Journal of Lie Theory
Volume **10** (2000) 111–126

Berezin transform on line bundles over bounded symmetric domains

Genkai Zhang

Communicated by B. Ørsted

Abstract. Let $D = G/K$ be a bounded symmetric domain. We consider the Berezin transform on line bundles over D. We find the symbol of the Berezin transform as a function of the G-invariant differential operators on the line bundles. We further calculate the symbol of the Berezin transform on the compact dual of D.

1. Introduction

Let D be a bounded domain in a complex space and let H be a Hilbert space of analytic functions on D with a reproducing kernel. The Wick quantization associates to each function on D an operator on H, so that roughly speaking the delta function at a point z gives to the projection onto the one-dimensional space generated by the reproducing kernel at z. Its adjoint then associates to each operator on D a function on D, called the covariant symbol of the operator. The Berezin transform, also called Berezin quantization, maps a function on D to the covariant symbol of its Wick quantization.

We consider now two Hilbert spaces H_1 and H_2 of holomorphic functions on a bounded symmetric domain $D = G/K$ and the corresponding Wick quantization, so that it associates a point $z \in D$ to the rank one operator that maps the reproducing kernel of H_1 at z to the reproducing kernel of H_2 at z. Thus we get similarly the Berezin transform. The Hilbert spaces we take will be the weighted Bergman spaces, or the holomorphic discrete series on D. The Berezin transform now can be viewed as acting on a weighted L^2-space on D, which can further be viewed as a trivialization of the L^2-space of sections of a line bundle over D. This explains our title. The group G of biholomorphic mappings of D acts on the weighted L^2-space, and the Berezin transform is invariant under the group action. Thus it is a function of the invariant differential operators. The function is also called the symbol of the Berezin transform. See the remark 1 after Theorem 4.6 and [22] for the precise formulation in the case of trivial line bundle. In this paper we will calculate this symbol.

ISSN 0949–5932 / \$2.50

We remark that the decomposition of the weighted L^2-space under G has been given by Shimeno [19]. Our result can also be interpreted as a decomposition of the Berezin transform under G.

When D is the unit ball in $\mathbb{C}^n$ the symbol has been calculated in our previous paper [15]. The result was also used to find a product formula for higher order Laplacian operators and to study the relative discrete series on the line bundle. The Berezin transform is generally an integral operator; its symbol is then the spherical transform of its integral kernel. In the case of unit ball in $\mathbb{C}^n$ the transform can be found by direct calculation, since the spherical function there can be written as hypergeometric series. The spherical function for higher rank domain is an integral over K of the generalized Harish-Chandra e-function (see (9)). However for higher rank domain no explicit formula for the spherical functions is known. One of our main contributions in this paper is an explicit formula for the analogues of the Harish-Chandra e-functions; see Proposition 4.3. The symbol is then obtained by directed computation of certain integral. This is similar to the calculation in [22]; see also [1]. The paper is organized as follows. In §2 we briefly recall the Jordan algebraic characterization of bounded symmetric domain. In §3 we introduce the Berezin transform on a bounded symmetric domain. We prove that it is bounded in certain L^p-spaces. Using the Siegel domain realization we give an explicit formula for the generalized Harish-Chandra e-functions, which are eigenfunctions of invariant differential operators with respect to the weighted action of G. The symbol of the Berezin transform is calculated in §4. Finally in §5 we consider similar problem for the Berezin transform on compact symmetric spaces. We find the corresponding symbol and give some application to tensor product decomposition of representations. We would like to thank the referee for his or her remarks on an early version of the paper.

2. Preliminaries

The general references in this section are [13] and [23].

Let D be an irreducible bounded symmetric domain in a complex n-dimensional space V. Let $Aut(D)$ be the group of all biholomorphic automorphisms of D, let $G = Aut(D)_0$ be the connected component of the identity in $Aut(D)$, and let K be the isotropy subgroup of G at the point 0. Then, as a Hermitian symmetric space, $D = G/K$. The Lie algebra $\mathfrak{g}$ of G is identified with the Lie algebra $aut(D)$ of all completely integrable holomorphic vector fields on D, equipped with the Lie product

$$[X, Y](z) := X'(z)Y(z) - Y'(z)X(z), \quad X, Y \in aut(D), \, z \in D.$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the *Cartan decomposition* of $\mathfrak{g}$ with respect to the involution $\theta(X)(z) := -X(-z)$. There exists a quadratic form $Q : V \to End(\overline{V}, V)$ (where $\overline{V}$ is the complex conjugate of V), such that $\mathfrak{p} = \{\xi_v; v \in V\}$, where $\xi_v(z) := v - Q(z)\bar{v}$. Let $\{z, v, w\}$ be the polarization of $Q(z)\bar{v}$, i.e.,

$$\{z, v, w\} = Q(z + w)\bar{v} - Q(z)\bar{v} - Q(w)\bar{v}.$$

The space V with this triple product $V \times \overline{V} \times V$, is a JB*-triple; see [23]. Define $D(z, \bar{v})w = \{z, v, w\}$. Then $D(z, \bar{v}) \in \mathrm{End}(V, V)$. The space V carries the K-*invariant inner product* $(z|w) := \frac{1}{p}\mathrm{Tr}D(z, \bar{w})$, where "Tr" is the trace functional on

$End(V)$, and $p = p(D)$ is the genus of D (see below). The subgroup K acts on D by unitary transformations. We let dz be the Lebesgue measure corresponding to the inner product. Besides the Euclidean norm, V carries also the *spectral norm*

$$\|z\| := \|\frac{1}{2}D(z,\bar{z})\|^{1/2},$$

and the domain D is realized as the open unit ball of V with respect to the spectral norm, i.e. $D = \{z \in V : \|z\| < 1\}$. Let us choose and fix a frame $\{e_j\}_{j=1}^r$ of triponents in V, where r is the rank of D. Then $e := e_1 + \cdots + e_r$ is a *maximal tripotent.* Let

$$V = \sum_{0\le j\le k\le r} \oplus V_{j,k} \tag{1}$$

be the *joint Peirce decomposition* of V associated with $\{e_j\}_{j=1}^r$, where

$$V_{j,k} = \{v \in V; D(e_l, e_l)v = (\delta_{l,j} + \delta_{l,k})v,\ 1 \le l \le r\},$$

for $(j,k) \ne (0,0)$, $V_{0,0} = \{0\}$, and $V_{j,j} = \mathbb{C}e_j$, $1 \le j \le r$. The integers

$$a := \dim V_{j,k},\ (1 \le j < k \le r);\ b := \dim V_{0,j},\ (1 \le j \le r)$$

are independent of the choice of the frame and of $1 \le j < k \le r$.

Let us define

$$V_1 = \sum_{1\le j\le k\le r} V_{j,k} \text{ and } V_{\frac{1}{2}} = \sum_{j=1}^{r} V_{0,j},$$

and let $n_1 = \dim(V_1)$, and $n_2 = \dim(V_{\frac{1}{2}})$. Then we have

$$n_1 = \frac{r(r-1)}{2}a + r,\ n_2 = rb, \quad \text{and} \quad n = n_1 + n_2.$$

The *genus* $p = p(D)$ is defined by

$$p := \frac{1}{r}\mathrm{Tr}D(e,e) = (r-1)a + b + 2.$$

Thus $(e_j|e_j) = \frac{1}{p}\mathrm{Tr}D(e_j,e_j) = \frac{1}{rp}\mathrm{Tr}D(e,e) = 1$, and this is true for every minimal tripotent in V.

Let $\mathfrak{a} = \mathbb{R}\xi_{e_1} + \cdots + \mathbb{R}\xi_{e_r}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Let $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ determined by

$$\beta_j(\xi_{e_k}) = 2\delta_{j,k},\ 1 \le j,k \le r,$$

and define an ordering on $\mathfrak{a}^*$ via

$$\beta_r > \beta_{r-1} > \cdots > \beta_1 > 0.$$

We will write an element $\underline{\lambda} \in (\mathfrak{a}^*)^{\mathbf{C}}$ as

$$\underline{\lambda} = \sum_{j=1}^{r} \lambda_j \beta_j.$$

The *positive root system* $\sigma^+(\mathfrak{g}, \mathfrak{a})$ consists of $\{\beta_j; 1 \le j \le r\}$, $\{(\beta_j \pm \beta_k)/2; 1 \le k < j \le r\}$, and $\{\beta_j/2; 1 \le j \le r\}$, with multiplicities 1, a, and $2b$ respectively. It follows that $\underline{\rho}$, the *half sum of the positive roots*, is given by

$$\underline{\rho} = \sum_{j=1}^{r} \rho_j \beta_j = \sum_{j=1}^{r} \frac{b+1+a(j-1)}{2} \beta_j. \tag{2}$$

Let $\mathfrak{n}$ be the sum of the positive roots spaces. Then we have the *Iwasawa decompositions* $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ and $G = NAK$. For $g \in G$ we let $g = n \exp(A(g)) \kappa(g)$ be its cooresponding decomposition.

We now introduce Siegel domain realization of the Hermitian symmetric space G/K and the conical functions. In that realization the A-part in the above Iwasawa decomposition and the Harish-Chandra e-function will have a rather explicit form; see (9) and (13) below.

The subspace $V_{\frac{1}{2}}$ is not a Jordan subalgebra of V, but rather a module over V_1. Precisely, the map $R : V_1 \to End(V_{\frac{1}{2}})$ defined by $R(z)w := \{z\bar{e}w\}$, $z \in V_1, w \in V_{\frac{1}{2}}$ is a monomorphism of Jordan $\star$-algebras, where the involution in $End(V_{\frac{1}{2}})$ corresponds to the K-invariant inner product $(\xi|\eta)$. Let $J = \{z \in V_1; Q(e)\bar{z} = z\}$. Then $V_1 = J + iJ$ and J is a real Jordan algebra. The *Cayley transform* γ is defined for $z = z_1 \oplus z_2 \in V$ (with $z_1 \in V_1$ and $z_2 \in V_{\frac{1}{2}}$) by

$$\gamma(z) = \frac{e + z_1}{e - z_1} \oplus R((e - z_1)^{-1}) z_2.$$

Its inverse is given for $w = w_1 + w_2$, $w_1 \in V_1$, $w_2 \in V_{\frac{1}{2}}$ by

$$\gamma^{-1}(w) = \frac{w_1 - e}{w_1 + e} \oplus 2R((w_1 + e)^{-1}) w_2.$$

Let $F : V_{\frac{1}{2}} \times V_{\frac{1}{2}} \to V_1$ be the V_1-valued Hermitian quadratic form

$$F(u, v) := \{u, v, e\}.$$

It satisfies

$$Q(x)(F(z, w)) = F(R(x)z, R(x)w), \ \forall x \in J, \forall z, w \in V_{\frac{1}{2}}.$$

Define Ω to be the cone of positive elements in J, $\Omega = \{x^2; x \in J, \det(x) \neq 0\}$. Then Ω is a symmetric convex cone. The bilinear map F is then positive with respect to Ω, i.e., $F(u, u) \in \overline{\Omega}$ for all $u \in V_{\frac{1}{2}}$. Define for $z = z_1 + z_2, w = w_1 + w_2 \in V$ (where z_1, $w_1 \in V_1$, and $z_2, w_2 \in V_{\frac{1}{2}}$)

$$\tau(z, w) := \frac{z_1 + w_1^\star}{2} - F(z_2, w_2) \quad \text{and} \quad \tau(z) := \tau(z, z) = \Re z_1 - F(z_2, z_2), \tag{3}$$

where $\Re z_1 = \frac{z_1 + z_1^\star}{2}$ is the real part of z_1 with respect to the spliting $V_1 = J + iJ$. The Cayley transform γ is then a biholomorphic transformation from D into the *Siegel domain*

$$T(\Omega) = T(\Omega, F) := \{w \in V; \tau(w) \in \Omega\}.$$

Finally we introduce the conical functions and the Gindikin-Koecher Gamma function, see [8]. Let $\{e_j\}_{j=1}^r$ be the fixed frame, put $u_j := \sum_{k=1}^j e_k$, $j = 1, \cdots, r$. Let $U_j := \{z \in V; D(u_j, u_j)z = z\}$. Then U_j is a Jordan $\star$-subalgebra of V_1 with a determinant polynomial Δ_j. We extend Δ_j to all of V via $\Delta_j(z) := \Delta_j(P_{U_j}(z))$, where P_{U_j} is the orthogonal projection onto U_j. The polynomials Δ_j are called (principal) *minors*. Notice that $\Delta_r(w) = \Delta(w) = \det(w)$. For any $\underline{\alpha} = (\alpha_1, \cdots, \alpha_r) \in \mathbb{C}^r$ consider the associated *conical function*

$$\Delta_{\underline{\alpha}}(w) := \Delta_1^{\alpha_1-\alpha_2}(w)\Delta_2^{\alpha_2-\alpha_3}(w)\cdots\Delta_r^{\alpha_r}(w),\ w \in V.$$

Notice that if $w = \sum_{j=1}^r w_j e_j$ then $\Delta_{\underline{\alpha}}(w) = \prod_{j=1}^r w_j^{\alpha_j}$. Thus the conical functions are generalizations of the power functions. The *Gindikin-Koecher Gamma function* associated with the convex, symmetric cone Ω is defined by

$$\Gamma_\Omega(\underline{\alpha}) := \int_\Omega e^{-tr(x)}\Delta_{\underline{\alpha}}(x)\Delta(x)^{-\frac{n_1}{r}}dx,$$

where the integral converges if and only if $\Re(\alpha_j) > (j-1)a/2$ for $j = 1, 2, \cdots, r$. Moreover, the convergence is absolute, and uniform on compact sets of $\underline{\alpha}$. We remark that $\Delta(x)^{-\frac{n_1}{r}}dx$ is the $Aut(\Omega)$-invariant measure on Ω. This allows one to get for $w \in \Omega + iJ$

$$\int_\Omega e^{-(w|x)}\Delta_{\underline{\alpha}}(x)\Delta(x)^{-\frac{n_1}{r}}dx = \Gamma_\Omega(\underline{\alpha})\Delta_{\underline{\alpha}}(w^{-1}) = \Gamma_\Omega(\underline{\alpha})\Delta^\star_{\underline{\alpha}^\star}(w),$$

where for $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_r) \in \mathbb{C}^r$ we denote $\underline{\alpha}^\star := (-\alpha_r, -\alpha_{r-1}, \cdots, -\alpha_1)$, and $\Delta^\star_{\underline{\beta}}$ is the conical function with respect to the frame $\{e_r, e_{r-1}, e_{r-2}, \cdots, e_1\}$. Moreover, the Gamma function Γ_Ω is expressed in terms of the ordinary Gamma function

$$\Gamma_\Omega(\underline{\alpha}) = (2\pi)^{(n_1-r)/2}\prod_{j=1}^r \Gamma(\alpha_j - (j-1)a/2).$$

Thus Γ_Ω extends to a meromorphic function in all of $\mathbb{C}^r$. We also adopt the notation $\Gamma_\Omega(\lambda) = \Gamma_\Omega(\lambda, \lambda, \cdots, \lambda)$.

3. Weighted L^2-space on bounded symmetric domains and tensor products of Bergman spaces

Let $h(z)$ be the unique K-invariant polynomial on V whose restriction to $\mathbb{R}e_1 + \cdots + \mathbb{R}e_r$ is given by

$$h(\sum_{j=1}^r a_j e_j) = \prod_{j=1}^r (1 - a_j^2).$$

Let $h(z, w)$ be the sesqui-holomorphic extension of $h(z)$ to $V \times V$, i.e.

$$h(z, w) = \exp\sum_{j=1}^n z_j\frac{\partial}{\partial\xi_j}\exp\sum_{j=1}^n \overline{w}_j\frac{\partial}{\partial\overline{\xi}_j}h(\xi)|_{\xi=0}.$$

Denote

$$d\mu_\sigma(z) = C_\sigma h^{\sigma-p}(z)dm(z)$$

where

$$C_\sigma = \frac{1}{\pi^n}\frac{\Gamma_\Omega(\sigma)}{\Gamma_\Omega(\sigma-\frac{n}{r})}$$

is a normalization constant. We let $L^2(D,\mu_\sigma)$ be the corresponding L^2-space of functions on D. The group G acts unitarily on $L^2(D,\mu_\sigma)$ via the following

$$U_g^{(\sigma)} f(z) = J_{g^{-1}}(z)^{\frac{\sigma}{p}} f(g^{-1}z), \tag{4}$$

where $J_{g^{-1}}$ is the complex Jacobian of g^{-1}. We will hereafter drop the supscript σ. To avoid some technical difficulty involving the universal covering group of G we will assume throughout this paper that σ is an integer.

Notice that the restriction of the Jacobian to K gives rise to a character of K:

$$\tau_\sigma(k) := J_k(0)^{\frac{\sigma}{p}} = J_k(z)^{\frac{\sigma}{p}}, \qquad k\in K, \tag{5}$$

since K acts on D by linear transformation. The space $L^2(D,\mu_\sigma)$ is a trivialization of the L^2-space of sections of the homogeneous line bundle over D induced from the one-dimensional character of K.

Let $\sigma > p-1$ and let $A^{\sigma,2}$ be the weighted Bergman space of holomorphic functions in $L^2(D,\mu_\sigma)$. It is invariant under the action of G. Let $\nu > p-1$. We consider the Bergman spaces $A^{\nu,2}$, $A^{\nu+\sigma,2}$ and their tensor product $A^{\nu+\sigma,2}\otimes\overline{A^{\nu,2}}$. Here $\overline{A^{\nu,2}}$ is space consisting of the complex conjugates of the functions in $A^{\nu,2}$ and $A^{\nu+\sigma,2}\otimes\overline{A^{\nu,2}}$ is realized as the space of functions $F(z,w)$ holomorphic in z and anti-holomorphic in w that are in $L^2(D\times D,\mu_{\nu+\sigma}\times\mu_\nu)$, with the corresponding G-actions. It is proved in [15] that the operator $R: A^{\nu+\sigma,2}\otimes\overline{A^{\nu,2}}\to L^2(D,\mu_\sigma)$:

$$RF(z) = F(z,z)h(z,z)^\nu$$

is a bounded G-intertwining operator. The operator R^*, the Hilbert space conjugate of R, is then the Wick quantization operator, see [22], [25]. The Berezin transform is defined by

$$B_{\nu,\sigma} = \frac{C_{\nu+\sigma}}{C_\sigma} RR^* \tag{6}$$

normalized so that $B_{\nu,\sigma}1 = 1$. Explicitly it is given by

$$B_{\nu,\sigma} f(z) = \frac{C_{\nu+\sigma}}{C_\sigma}\int_D f(w)\frac{h(z,z)^\nu h(w,w)^\nu}{|h(z,w)^\nu|^2 h(z,w)^\sigma} d\mu_\sigma(w). \tag{7}$$

The Berezin transform is G-invariant:

$$U_g B_{\nu,\sigma} = B_{\nu,\sigma} U_g. \tag{8}$$

We study first the boundedness property of B_σ.

Proposition 3.1. *Suppose $\nu,\sigma > p-1$. Then $B_{\nu,\sigma}$ is a bounded operator on $L^p(D,d\mu_\sigma)$ for $1\le p\le\infty$.*

Proof. We prove first that $B_{\nu,\sigma}$ is a bounded operator on $L^1(D, d\mu_\sigma)$. Let $f \in L^1 = L^1(D, d\mu_\sigma)$. Then

$$\|B_{\nu,\sigma}f\|_{L^1} \leq C_{\nu,\sigma}\int_D \left|\int_D f(w)\frac{h(z,z)^\nu h(w,w)^\nu}{|h(z,w)^\nu|^2 h(z,w)^\sigma}d\mu_\sigma(w)\right| d\mu_\sigma(z)$$
$$\leq C_{\nu,\sigma}\int_D |f(w)|h(w,w)^\nu \left(\int_D \frac{h(z,z)^\nu}{|h(z,w)^\nu|^2|h(z,w)^\sigma|}d\mu_\sigma(z)\right) d\mu_\sigma(w)$$

Now by the Faraut-Koranyi's generalization [7] of the Forelli-Rudin inequality, the inner integral is, up to a constant, dominanted by

$$h(w,w)^{-\nu},$$

Therefore, we obtain

$$\|B_{\nu,\sigma}f\|_{L^1} \leq C\int_D |f(w)|d\mu_\sigma(w) = C\|f\|_{L^1},$$

and $B_{\nu,\sigma}$ is bounded on L^1. Since $B_{\nu,\sigma}$ is a formally self-adjoint operator it is bounded on $(L^1)^* = L^\infty$. Our claim now follows from interpolation. ∎

Remark 3.2. The above proposition is still valid if the condition on ν and σ and if the measure $d\mu_\sigma$ and $d\mu_\nu$ are not normalized. (Note that the normalizing constant C_σ has sigularity when $\sigma \leq \frac{n}{r}$.) The same applies to Theorem 4.6 in the next section. We thank the referee for pointing out this to us.

For $g \in G$ let $g = n\exp(A(g))k(g)$ be its Iwasawa decomposition in $G = NAK$, with $n \in N, \exp(A(g)) \in A$, and $k(g) \in K$. We define the following generalized Harish-Chandra e-function on D

$$e_{\underline{\lambda},\sigma}(z) = e^{(\underline{\lambda}+\underline{\rho})(A(g))}\tau_\sigma(\kappa(g))(J_g(0))^{-\frac{\sigma}{p}}, z = g\cdot 0 \quad \text{and} \quad g \in G, \tag{9}$$

where τ_σ is defined in (5). It is clear that the right hand side is independent of the representative $g \in G$, that is, the function is well-defined. (Notice that in our formula we use $\underline{\lambda}+\underline{\rho}$ instead of $i\underline{\lambda}+\underline{\rho}$ as in the standard formula for Harish-Chandra e-function.) We note that it transforms under NA-group according to

$$e_{\underline{\lambda},\sigma}(gz) = e^{(\underline{\lambda}+\underline{\rho})A(g)}(J_g(z))^{-\frac{\sigma}{p}}e_{\underline{\lambda},\sigma}(z), \qquad g \in NA. \tag{10}$$

When $\sigma = 0$ this is the Harish-Chandra e-function. Denote $\mathcal{D}_\sigma(D)$ the algebra of invariant differential operators on $C^\infty(D)$ with respect to the G-action (4). It is commutative; see [21] and [20].

Theorem 3.3. *The functions $e_{\underline{\lambda},\sigma}$ defined in (9) are eigenfunctions of all invariant differential operators $\mathcal{D}_\sigma(D)$.*

Proof. We use the Helgason's idea and prove the following functional equality for $f(z) = e_{\underline{\lambda},\sigma}(z)$,

$$\int_K (U_{x^{-1}}f)(kz)J_{x^{-1}}(0)^{-\frac{\sigma}{p}}dk = f(x\cdot 0)\int_K (U_k^{-1}f)(z)J_{k^{-1}}(0)^{-\frac{\sigma}{p}}dk, \qquad x \in G. \tag{11}$$

(See also [11], Chapter IV, Proposition 2.4.) Write $z = y \cdot 0$, $y \in G$. Then

$$(U_x f)(kz) = f(xky \cdot 0) J_x(kz)^{\frac{\sigma}{p}} = e^{(\underline{\lambda}+\underline{\rho})A(xky)} \tau_\sigma(\kappa(xky)) J_{xky}(0)^{-\frac{\sigma}{p}} J_x(kz)^{\frac{\sigma}{p}}.$$

Let $x = n \exp(A(x))\kappa(x)$ be the NAK-decomposition of x and let $k_1 = \kappa(x)k$. We rewrite the first two factors. We have

$$A(xky) = A(x) + A(\kappa(x)ky) = A(x) + A(k_1 y)$$

since A normalizes N, and

$$\kappa(xky) = \kappa(\kappa(x)ky) = \kappa(k_1 y).$$

The last two factors can be written as, using $k = \kappa(x)^{-1}k_1$,

$$\begin{aligned} J_{xky}(0)^{-\frac{\sigma}{p}} J_x(kz)^{\frac{\sigma}{p}} &= J_{ky}(0)^{-\frac{\sigma}{p}} \\ &= J_{\kappa(x)^{-1}}(k_1 y \cdot 0)^{-\frac{\sigma}{p}} J_{k_1 y}(0)^{-\frac{\sigma}{p}} \\ &= \tau_\sigma(\kappa(x)) J_{k_1 y}(0)^{-\frac{\sigma}{p}}. \end{aligned}$$

Thus

$$(U_x f)(kz) = e^{(\underline{\lambda}+\underline{\rho})A(x)} e^{(\underline{\lambda}+\underline{\rho})A(k_1 y)} \tau_\sigma(\kappa(x)) J_{k_1 y}(0)^{-\frac{\sigma}{p}} \tau_\sigma(\kappa(k_1 y)).$$

Performing the change of variables $k = \kappa(x)^{-1}k_1$, the integral on the left hand side of (11) is

$$\begin{aligned} & e^{(\underline{\lambda}+\underline{\rho})A(x)} \tau_\sigma(\kappa(x)) J_x(0)^{-\frac{\sigma}{p}} \int_K e^{(\underline{\lambda}+\underline{\rho})A(k_1 y)} \tau_\sigma(\kappa(k_1 y)) J_{k_1 y}(0)^{-\frac{\sigma}{p}} dk_1 \\ &= f(x \cdot 0) \int_K f(k_1 z) dk_1 \qquad (12) \\ &= f(x \cdot 0) \int_K (U_{k_1} f)(z) J_{k_1}(0)^{-\frac{\sigma}{p}} dk_1, \end{aligned}$$

i.e., the right hand side of (11). Now for any differential operator L in $\mathcal{D}_\sigma(D)$ we let L act on the equality (11) with respect to the variable z. Using the invariance we get

$$\begin{aligned} & \int_K (U_{x^{-1}} Lf)(kz) J_{x^{-1}}(0)^{-\frac{\sigma}{p}} dk \\ &= f(x \cdot 0) \int_K (U_{k^{-1}} Lf)(z) J_{k^{-1}}(0)^{-\frac{\sigma}{p}} dk. \end{aligned}$$

Letting $z = 0$ we get

$$Lf(x \cdot 0) = Lf(0) f(x \cdot 0);$$

that is, f is an eigenfunction of L with eigenvalue $Lf(0)$. ∎

4. Berezin transform on bounded symmetric domains

The next result follows from the invariance of the Berezin transform (8) and the transformation formula of $e_{\underline{\lambda},\sigma}$. We omit the proof; see [1], Lemma 1.1 for the case $\sigma = 0$.

Proposition 4.1. *Suppose $e_{\underline{\lambda},\sigma} \in L^1(D, h^\nu d\mu_\sigma)$ then it is an eigenfunction of $B_{\nu,\sigma}$ and the eigenvalue is $B_{\nu,\sigma} e_{\underline{\lambda},\sigma}(0)$*

We note that, by (7),

$$B_{\nu,\sigma} e_{\underline{\lambda},\sigma}(0) = \frac{C_{\nu+\sigma}}{C_\sigma} \int_D e_{\underline{\lambda},\sigma}(w) h^\nu(w) d\mu_\sigma(w).$$

Thus $e_{\underline{\lambda},\sigma} \in L^1(D, h^\nu d\mu_\sigma)$ if and only if the integral $B_{\nu,\sigma} e_{\underline{\lambda},\sigma}(0)$ is absolutely convergent. The eigenvalue $B_{\nu,\sigma} e_{\underline{\lambda},\sigma}(0)$ is then the symbol of the Berezin transform. We have charaterized in [1] those $e_{\underline{\lambda},\sigma}$ in $L^1(D, d\mu_\sigma)$. To find those $\underline{\lambda}$ for which $e_{\underline{\lambda},\sigma} \in L^1(D, h^\nu d\mu_\sigma)$ and to calculate the symbol we will use the Siegel domain realization of D. Recall that the Cayley transform γ then maps D biholomorphically onto the Siegel domain $T(\Omega)$. The group $\gamma G \gamma^{-1}$ then acts on $T(\Omega)$. With some abuse of notation we write this group also by G and its corresponding Iwasawa decomposition by NAK. One can similarly formulate the Berezin transform in terms of the Bergman spaces on the Siegel domain. We give now the corresponding formulas. Recall also the definition of τ in (3). The measure μ_σ on D now corresponds to

$$d\widetilde{\mu_\sigma}(w) = \tilde{C}_\sigma \Delta(\tau(w))^{\sigma-p} dw,$$

on $T(\Omega)$, where $\tilde{C}_\sigma = 4^{n-rp} C_\sigma$. The group G acts on $L^2(T(\Omega), \widetilde{\mu_\sigma})$ by the same formula (4). Using the transformation formulas

$$\det(\gamma^{-1})'(\xi) = 2^n \Delta(e+\xi_1)^{-p}$$

and

$$h(\gamma^{-1}(\xi), \gamma^{-1}(\eta)) = 4^r \Delta(e+\xi_1)^{-1} \Delta(\tau(\xi,\eta)) \overline{\Delta(e+\xi_1)^{-1}}.$$

we can now prove the following

Lemma 4.2. *The operator*

$$Uf(w) = f(\gamma^{-1}(w)) J_{\gamma^{-1}}(w)^{\frac{\sigma}{p}}$$

is a unitary intertwining operator from $L^2(D, \mu_\sigma)$ onto $L^2(T(\Omega), \widetilde{\mu_\sigma})$

The Berezin transform on $T(\Omega)$ is formally $UB_{\nu,\sigma}U^{-1}$ and it takes the form

$$\widetilde{B_{\nu,\sigma}} f(z) = \frac{C_{\nu+\sigma}}{C_\sigma} \int_{T(\Omega)} f(w) \left(\frac{\Delta(\tau(z))\Delta(\tau(w))}{|\Delta(\tau(z,w))|^2} \right)^\nu \frac{1}{\Delta(\tau(z,w))^\sigma} d\widetilde{\mu_\sigma}(w).$$

Let $\widetilde{e_{\underline{\lambda},\sigma}}(w) = U e_{\underline{\lambda},\sigma}(w)$. Thus by Proposition 4.1 $\widetilde{e_{\underline{\lambda},\sigma}}$ is an eigenfunction of $\widetilde{B_{\nu,\sigma}}$ if the integral $B_{\nu,\sigma} e_{\underline{\lambda},\sigma}(0) = \widetilde{B_{\nu,\sigma}} \widetilde{e_{\underline{\lambda},\sigma}}(e)$ is convergent; the later is

$$\frac{C_{\nu+\sigma}}{C_\sigma} \int_{T(\Omega)} \widetilde{e_{\underline{\lambda},\sigma}}(w) \frac{\Delta(\tau(w))^\nu}{|\Delta(\tau(e,w))|^{2\nu}} \frac{1}{\Delta(\tau(e,w))^\sigma} d\widetilde{\mu_\sigma}(w).$$

We will evaluate the integral and at the same time find those $\underline{\lambda}$ for which the integral is absolutely convergent. For that purpose we will first find an explict formula for the function $\widetilde{e_{\underline{\lambda},\sigma}}$. On the Siegel domain $T(\Omega)$, where $w=e$ instead of $z=0$ is the base point, the formula (9) now takes the form

$$\widetilde{e_{\underline{\lambda},\sigma}}(w)=e^{(\underline{\lambda}+\underline{\rho})A(g)}\tau_\sigma(\kappa(g))J_g(e)^{-\frac{\sigma}{p}},\qquad w=g\cdot e,\qquad g\in G.$$

The first factor is

$$e^{(\underline{\lambda}+\underline{\rho})A(g)}=\Delta_{\underline{\lambda}+\underline{\rho}}(\tau(w)),$$

see [22]. We take $g\in NA$ such that $g\cdot e=w$, so that $\kappa(g)=1$ and $\tau_\sigma(\kappa(g))=1$. Recall the transformation formula of the function $\Delta(\tau(v))^{-\sigma}$

$$\Delta(\tau(g\cdot v))^{-\sigma}=\Delta(\tau(v))^{-\sigma}|J_g(e)^{-\frac{\sigma}{p}}|^2;$$

see [22]. In particular taking $v=e$ we find

$$\Delta(\tau(w))^{-\sigma}=|J_g(e)^{-\frac{\sigma}{p}}|^2.$$

However for $g\in NA$ the Jacobian J_g is positive since NA acts on $T(\Omega)$ by translation and multiplication by elements in Ω; see e.g. [22]. Thus

$$J_g(e)^{-\frac{\sigma}{p}}=\Delta(\tau(w))^{-\frac{\sigma}{2}}.$$

Consequently we obtain

Proposition 4.3. *In the Siegel domain realization the function $e_{\underline{\lambda},\sigma}$ is given by*

$$\widetilde{e_{\underline{\lambda},\sigma}}(w)=\Delta_{\underline{\lambda}+\underline{\rho}}(w)\Delta(\tau(w))^{-\frac{\sigma}{2}}.\tag{13}$$

Theorem 4.4. *Let $\nu,\sigma>p-1$. Suppose $\underline{\lambda}\in(\mathfrak{a}^*)^{\mathbf{C}}$ satisfies*

$$|\Re(\lambda_j)|<\nu+\frac{\sigma}{2}-\frac{p-1}{2}.$$

Then $e_{\underline{\lambda},\sigma}$ (and $\widetilde{e_{\underline{\lambda},\sigma}}$) is an eigenfunction of $B_{\nu,\sigma}$ (respectively $\widetilde{B_{\nu,\sigma}}$) with eigenvalue

$$\frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-\frac{n}{r})\Gamma_\Omega(\underline{\lambda}^*+\underline{\rho}^*+\nu+\frac{\sigma}{2})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu+\sigma-\frac{n}{r})}.\tag{14}$$

Proof. We calculate $\widetilde{B_{\nu,\sigma}}\widetilde{e_{\underline{\lambda},\sigma}}(e)$. Disregarding the constant $4^{n-rp}C_{\nu+\sigma}$, it is

$$\begin{aligned}&\int_{T(\Omega)}\Delta_{\underline{\lambda}+\underline{\rho}}(\tau(w))\Delta(\tau(w))^{-\frac{\sigma}{2}}\frac{\Delta(\tau(w))^{\nu+\sigma-p}}{|\Delta(\tau(e,w))|^{2\nu}\Delta(\tau(e,w))^{\sigma}}dw\\&=\int_{T(\Omega)}\Delta_{\underline{\lambda}+\underline{\rho}}(\tau(w))\Delta(\tau(w))^{-\frac{\sigma}{2}}\frac{\Delta(\tau(w))^{\nu+\sigma-p}}{\Delta(\tau(ie,w))^{\nu+\sigma}\overline{\Delta(\tau(ie,w))^{\nu}}}dw.\end{aligned}$$

Now $w\in T(\Omega)$ can be written as $w=w_1+w_2\in V=V_1+V_{\frac{1}{2}}$, with $w_1=\xi+i\eta\in V_1=J+iJ$, and $y=\xi-F(w_2,w_2)\in\Omega$, and $dw=dy\,d\eta\,dw_2$, where dw, dy, $d\eta$ and dw_2 are the Lebesgue measure on V, Ω, J and $V_{\frac{1}{2}}$ respectively. The above integral is

$$\begin{aligned}&2^{2r\nu+r\sigma}\int_\Omega dy\Delta_{\underline{\lambda}+\underline{\rho}}(y)\Delta(y)^{\nu+\frac{\sigma}{2}-p}\int_{V_{1/2}}dw_2\\&\times\int_J\Delta(e+y+F(w_2,w_2)-i\eta)^{-\nu-\sigma}\overline{\Delta(e+y+F(w_2,w_2)-i\eta)^{-\nu}}d\eta.\end{aligned}\tag{15}$$

We change the variable in the inner integral, $\eta = Q((y+e+F(w_2,w_2))^{1/2})(t)$. Then $\Delta(Q(y)(t)) = \Delta(y)^2\Delta(t)$ and $\det(Q(y)) = \Delta(y)^{\frac{2n_1}{r}}$, for $t, y \in V_1$ (see [8], Chapter III). The inner integral is

$$\Delta(y+e+F(w_2,w_2))^{\frac{n_1}{r}-2\nu}\int_J \Delta(e-it)^{-\nu-\sigma}\overline{\Delta(e-it)^{-\nu}}dt.$$

Since $\Delta(e+it)^{-\nu}$ is the Fourier transform of

$$\frac{1}{\Gamma_\Omega(\nu)}e^{-tr(x)}\Delta(x)^{\nu-\frac{n_1}{r}}\chi_\Omega(x),$$

where χ_Ω is the characteristic function of Ω, we get by Parseval's formula

$$\begin{aligned}
&\int_J \frac{1}{\Delta(e-it)^{\nu+\sigma}\overline{\Delta(e-it)^{\nu}}}dt \\
&= \frac{(2\pi)^{n_1}}{\Gamma_\Omega(\nu+\sigma)\Gamma_\Omega(\nu)}\int_\Omega e^{-2tr(x)}\Delta(x)^{2\nu+\sigma-\frac{2n_1}{r}}dx \\
&= \frac{\pi^{n_1}}{2^{2r\nu+r\sigma-2n_1}}\frac{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}{\Gamma_\Omega(\nu+\sigma)\Gamma_\Omega(\nu)};
\end{aligned}$$

see [8], Chapter VII. Thus the integral (15) is

$$\begin{aligned}
&\pi^{n_1}2^{2n_1}\frac{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}{\Gamma_\Omega(\nu+\sigma)\Gamma_\Omega(\nu)} \\
&\times\int_\Omega \Delta_{\underline{\lambda}+\underline{\rho}}(y)\Delta(y)^{\nu+\frac{\sigma}{2}-p}dy\left(\int_{V_{\frac{1}{2}}}\Delta(y+e+F(w_2,w_2))^{\frac{n_1}{r}-2\nu-\sigma}dw_2\right).
\end{aligned}\tag{16}$$

The rest of the calculation is similar to that in [22] and [1]. We write

$$\begin{aligned}
&\Delta(y+e+F(w_2,w_2))^{\frac{n_1}{r}-2\nu-\sigma} \\
&= \frac{1}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\int_\Omega e^{-(y+e+F(w_2,w_2)|x)}\Delta(x)^{2\nu+\sigma-\frac{2n_1}{r}}dx.
\end{aligned}\tag{17}$$

Therefore, using Fubini's theorem, the integral (16), disregarding the constant in front of the integral, is

$$\begin{aligned}
&\frac{1}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\int_\Omega e^{-tr(x)}\Delta(x)^{2\nu+\sigma-\frac{2n_1}{r}}dx \\
&\times\left(\int_\Omega e^{-(y|x)}\Delta_{\underline{\beta}+\nu-p}(y)dy\left(\int_{V_{\frac{1}{2}}}e^{-(F(w_2,w_2)|x)}dw_2\right)\right).
\end{aligned}\tag{18}$$

Now,

$$(F(w_2,w_2)|x) = (Q(x^{1/2})F(w_2,w_2)|e) = (F(R(x^{1/2})w_2, R(x^{1/2})w_2)|e).$$

Since for $x\in\Omega$, $\det_{\mathbb{R}}(R(x^{-1/2})) = \Delta(x)^{-\frac{n_2}{r}}$, we get

$$\int_{V_{\frac{1}{2}}} e^{-(F(w_2,w_2)|x)}dw_2 = \pi^{n_2}\Delta(x)^{-\frac{n_2}{r}}.\tag{19}$$

Thus, (18) is

$$\frac{\pi^{n_2}}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\int_\Omega e^{-tr(x)}\Delta(x)^{2\nu+\sigma-p}dx\left(\int_\Omega e^{-(y|x)}\Delta_{\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-p}(y)dy\right). \quad (20)$$

The inner integral is absolutly convergent if and only if

$$\Re(\lambda_j)+\rho_j > -\nu-\frac{\sigma}{2}+\frac{n}{r}+\frac{a}{2}(j-1), \quad j=1,2,\cdots,r, \quad (21)$$

and its value is

$$\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-\frac{n}{r})\Delta^\star_{\underline{\lambda}^\star+\underline{\rho}^\star-\nu-\frac{\sigma}{2}+\frac{n}{r}}(x). \quad (22)$$

Therefore, (20) is

$$\pi^{n_2}\frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-\frac{n}{r})}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\int_\Omega e^{-tr(x)}\Delta^\star_{\underline{\lambda}^\star+\underline{\rho}^\star-\nu-\frac{\sigma}{2}+\frac{n}{r}}(x)\Delta(x)^{2\nu+\sigma-p}(x)dx$$
$$=\frac{\pi^{n_2}}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\int_\Omega e^{-tr(x)}\Delta^\star_{\underline{\lambda}^\star+\underline{\rho}^\star+\nu+\frac{\sigma}{2}-\frac{n_1}{r}}(x)dx.$$

This integral is absolutely convergent if and only if

$$\Re(\lambda_j^\star)+\underline{\rho}_j^\star+\nu+\frac{\sigma}{2} > \frac{a}{2}(j-1), \quad ,j=1,\cdots,r \quad (23)$$

and in that case its value is

$$\Gamma_\Omega(\underline{\lambda}^\star+\underline{\rho}^\star+\nu+\frac{\sigma}{2})$$

The inequalities (21) and (23) are equivalent to the condition in our Theorem, after a simplication. Taking into the account of the disregarded constants, we find now that the integral $B_{\nu,\sigma}e_{\underline{\lambda},\sigma}(e)$ is

$$4^{n-rp}C_{\nu+\sigma}2^{2n_1}\pi^{n_1+n_2}\frac{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}{\Gamma_\Omega(\nu+\sigma)\Gamma_\Omega(\nu)}\frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-\frac{n}{r})}{\Gamma_\Omega(2\nu+\sigma-\frac{n_1}{r})}\Gamma_\Omega(\underline{\lambda}^\star+\underline{\rho}^\star+\nu+\frac{\sigma}{2})$$
$$=\frac{\Gamma_\Omega(\underline{\lambda}+\underline{\rho}+\nu+\frac{\sigma}{2}-\frac{n}{r})\Gamma_\Omega(\underline{\lambda}^\star+\underline{\rho}^\star+\nu+\frac{\sigma}{2})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu+\sigma-\frac{n}{r})}.$$

This proves the theorem ■

Denote $b_{\nu,\sigma}(\underline{\lambda})$ the eigenvalue (14) of $B_{\nu,\sigma}$. Note that the function $b_{\nu,\sigma}(\underline{\lambda})$ can be written as

$$b_{\nu,\sigma}(\underline{\lambda})=\prod_{j=1}^r\frac{\Gamma(\nu+\frac{\sigma}{2}-\frac{p-1}{2}+\lambda_j)\Gamma(\nu+\frac{\sigma}{2}-\frac{p-1}{2}-\lambda_j)}{\Gamma(\nu-\frac{p-1}{2}+\rho_j)\Gamma(\nu+\sigma-\frac{p-1}{2}-\rho_j)}$$

using the usual Gamma-function. This should be compared with the case $\sigma=0$ in [22]. We define as usual the spherical function corresponding to character τ_σ,

$$\phi_{\underline{\lambda},\sigma}(z)=\int_K e_{\underline{\lambda},\sigma}(kz)dk.$$

Corollary 4.5. *Let ν,σ and $\underline{\lambda}\in(\mathfrak{a}^*)^{\mathbf{C}}$ as in Theorem 4.4. Then $\phi_{\underline{\lambda},\sigma}$ is an eigenfunction of $B_{\nu,\sigma}$ with eigenvalue $b_{\nu,\sigma}(\underline{\lambda})$*

We can then reformulate the previous calculation in terms of operator calculus.

Theorem 4.6. *Let $\nu, \sigma > p-1$. The Berezin operator $B_{\nu,\sigma}$ acts on $L^2(D, \mu_\sigma)$ as a bounded operator and its spectral symbol is $b_{\nu,\sigma}(\underline{\lambda})$*

Proof. The boundedness of $B_{\nu,\sigma}$ on $L^2(D, \mu_\sigma)$ is proved in Proposition 3.1. Now under the G-action U^σ the space $L^2(D, \mu_\sigma)$ is decomposed into a direct sum of integrals of irreducible representations of G and relative discrete series, see [19]. The representations appearing in the decomposition are generated by certain $\phi_{\underline{\lambda}}$; the corresponding parameter $\underline{\lambda} = \sum_{j=1}^r \lambda_j \beta_j$ satisfies

$$0 < \lambda_j < \frac{\sigma - b - 1}{2}.$$

See [19] Theorem 6.7. (Note that our λ_j are $2\lambda_j$ there, and the our σ is their $-l$; see also [5] for the identification of the space $L^2(D, \mu_\sigma)$ with the representation space of G induced from an one-dimensional representation of K studied in [19].) Now by Corollary 4.5 we see that for those $\underline{\lambda}$ the function $\phi_{\underline{\lambda},\sigma}$ is an eigenfunction of $B_{\nu,\sigma}$ with the corresponding eigenvalue $b_{\nu,\sigma}(\underline{\lambda})$. ■

Remark 4.7. The function $b_{\nu,\sigma}(\underline{\lambda})$ is a Weyl group invariant function of $\underline{\lambda}$. Now a basis of the G-invariant differential operators on $L^2(D, \mu_\sigma)$ can be chosen so that their symbols are the fundamental symmetric polynomials in $\underline{\lambda}$. In this way $B_{\nu,\sigma}$ can be viewed a function of the invariant differential operators. See [21] and [20] for the study of invariant differential operators on line bundle over D.

Remark 4.8. When $\sigma = 0$, namely the case of trivial line bundle, the above result is proved in [22].

Remark 4.9. When the domain D is the unit ball in $\mathbb{C}^n$ the above result is proved in [16] by direct integration; in that case the spherical function is the classical hypergeometric function; see also [12].

Remark 4.10. Similar to The above proposition is still valid if the condition on ν and σ and if the measure $d\mu_\sigma$ and $d\mu_\nu$ are not normalized. (Note that the normalizing constant C_σ has sigularity when $\sigma \leq \frac{n}{r}$.)

5. Berezin transform on line bundle over compact Hermitian symmetric spaces

We consider the Berezin transform on the compact Hermitian symmetric spaces. In [25] the symbol of the Berezin transform is obtained for the compact case from the non-compact case. We can apply the same method to derive the following results, but omit their proofs. We refer to [25] for the exact formulation. Let $X = G^*/K$ be the compact dual Hermitian symmetric space of the domain $D = G/K$. The vector space V can be realized as a dense subspace of X. For σ a positive integer we consider the space $L^2_\sigma(V)$ of L^2-functions on V with respect to the measure

$$c_\sigma h(z, -z)^{-\sigma - p} dm(z),$$

where
$$c_\sigma = \frac{\Gamma_\Omega(\sigma+p)}{\Gamma_\Omega(\sigma+p-n/r)}.$$
The group G^* acts on $L^2_\sigma(V)$ via the following
$$U_g^{(-\sigma)} f(z) = J_{g^{-1}}(z)^{-\frac{\sigma}{p}} f(g^{-1}z).$$
$L^2_\sigma(V)$ can also be realized as a space of sections of a line bundle over X.

The representations of G^* that appear in $L^2_\sigma(V)$ have been described in [18], which we recall below.

Extend the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ to a Cartan subalgebra $\mathfrak{a}+\mathfrak{q}$ of $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$. Let $iZ \in \mathfrak{q}$ be in the center of $\mathfrak{k}$ chosen as in [18]. Let V^Λ be an irreducible representation of G^* with highest weight Λ. If V^Λ appears in $L^2_\sigma(V)$ then the restriction of Λ on $\mathfrak{q}$ is σZ^*, where Z^* is the dual of Z. Suppose now Λ satisfies this condition. Let
$$\underline{\mathbf{m}} = \underline{\mathbf{m}}(\Lambda) = m_1\beta_1 + \cdots + m_r\beta_r$$
be its restriction to $\mathfrak{a}$. Then Λ appears in $L^2_\sigma(V)$ if and only if the following conditions hold:
$$m_1 - \frac{\sigma}{2} \in \mathbf{N}$$
and
$$m_j - m_i \in 2\mathbf{N}, \quad 1 \le i < j \le r.$$
Here $\mathbf{N}$ is the set of nonnegative integers. Define similarly the Berezin transform $B_{\nu,\sigma}$ on $L^2_\sigma(V)$, with the weighted Bergman spaces $A^{\nu+\sigma,2}$ and $A^{\nu,2}$ being replaced by the subspaces $A^{\nu+\sigma,2}(X)$ and $A^{\nu,2}(X)$ of analytic functions in $L^2_{\nu+\sigma}(V)$ and $L^2_\nu(V)$, respectively. (Actually they contain holomorphic polynomials on V.)

Theorem 5.1. *The Berezin operator $B_{\nu,\sigma}$ on $L^2_\sigma(V)$ is a diagonal operator under the decomposition of $L^2_\sigma(V)$ into G^*-irreducible subspaces V^Λ. Its eigenvalue on V^Λ is given by*
$$\frac{\Gamma_\Omega(\nu+\sigma+p)\Gamma_\Omega(\nu+p-\frac{n}{r})}{\Gamma_\Omega(\nu+\frac{\sigma}{2}+p-\underline{\mathbf{m}}^\star)\Gamma_\Omega(\nu+\frac{\sigma}{2}+p-n/r-\underline{\mathbf{m}})}.$$

Remark 5.2. We note that the above theorem gives an evaluation for the integration over X of the K-invariant function $h(z,-z)^{-\nu}$ against the spherical polynomials $\Phi_{\underline{\mathbf{m}},\sigma}$, corresponding to the one dimensional character $\tau_{-\sigma}$ of K, realized as a function on $\mathfrak{p}^+$ (also called the generalized Jacobi polynomials, see [9], [10]); explicitly we have
$$\begin{aligned} &c_{\nu+\sigma}\int_{\mathfrak{p}^+} h(z,-z)^{-\sigma}\Phi_{\underline{\mathbf{m}},\sigma}(z)h(z,-z)^{-\nu}dm(z) \\ &= \frac{\Gamma_\Omega(\nu+\sigma+p)\Gamma_\Omega(\nu+p-\frac{n}{r})}{\Gamma_\Omega(\nu+\frac{\sigma}{2}+p-\underline{\mathbf{m}}^\star)\Gamma_\Omega(\nu+\frac{\sigma}{2}+p-n/r-\underline{\mathbf{m}})}. \end{aligned} \tag{24}$$
This might be of independent interests in the theory of special functions.

We apply the above theorem and get a decomposition formula for $A^{\nu+\sigma,2}(X) \otimes \overline{A^{\nu,2}(X)}$; see [25] for the proof when $\sigma = 0$.

Theorem 5.3. *Suppose ν, σ are positive integers. Then the tensor product $A^{\nu+\sigma,2}(X) \otimes \overline{A^{\nu,2}(X)}$ admits the orthogonal decomposition*

$$A^{\nu+\sigma,2}(X) \otimes \overline{A^{\nu,2}(X)} = \sum_{m_r \leq \nu + \frac{\sigma}{2}} V^{\Lambda}$$

with multiplicity one.

Theorem 5.1 can now be interpreted as a formula for certain Clebsch-Gordan coefficients. The vector $h^{\nu}(z, -z)$ in $A^{\nu+\sigma,2}(X) \otimes \overline{A^{\nu,2}(X)}$ is K-invariant, each V^{Λ} contains a (up to a constant) unique K-invariant vector, namely the spherical polynomial $\Phi_{\underline{\mathbf{m}},\sigma}$. Theorem 5.1 gives the coefficient of the expansion of $h^{\nu}(z, -z)$ in terms of the spherical polynomials; see also [25] for the case $\sigma = 0$.

References

[1] Arazy, J., and G. Zhang, *Invariant Mean Value and Harmoncity in Cartan and Siegel Domains*, Lecture Notes in Pure and Applied Mathematics, **175**, 19–40.

[2] Berezin, F. A., *General concept of quantization*, Commun. Math. Phys., **40** (1975), 153–174.

[3] Cahen, M., S. Gutt, and J. Rawnsley, *Quantization of Kähler manifolds II* , Trans. Amer. Math. Soc. **337** (1993), 73–98.

[4] Debiard, A., *Système Différentiel hypergéométrique et parties radiales des opérateurs invariants des espaces symétriques de type BC_p*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, Springer Lecture Notes in Math. **1296** 1987, 42–123.

[5] Dooley, A. H., B. Ørsted, and G. Zhang, *Relative discrete series of line bundles over bounded symmetric domains*, Annales Institut Fourier, **46** (1996), 1011–1026.

[6] Erdelyi et al, "Higher transcendental functions," Vol. 1, McGraw-Hill New York - Toronto - London (1953).

[7] Faraut, J., and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Func. Anal., **88** (1990), 64–89.

[8] —, "Analysis on symmetric cones," Oxford University Press, New York, 1993.

[9] Heckman, G. J., *Root systems and hypergeometric functions, II*, Compositio Math. **64** (1987), 353–373.

[10] Heckman, G. J., and E. M. Opdam, *Root systems and hypergeometric functions I*, Compositio Math. **64** (1987), 329–352.

[11] Helgason, S., "Groups and geometric analysis," Academic Press, London, 1984.

[12] Hille, S. C., *Canonical Representations Associated to a Character on a Hermitian Symmetric Space of Rank One*, preprint, 1996.

[13] Loos, O., "Bounded Symmetric Domains and Jordan Pairs," University of California, Irvine, 1977.

[14] Peetre, J., *Berezin transform and Ha-plitz opertors*, J. Oper. Theory **24** (1990), 165–168.

[15] Peetre J., and G. Zhang, *A weighted Plancherel formula III. The case of a hyperbolic matrix ball*, Collect. Math. **43** (1992), 273–301.

[16] —, *Invariant Cauchy-Riemann operators and realization of relative discrete series*, Michigan Math. J. **45** (1998), 387–397.

[17] Satake, I., "Algebraic structures of symmetric domains," Iwanami Shoten and Princeton Univ. Press, Tokyo and Princeton, New Jersey, 1980.

[18] Schlichtkrull, H., *One-dimensional K-types in finite dimensional representations of semisimple Lie groups: A generalization of Helgason's theorem*, Math. Scand. **54** (1984), 279–294.

[19] Shimeno, N., *The Plancherel formula for spherical functions with one-dimensional K-type on a simply connected simple Lie group of Hermitian type*, J. Funct. Anal. **121** (1994), 331–388.

[20] —, *Eigenspaces of invariant differential operatros on a homogeneous line bundle on a Riemannian symmetric space*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **37** (1990), 201–234.

[21] Shimura, G., *Invariant Differential operators on Hermitian symmetric spaces*, Ann. Math. **132** (1990), 232–272.

[22] Unterberger, A., and H. Upmeier, *Berezin transform and invariant differential operators*, Comm. Math. Phy. **164** (1994), 563–597.

[23] Upmeier, H., "Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics"," CBMS Conference Series No. 67, Amer. Math. Soc., 1987.

[24] Zelobenko, D. P., "Compact Lie groups and their representations," Amer. Math. Soc., Transl. Math. Monographs **40**, Providence, Rhode Island, 1973.

[25] Zhang, G., *Berezin transform on compact Hermitian symmetric spaces*, Mauscripta Mathematica **49** (1997), 371–388.

Department of Mathematics
University of Karlstad
S-651 88 Karlstad
Sweden
genkai.zhang@kau.se

Received October 1, 1998