O}^9)=\mathrm{Sp}_6\cdot\pi(\mathrm{SL}_2\cdot E)\subset\mathfrak{p}^+\cap\mathcal{O}^2$$

and $\mathfrak{p}_2^+(H^{19})=\langle X_{23}\rangle$ and $X_{23}\in\mathcal{O}^1$, we conclude that $\mathcal{O}^9\cap\mathfrak{p}_2(H^{19})=\emptyset$. Hence $\mathcal{O}^9\not\subset\overline{\mathcal{O}^{19}}$ by Theorem 3.1, and so $\mathcal{O}^{19}\not>\mathcal{O}^9$.

Finally we consider the pair $(24,19)$. Write an arbitrary $X\in\mathfrak{p}_2(H^{24})$ as

$$X=\xi_1X_{20}+\xi_2X_{-18}+\xi_3X_{-14}+\xi_4X_{19}+Y$$

where $Y\in\mathfrak{p}_3(H^{24})$. To simplify the notations, we set $H=H^{24}$. In this case, the parabolic subgroup Q_H (see section 3) is a Borel subgroup of K.

By taking $\xi_1=\cdots=\xi_4=1$ and $Y=0$, the above element X becomes the representative E of $\mathcal{O}^{24}$ (see Table 2). The generic orbit $Q_H\cdot E$ is open and dense in $\mathfrak{p}_2(H)$. Recall that $(Q_H,\mathfrak{p}_2(H))$ is a prehomogeneous vector space. Its singular set is the union of the hyperplanes S_i defined by $\xi_i=0$ $(i=1,2,3,4)$. A computation shows that each of the intersections

$$S_1\cap\mathcal{O}^{22},\quad S_2\cap\mathcal{O}^{23},\quad S_3\cap\mathcal{O}^{16},\quad S_4\cap\mathcal{O}^{20}$$

is an open dense set in the corresponding hyperplane. Since

$$\dim\mathcal{O}^{19}=\dim\mathcal{O}^{20}>\dim\mathcal{O}^{16},$$

it is clear that $\mathcal{O}^{19}$ does not meet $S_3\cap\overline{\mathcal{O}^{16}}$ or $S_4\cap\overline{\mathcal{O}^{20}}$. Since $\mathcal{O}^{19}$ is not contained in $\overline{\mathcal{O}^{22}}$ or $\overline{\mathcal{O}^{23}}$ (see Table 7), we infer that

$$\mathcal{O}^{19}\cap\mathfrak{p}_2(H)=\mathcal{O}^{19}\cap S=\emptyset.$$

By Theorem 3.1, $\mathcal{O}^{24}\not>\mathcal{O}^{19}$.

This completes the proof of Theorem 2.1.

6. Appendix

In Table 9 we give our enumeration of the positive roots of F_4 and their corresponding coroots.

Table 9: The positive roots and coroots of $\mathbf{F}_4$

i	α_i	H_i	i	α_i	H_i	i	α_i	H_i
1	1000	1000	9	0120	0110	17	1221	2421
2	0100	0100	10	0111	0211	18	1122	1111
3	0010	0010	11	1120	1110	19	1231	2431
4	0001	0001	12	1111	2211	20	1222	1211
5	1100	1100	13	0121	0221	21	1232	2432
6	0110	0210	14	1220	1210	22	1242	1221
7	0011	0011	15	1121	2221	23	1342	1321
8	1110	2210	16	0122	0111	24	2342	2321

If α_i and α_j are two roots of F_4 such that their sum $\alpha_i + \alpha_j$ is also a root, say α_k, then the corresponding root vectors from our Chevalley basis satisfy $[X_i, X_j] = N(i,j)X_k$ where $N(i,j)$ is an integer. If $i + j \neq 0$ and $\alpha_i + \alpha_j$ is not a root then $[X_i, X_j] = 0$. Finally if $i + j = 0$ then $[X_{-i}, X_i] = H_i$. In Table 10, for each positive root α_i, i.e., for $1 \leq i \leq 24$, we list the indices j of all roots α_j such that $\alpha_i + \alpha_j$ is a root. Each of these indices j is followed by a semi-colon and the expression $N(i,j)X_k$. For instance, if $i = 4$ we have

$$[X_4, X_3] = -X_7, \quad [X_4, X_{17}] = 2X_{20}, \quad [X_4, X_{-15}] = -2X_{-11}.$$

Table 10: The structure constants of F_4

1	$2:X_5,\ 6:X_8,\ 9:X_{11},\ 10:X_{12},\ 13:X_{15},\ 16:X_{18},\ 23:X_{24}$ $-5:-X_{-2},\ -8:-X_{-6},\ -11:-X_{-9},\ -12:-X_{-10},\ -15:-X_{-13}$ $-18:-X_{-16},\ -24:-X_{-23}$
2	$1:-X_5,\ 3:-X_6,\ 7:X_{10},\ 11:X_{14},\ 15:X_{17},\ 18:X_{20},\ 22:X_{23}$ $-5:X_{-1},\ -6:X_{-3},\ -10:-X_{-7},\ -14:-X_{-11},\ -17:-X_{-15}$ $-20:-X_{-18},\ -23:-X_{-22}$
3	$2:X_6,\ 4:X_7,\ 5:X_8,\ 6:2X_9,\ 8:2X_{11},\ 10:X_{13},\ 12:X_{15}$ $17:X_{19},\ 20:X_{21},\ 21:2X_{22},\ -6:-2X_{-2},\ -7:-X_{-4}$ $-8:-2X_{-5},\ -9:-X_{-6},\ -11:-X_{-8},\ -13:-X_{-10}$ $-15:-X_{-12},\ -19:-X_{-17},\ -21:-2X_{-20},\ -22:-X_{-21}$
4	$3:-X_7,\ 6:X_{10},\ 8:X_{12},\ 9:X_{13},\ 11:X_{15},\ 13:2X_{16},\ 14:X_{17}$ $15:2X_{18},\ 17:2X_{20},\ 19:X_{21},\ -7:X_{-3},\ -10:-X_{-6}$ $-12:-X_{-8},\ -13:-2X_{-9},\ -15:-2X_{-11},\ -16:-X_{-13}$ $-17:-2X_{-14},\ -18:-X_{-15},\ -20:-X_{-17},\ -21:-X_{-19}$
5	$3:-X_8,\ 7:X_{12},\ 9:-X_{14},\ 13:-X_{17},\ 16:-X_{20},\ 22:X_{24}$ $-1:X_2,\ -2:-X_1,\ -8:X_{-3},\ -12:-X_{-7},\ -14:X_{-9}$ $-17:X_{-13},\ -20:X_{-16},\ -24:-X_{-22}$
6	$1:-X_8,\ 3:-2X_9,\ 4:-X_{10},\ 7:X_{13},\ 8:-2X_{14},\ 12:-X_{17}$ $15:X_{19},\ 18:X_{21},\ 21:-2X_{23},\ -2:-X_3,\ -3:2X_2$ $-8:2X_{-1},\ -9:X_{-3},\ -10:X_{-4},\ -13:-X_{-7},\ -14:X_{-8}$ $-17:X_{-12},\ -19:-X_{-15},\ -21:-2X_{-18},\ -23:X_{-21}$
7	$2:-X_{10},\ 5:-X_{12},\ 6:-X_{13},\ 8:-X_{15},\ 10:-2X_{16},\ 12:-2X_{18}$ $14:X_{19},\ 17:X_{21},\ 19:2X_{22},\ -3:X_4,\ -4:-X_3,\ -10:2X_{-2}$ $-12:2X_{-5},\ -13:X_{-6},\ -15:X_{-8},\ -16:X_{-10}$ $-18:X_{-12},\ -19:-2X_{-14},\ -21:-X_{-17},\ -22:-X_{-19}$
8	$3:-2X_{11},\ 4:-X_{12},\ 6:2X_{14},\ 7:X_{15},\ 10:X_{17},\ 13:-X_{19}$ $16:-X_{21},\ 21:-2X_{24},\ -1:X_6,\ -3:2X_5,\ -5:-X_3,\ -6:-2X_1$ $-11:X_{-3},\ -12:X_{-4},\ -14:-X_{-6},\ -15:-X_{-7}$ $-17:-X_{-10},\ -19:X_{-13},\ -21:2X_{-16},\ -24:X_{-21}$
9	$1:-X_{11},\ 4:-X_{13},\ 5:X_{14},\ 12:-X_{19},\ 18:X_{22},\ 20:X_{23},\ -3:X_6$ $-6:-X_3,\ -11:X_{-1},\ -13:X_{-4},\ -14:-X_{-5},\ -19:X_{-12}$ $-22:-X_{-18},\ -23:-X_{-20}$
10	$1:-X_{12},\ 3:-X_{13},\ 7:2X_{16},\ 8:-X_{17},\ 11:-X_{19},\ 12:-2X_{20}$ $15:-X_{21},\ 19:2X_{23},\ -2:X_7,\ -4:X_6,\ -6:-X_4,\ -7:-2X_2$ $-12:2X_{-1},\ -13:X_{-3},\ -16:-X_{-7},\ -17:X_{-8}$ $-19:2X_{-11},\ -20:X_{-12},\ -21:X_{-15},\ -23:-X_{-19}$
11	$2:-X_{14},\ 4:-X_{15},\ 10:X_{19},\ 16:-X_{22},\ 20:X_{24},\ -1:X_9$ $-3:X_8,\ -8:-X_3,\ -9:-X_1,\ -14:X_{-2},\ -15:X_{-4}$ $-19:-X_{-10},\ -22:X_{-16},\ -24:-X_{-20}$
12	$3:-X_{15},\ 6:X_{17},\ 7:2X_{18},\ 9:X_{19},\ 10:2X_{20},\ 13:X_{21}$ $19:2X_{24},\ -1:X_{10},\ -4:X_8,\ -5:X_7,\ -7:-2X_5$ $-8:-X_4,\ -10:-2X_1,\ -15:X_{-3},\ -17:-X_{-6},\ -18:-X_{-7}$ $-19:-2X_{-9},\ -20:-X_{-10},\ -21:-X_{-13},\ -24:-X_{-19}$

Table 10 (continued)

13	$1:-X_{15},\ 4:-2X_{16},\ 5:X_{17},\ 8:X_{19},\ 12:-X_{21},\ 15:-2X_{22}$ $17:-2X_{23},\ -3:X_{10},\ -4:2X_9,\ -6:X_7,\ -7:-X_6$ $-9:-X_4,\ -10:-X_3,\ -15:2X_{-1},\ -16:X_{-4},\ -17:-2X_{-5}$ $-19:-X_{-8},\ -21:X_{-12},\ -22:X_{-15},\ -23:X_{-17}$
14	$4:-X_{17},\ 7:-X_{19},\ 16:-X_{23},\ 18:-X_{24},\ -2:X_{11},\ -5:-X_9$ $-6:-X_8,\ -8:X_6,\ -9:X_5,\ -11:-X_2,\ -17:X_{-4}$ $-19:X_{-7},\ -23:X_{-16},\ -24:X_{-18}$
15	$2:-X_{17},\ 4:-2X_{18},\ 6:-X_{19},\ 10:X_{21},\ 13:2X_{22},\ 17:-2X_{24}$ $-1:X_{13},\ -3:X_{12},\ -4:2X_{11},\ -7:-X_8,\ -8:X_7$ $-11:-X_4,\ -12:-X_3,\ -13:-2X_1,\ -17:2X_{-2},\ -18:X_{-4}$ $-19:X_{-6},\ -21:-X_{-10},\ -22:-X_{-13},\ -24:X_{-17}$
16	$1:-X_{18},\ 5:X_{20},\ 8:X_{21},\ 11:X_{22},\ 14:X_{23},\ -4:X_{13},$ $-7:-X_{10},\ -10:X_7,\ -13:-X_4,\ -18:X_{-1}$ $-20:-X_{-5},\ -21:-X_{-8},\ -22:-X_{-11},\ -23:-X_{-14}$
17	$3:-X_{19},\ 4:-2X_{20},\ 7:-X_{21},\ 13:2X_{23},\ 15:2X_{24},\ -2:X_{15}$ $-4:2X_{14},\ -5:-X_{13},\ -6:-X_{12},\ -8:X_{10},\ -10:-X_8$ $-12:X_6,\ -13:2X_5,\ -14:-X_4,\ -15:-2X_2,\ -19:X_{-3}$ $-20:X_{-4},\ -21:X_{-7},\ -23:-X_{-13},\ -24:-X_{-15}$
18	$2:-X_{20},\ 6:-X_{21},\ 9:-X_{22},\ 14:X_{24},\ -1:X_{16},\ -4:X_{15}$ $-7:-X_{12},\ -12:X_7,\ -15:-X_4,\ -16:-X_1,\ -20:X_{-2}$ $-21:X_{-6},\ -22:X_{-9},\ -24:-X_{-14}$
19	$4:-X_{21},\ 7:-2X_{22},\ 10:-2X_{23},\ 12:-2X_{24},\ -3:X_{17}$ $-6:X_{15},\ -7:2X_{14},\ -8:-X_{13},\ -9:-X_{12},\ -10:-2X_{11}$ $-11:X_{10},-12:2X_9,\ -13:X_8,\ -14:-X_7,\ -15:-X_6$ $-17:-X_3,\ -21:X_{-4},\ -22:X_{-7},\ -23:X_{-10},\ -24:X_{-12}$
20	$3:-X_{21},\ 9:-X_{23},\ 11:-X_{24},\ -2:X_{18},\ -4:X_{17},\ -5:-X_{16}$ $-10:-X_{12},\ -12:X_{10},\ -16:X_5,\ -17:-X_4$ $-18:-X_2,\ -21:X_{-3},\ -23:X_{-9},\ -24:X_{-11}$
21	$3:-2X_{22},\ 6:2X_{23},\ 8:2X_{24},\ -3:2X_{20},\ -4:X_{19},\ -6:2X_{18}$ $-7:X_{17},\ -8:-2X_{16},\ -10:-X_{15},\ -12:X_{13},\ -13:-X_{12}$ $-15:X_{10},\ -16:X_8,\ -17:-X_7,\ -18:-X_6$ $-19:-X_4,\ -20:-X_3,\ -22:X_{-3},\ -23:-X_{-6},\ -24:-X_{-8}$
22	$2:-X_{23},\ 5:-X_{24},\ -3:X_{21},\ -7:X_{19},\ -9:X_{18},\ -11:-X_{16}$ $-13:-X_{15},\ -15:X_{13},\ -16:X_{11},\ -18:-X_9$ $-19:-X_7,\ -21:-X_3,\ -23:X_{-2},\ -24:X_{-5}$
23	$1:-X_{24},\ -2:X_{22},\ -6:-X_{21},\ -9:X_{20},\ -10:X_{19},\ -13:-X_{17}$ $-14:-X_{16},\ -16:X_{14},\ -17:X_{13},\ -19:-X_{10}$ $-20:-X_9,\ -21:X_6,\ -22:-X_2,\ -24:X_{-1}$
24	$-1:X_{23},\ -5:X_{22},\ -8:-X_{21},\ -11:X_{20},\ -12:X_{19}\ -14:-X_{18}$ $-15:-X_{17},\ -17:X_{15},\ -18:X_{14},\ -19:-X_{12},\ -20:-X_{11}$ $-21:X_8,\ -22:-X_5,\ -23:-X_1$

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