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On an analog of Hermite's constant

Takao Watanabe*

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Abstract. To any strongly k-rational representation π of a connected reductive algebraic group G defined over a number field k, we attach a constant γ_π as an analog of Hermite's constant, and we give a lower estimate of γ_π in the case that the stabilier of the highest weight space of π in G is a maximal parabolic subgroup.

Introduction

In the theory of quadratic forms, the number

$$\gamma_n = \max_L \min_{x \in L \setminus \{0\}} {}^t x x\,, \qquad (L \text{ runs over unimodular lattices in } \mathbb{R}^n)$$

is called Hermite's constant and its exact value is known only for $n \leq 8$ (cf.[10]). An estimate of γ_n is one of interesting problems in the geometry of numbers. Minkowski's first convex-bodies theorem and Minkowski-Hlawka theorem show that

$$\left(\frac{2\zeta(n)}{V(n)}\right)^{2/n} \leq \gamma_n \leq 4\left(\frac{1}{V(n)}\right)^{2/n}, \tag{0.1}$$

where $V(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $\zeta(\cdot)$ the Riemann zeta function.

As a generalization of Hermite's constant, Rankin [16] defined the constant $\gamma_{n,d}$, $(1 \leq d \leq n-1)$, by

$$\gamma_{n,d} = \max_L \min_{\substack{x_1,\cdots,x_d \in L \\ x_1 \wedge \cdots \wedge x_d \neq 0}} \det({}^t x_i x_j)_{i,j}\,, \quad (L \text{ runs over unimodular lattices in } \mathbb{R}^n)$$

and proved the inequality

$$\gamma_{n,d} \leq \gamma_{m,d}(\gamma_{n,m})^{d/m}$$

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for $1 \leq d < m \leq n-1$. On the other hand, by extending a base field, Icaza [9] defined Hermite-Humbert's constant $\gamma_n(k)$ as an analog of γ_n for any number field k and gave an upper bound analogous to (0.1). These two directions of genralizations of Hermite's constant are unified by Thunder [17]. Namely, Thunder defined the constant $\gamma_{n,d}(k)$ for any number field k by using the notion of height on flag varieties. He also extend the inequality (0.1) and Rankin's inequality to the case of $\gamma_{n,d}(k)$ by the adelic variant of Minkowski's first convex-bodies theorem and Minkowski-Hlawka theorem. Thunder's lower bound is given as follows;

$$\gamma_{n,d}(k)^{n[k:\mathbb{Q}]/2} \geq \frac{w_k n}{h_k R_k} \frac{\displaystyle\prod_{j=n-d+1}^{n} \frac{\zeta_k(j)|D_k|^{j/2}}{j^{r_1+r_2}2^{jr_2}V(j)^{r_1}V(2j)^{r_2}}}{\displaystyle\prod_{\ell=2}^{d} \frac{\zeta_k(\ell)|D_k|^{\ell/2}}{\ell^{r_1+r_2}2^{\ell r_2}V(\ell)^{r_1}V(2\ell)^{r_2}}}, \tag{0.2}$$

where ζ_k is the Dedekind zeta function, h_k the class number, R_k the regulator, D_k the discriminant, r_1 and r_2 the number of real and imaginary places of k, respectively.

In this paper, we generalize the constnat $\gamma_{n,d}(k)$ and its estimate (0.2) within the framework of Borel's reduction theory of reductive algebraic groups ([1],[8]). Observe that Rankin's constant $\gamma_{n,d}$ is written as

$$\gamma_{n,d} = \max_{g \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \min_{\substack{x_1,\cdots x_d \in \mathbb{Z}^n \\ x_1 \wedge \cdots \wedge x_d \neq 0}} ||\pi_d(g)(x_1 \wedge \cdots \wedge x_d)||_\infty^2,$$

where (π_d, V_d) denotes the d-th exterior representation of GL_n and $||\cdot||_\infty$ the usual Euclidean metric on $V_d(\mathbb{R})$. If we choose an appropriate height function $||\cdot||_{\mathbb{A}_\mathbb{Q}}$ on $GL(V_d, \mathbb{A}_\mathbb{Q})V_d(\mathbb{Q})$ and a highest weight vector $v_d \in V_d(\mathbb{Q})$, then $\gamma_{n,d}$ is adelically written as

$$\gamma_{n,d} = \max_{g \in SL_n(\mathbb{A}_\mathbb{Q})/SL_n(\mathbb{Q})} \min_{\gamma \in SL_n(\mathbb{Q})} ||\pi_d(g\gamma)v_d||_{\mathbb{A}_\mathbb{Q}}^2.$$

In this form, we replace SL_n, (π_d, V_d) and v_d by an arbitrary connected reductive algebraic group G over k, its strongly k-rational representation (π, V_π) and a highest weight vector x_0 in $V_\pi(k)$, respectively. Then it is known by Borel's reduction theory that the constant

$$\gamma_\pi = \max_{g \in G(\mathbb{A})^1/G(k)} \min_{\gamma \in G(k)} ||\pi(g\gamma)x_0||_\mathbb{A}^{2/[k:\mathbb{Q}]}$$

exists (Proposition 2). If $G = GL_n$ and $\pi = \pi_d$, then γ_π coincides with $\gamma_{n,d}(k)$. In order to estimate γ_π, we can use Thunder's technique based on the mean value argument. Let Q be the stabilizer of the highest weight space of π in G, which is a k-parabolic subgroup of G. If Q is maximal, we shall prove the estimate of the form

$$\gamma_\pi \geq \left(\frac{C_Q d_G e_Q \tau(G)}{C_G d_Q \tau(Q)}\right)^{\frac{2e_\xi}{[k:\mathbb{Q}]e_Q}}. \tag{0.3}$$

Here $\tau(G)$ and $\tau(Q)$ denote the Tamagawa numbers of G and Q, respectively, and other constants C_Q, d_Q, e_Q, e_ξ and similarly C_G, d_G are defined in (1.1), (1.2), (3.1) and (3.2) below. We note that this lower bound depends only on the equivalence class ξ of π. If G is split over k, then the constants $C_G, C_Q, \cdots$ are explicitly described in terms of the Dedekind zeta function and the Gamma function (Theorem 2). As an example, we shall compute the lower bound in the case that G is a classical split group and observe that (0.3) coincides with (0.2) if $G = GL_n$ and $\pi = \pi_d$. Although Thunder applied Weil's method to compute the lower bound of (0.2), we make use Langlands' method to compute the lower bound of (0.3). At present, we have no any result analogous to Minkowsi's first convex-bodies theorem except for the case of $G = GL_n$. This is a reason of the lack of an upper bound of γ_π. In a subsequent paper, we will study this problem when G is an orthogonal group.

Notations. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of rational, real and complex numbers, respectively. The group of positive real numbers is denoted by $\mathbb{R}_+^\times$.

Let k be an algebraic number field of finite degree over $\mathbb{Q}$, $\mathfrak{O}$ the ring of integers in k and $\mathfrak{V}$ the set of all places of k. We write $\mathfrak{V}_\infty$ and $\mathfrak{V}_f$ for the sets of all infinite places and all finite places of k, respectively. For $v \in \mathfrak{V}$, k_v denotes the completion of k at v with the normalized multiplicative valuation $|\cdot|_v$. If v is finite, $\mathfrak{O}_v$ denotes the ring of integers in k_v. We set $k_\infty = \prod_{v\in\mathfrak{V}_\infty} k_v$ and denote by $\mathbb{A}_f$ the ring of finite adeles of k. Thus the adele ring of k is $\mathbb{A} = k_\infty \times \mathbb{A}_f$. The idele norm on $\mathbb{A}^\times$ is denoted by $|\cdot|_\mathbb{A}$. We fix an algebraic closure $\overline{k}$ of k and write Γ_k for the absolute Galois group.

Let G be a connected affine algebraic group defined over k. For any k-algebra R, $G(R)$ stands for the set of R-rational points of G. We denote by $\mathbf{X}^*(G)$ and by $\mathbf{X}_k^*(G)$ the free $\mathbb{Z}$-modules consisting of all rational characters and all k-rational characters of G, respectively. The absolute Galois group Γ_k acts on $\mathbf{X}^*(G)$. The representation of Γ_k in the space $\mathbf{X}^*(G) \otimes_\mathbb{Z} \mathbb{Q}$ is denoted by σ_G and the corresponding Artin L-function is denoted by $L(s, \sigma_G) = \prod_{v\in\mathfrak{V}_f} L_v(s, \sigma_G)$. We set $\sigma_k(G) = \lim_{s\to 1}(s-1)^n L(s, \sigma_G)$, where n is the rank of $\mathbf{X}_k^*(G)$. For a left invariant gauge form ω^G on G defined over k, we associate a left invariant Haar measure ω_v^G on $G(k_v)$. Then, the Tamagawa measure on $G(\mathbb{A})$ is well defined by $\omega_\mathbb{A}^G = |D_k|^{-\dim G/2}\omega_\infty^G \omega_f^G$, where $\omega_\infty^G = \prod_{v\in\mathfrak{V}_\infty} \omega_v^G$, $\omega_f^G = \sigma_k(G)^{-1}\prod_{v\in\mathfrak{V}_f} L_v(1, \sigma_G)\omega_v^G$ and $|D_k|$ is the absolute value of the discriminant of k. For $\chi \in \mathbf{X}_k^*(G)$, let $|\chi|_\mathbb{A}$ be the continuous homomorphism $G(\mathbb{A}) \to \mathbb{R}_+^\times$ defined by $|\chi|_\mathbb{A}(g) = |\chi(g)|_\mathbb{A}$. We write $G(\mathbb{A})^1$ for the intersection of kernels of all such $|\chi|_\mathbb{A}$'s. If $\chi_1, \cdots, \chi_n$ is a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$, then the mapping

$$g \mapsto (|\chi_1(g)|_\mathbb{A}, \cdots, |\chi_n(g)|_\mathbb{A})$$

yields an isomorphism from the quotient group $G(\mathbb{A})/G(\mathbb{A})^1$ to $(\mathbb{R}_+^\times)^n$. We put the Lebesgue measure dx on $\mathbb{R}$ and the invariant measure dx/x on $\mathbb{R}_+^\times$. Then there exists uniquely a Haar measure $\omega_{G(\mathbb{A})^1}$ of $G(\mathbb{A})^1$ such that the Haar measure on $G(\mathbb{A})/G(\mathbb{A})^1$ matching with $\omega_\mathbb{A}^G$ and $\omega_{G(\mathbb{A})^1}$ is equal to the pull-back of the measure $\prod_{i=1}^n dx_i/x_i$ on $(\mathbb{R}_+^\times)^n$ by the above isomorphism. The measure $\omega_{G(\mathbb{A})^1}$ is independent of the choice of a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$. Since $G(k)$

is a discrete subgroup of $G(\mathbb{A})^1$, we put the counting measure $\omega_{G(k)}$ on $G(k)$. Then the Tamagawa number $\tau(G)$ is defined to be the volume of the quotient space $G(\mathbb{A})^1/G(k)$ with respect to $\omega_{G(\mathbb{A})^1}/\omega_{G(k)}$. Here, in general, if μ_A and μ_B denote Haar measures on a locally compact unimodular group A and its closed unimodular subgroup B, respectively, then μ_A/μ_B denotes a unique left A-invariant measure on the homogeneous space A/B matching with μ_A and μ_B in the sense that $\mu_A = \mu_A/\mu_B \cdot \mu_B$ holds.

If X is a locally compact topological space, we denote by $C_0(X)$ the space of all compactly supported continuous functions on X.

1. Strongly k-rational representations

In this section, let G be a connected reductive group defined over k. We recall the notion of strongly k-rational representatons of G (cf.[3],[18]).

We fix a maximally k-split torus S of G and a maximal k-torus T of G containing S. We also fix a minimal k-parabolic subgroup P of G containing S and a Borel subgroup B of P containing T. Then, we denote by Φ (resp. Φ_k) the absolute (resp. relative) root system of G with respect to T (resp. S) and by Δ (resp. Δ_k) the set of simple roots of Φ (resp. Φ_k) corresponding to B (resp. P). The restriction homomorphism $\rho_k : \mathbf{X}^*(T) \to \mathbf{X}^*(S)$ maps Δ into $\Delta_k \cup \{0\}$. We fix an admissible and compatible inner product $(\,,\,)$ on $\mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbf{X}^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ ([3, (6.10)]). We define the $*$-action of Γ_k on $\mathbf{X}^*(T)$. For given $\sigma \in \Gamma_k$, there is an element w in the absolute Weyl group such that $w(\sigma(\Delta)) = \Delta$. Then we set $\sigma^* = w \circ \sigma$, which acts on $\mathbf{X}^*(T)$ by

$$\sigma^*\chi(t) = \sigma(\chi(\sigma^{-1}(w^{-1}(t)))), \qquad (\chi \in \mathbf{X}^*(T),\ t \in T)$$

This action is independent of the choice of w.

Let $\mathfrak{L}$ be the weight lattice of T and $\mathfrak{L}_+$ the set of dominant weights in $\mathfrak{L}$ with respect to B, i.e.,

$$\mathfrak{L} = \{\lambda \in \mathbf{X}^*(T) \ :\ (\lambda, a^\vee) \in \mathbb{Z} \ \text{for all}\ a \in \Delta\},$$
$$\mathfrak{L}_+ = \{\lambda \in \mathfrak{L} \ :\ (\lambda, a^\vee) \geq 0 \ \text{for all}\ a \in \Delta\},$$

where $a^\vee = 2(a,a)^{-1}a$ is the coroot of a. The set $\mathfrak{L}_+$ is stable under the $*$-action of Γ_k, so $\mathfrak{L}_+^{\Gamma_k}$ stands for the subset consisting of Γ_k-invariant elements in $\mathfrak{L}_+$. For $a \in \Delta$, the fundamental weight $\ell_a \in \mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by the conditions $(\ell_a, b^\vee) = \delta_{a,b}$, $(b \in \Delta)$. Any dominant weight λ is written as a non-negative integral linear combination of the ℓ_a's. If G is semisimple and simplyconnected, then the ℓ_a are $\mathbb{Z}$-basis of $\mathbf{X}^*(T)$. For k-root $\alpha \in \Delta_k$, we define the fundamental k-weight $m_\alpha \in \mathbf{X}^*(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$m_\alpha = \sum_{a \in \rho_k^{-1}(\alpha)} \rho_k(\ell_a).$$

Let $\mathfrak{R}(G)$ be the set of equivalent classes of all irreducible $\overline{k}$-rational representations of G. If λ_ξ denotes the highest weight of $\xi \in \mathfrak{R}(G)$, then the correspondence $\xi \to \lambda_\xi$ yields a bijection from $\mathfrak{R}(G)$ to $\mathfrak{L}_+$ (cf.[18]). For $\xi \in \mathfrak{R}(G)$, let π be a representation contained in ξ. We write V_π for the representation space of π. The highest weight space in V_π is denoted by D_π. Let Q_π be the stabilizer of D_π in G and let λ_π be the $\overline{k}$-rational character of Q_π by which Q_π acts on D_π. Then Q_π is a standard $\overline{k}$-parabolic subgroup of G and the restriction of λ_π to T is equal to λ_ξ. An equivalent class $\xi \in \mathfrak{R}(G)$ is said to be k-rational if ξ contains a k-rational representation $\pi : G \to GL(V_\pi)$, where V_π is defined over k. Then, such a k-rational representation π is unique up to k-equivalence ([3, (12.3)]). Moreover, $\xi \in \mathfrak{R}(G)$ is said to be strongly k-rational if ξ is k-rational and, for a k-rational representation $\pi \in \xi$, the highest weight space D_π is defined over k. Then Q_π is defined over k and $\lambda_\pi \in \mathbf{X}_k^*(Q_\pi)$. We denote by $\mathfrak{R}_k^s(G)$ the set of all strongly k-rational $\xi \in \mathfrak{R}(G)$. The next proposition is due to Borel and Tits ([3, §12], [18]).

Proposition 1. (1) *If $\xi \in \mathfrak{R}(G)$ is k-rational, then $\lambda_\xi \in \mathfrak{L}_+^{\Gamma_k}$. (Note that the converse statement is not true in general.)*

(2) *$\xi \in \mathfrak{R}(G)$ is strongly k-rational if and only if $\lambda_\xi \in \mathfrak{L}_+^{\Gamma_k}$ and $(\lambda_\xi, a^\vee) = 0$ for all $a \in \rho_k^{-1}(0) \cap \Delta$. Then λ_ξ is completely determined by $\rho_k(\lambda_\xi)$ and $\rho_k(\lambda_\xi)$ is a non-negative integral linear combination of the fundamental k-weights.*

(3) *For each $\alpha \in \Delta_k$, there exists a positive integer d_α such that $nd_\alpha m_\alpha \in \{\rho_k(\lambda_\xi) \ : \ \xi \in \mathfrak{R}_k^s(G)\}$ for all positive integers n.* ∎

By the statment (2), $\xi \mapsto \rho_k(\lambda_\xi)$ yields a bijection from $\mathfrak{R}_k^s(G)$ to $\rho_k(\mathfrak{L}_+^{\Gamma_k})$. If G is semisimple and simplyconnected, then one has

$$\rho_k(\mathfrak{L}_+^{\Gamma_k}) = \{ \sum_{\alpha \in \Delta_k} c_\alpha m_\alpha \ : \ 0 \le c_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \Delta_k \}.$$

We say that $\xi \in \mathfrak{R}_k^s(G)$ is maximal if $\rho_k(\lambda_\xi)$ is a positive integer multiple of some one fundamental k-weight m_α. Then, for a k-rational representation $\pi \in \xi$, Q_π is the standard maximal k-parabolic subgroup associated to $\Delta_k \backslash \{\alpha\}$, i.e., the centralizer of the torus $(\cap_{\beta \in \Delta_k \backslash \{\alpha\}} \mathrm{Ker}\,\beta)^0$ in G is a Levi subgroup of Q_π.

In the rest of this section, we define some notations. Let M be the centralizer of S in G. Then P has a Levi decomposition $P = MU$, where U is the unipotent radical of P. For every stanadard k-parabolic subgroup Q of G, Q has a unique Levi subgroup M_Q containing M. We denote by U_Q the unipotent radical of Q. Throughout this paper, we fix a maximal compact subgroup $K^G = \prod_{v \in \mathfrak{V}} K_v^G$ of $G(\mathbb{A})$ satisfying the following property; For every standard k-parabolic subgroup Q of G, $K^G \cap Q(\mathbb{A}) = (K^G \cap M_Q(\mathbb{A}))(K^G \cap U_Q(\mathbb{A}))$, $K^G \cap M_Q(\mathbb{A})$ is a maximal compact subgroup of $M_Q(\mathbb{A})$ and $M_Q(\mathbb{A})$ posseses an Iwasawa decomposition $(K^G \cap M_Q(\mathbb{A}))M(\mathbb{A})(M_Q(\mathbb{A}) \cap U(\mathbb{A}))$. It is known that such maximal compact subgroup of $G(\mathbb{A})$ exists. We set $K^{M_Q} = K^G \cap M_Q(\mathbb{A})$, $P^Q = M_Q \cap P$ and $U^Q = M_Q \cap U$.

Let Q be a standard k-parabolic subgroup of G. We include the case $Q = G$. Let Z_Q be the greatest central k-split torus in M_Q. The restriction map $\mathbf{X}_k^*(M_Q) \to \mathbf{X}^*(Z_Q)$ is injective. Since $\mathbf{X}_k^*(M_Q)$ has the same rank as $\mathbf{X}^*(Z_Q)$, the index

$$d_Q = [\mathbf{X}^*(Z_Q) : \mathbf{X}_k^*(M_Q)] \tag{1.1}$$

is finite. If $\chi_1, \cdots, \chi_r$ is a $\mathbb{Z}$-basis of $\mathbf{X}^*(Z_Q)$, the map $z \mapsto (\chi_1(z), \cdots, \chi_r(z))$ defines an isomorphism from $Z_Q(\mathbb{A})$ to $(\mathbb{A}^\times)^r$. We regard $\mathbb{R}_+^\times$ as a subgroup of $\mathbb{A}^\times$ by identifying $x \in \mathbb{R}_+^\times$ with the idele $x_{\mathbb{A}} = (x_v)$ such that $x_v = x$ if $v \in \mathfrak{V}_\infty$ and $x_v = 1$ if $v \in \mathfrak{V}_f$. Let Z_Q^+ denote the inverse image of $(\mathbb{R}_+^\times)^r$ by the isomorpism $Z_Q(\mathbb{A}) \to (\mathbb{A}^\times)^r$. Then $M_Q(\mathbb{A})$ has the direct product decomposition $M_Q(\mathbb{A}) = Z_Q^+ M_Q(\mathbb{A})^1$. We define the Haar measure $\mu_{Z_Q^+}$ on Z_Q^+ by the pull-back of the invariant measure $\prod_{i=1}^r dx_i/x_i$ on $(\mathbb{R}_+^\times)^r$ with respect to the isomorphsim $z \mapsto (|\chi_1(z)|_{\mathbb{A}}, \cdots, |\chi_r(z)|_{\mathbb{A}})$ from Z_Q^+ onto $(\mathbb{R}_+^\times)^r$. It follows from definition of $\omega_{M(\mathbb{A})^1}$ that the Tamagawa measure $\omega_{\mathbb{A}}^{M_Q}$ is decomposed to $d_Q \mu_{Z_Q^+} \cdot \omega_{M_Q(\mathbb{A})^1}$. We note that Z_Q^+ and $\mu_{Z_Q^+}$ are independent of the choice of a basis of $\mathbf{X}^*(Z_Q)$. In particular, Z_G^+ is a subgroup of Z_Q^+ for the surjectivity of the restriction map $\mathbf{X}^*(Z_Q) \to \mathbf{X}^*(Z_G)$.

We define another Haar measure $\nu_{M_Q(\mathbb{A})}$ of $M_Q(\mathbb{A})$ as follows. Let $\omega_{\mathbb{A}}^M$ and $\omega_{\mathbb{A}}^{U^Q}$ be the Tamagawa measures of $M(\mathbb{A})$ and $U^Q(\mathbb{A})$, respectively. There is the function δ_{PQ} on $M(\mathbb{A})$ such that the integration formula

$$\int_{U^Q(\mathbb{A})} f(mum^{-1}) d\omega_{\mathbb{A}}^{U^Q}(u) = \delta_{PQ}(m)^{-1} \int_{U^Q(\mathbb{A})} f(u) d\omega_{\mathbb{A}}^{U^Q}(u)$$

holds for any function $f \in C_0(U^Q(\mathbb{A}))$. In other words, δ_{PQ}^{-1} is the modular character of $P^Q(\mathbb{A})$. Let $\nu_{K^{M_Q}}$ be the Haar measure on K^{M_Q} normalized so that the total volume equals one. Then the mapping

$$f \mapsto \int_{K^{M_Q}} \int_{M(\mathbb{A})} \int_{U^Q(\mathbb{A})} f(hmu) \delta_{PQ}(m) d\nu_{K^{M_Q}}(h) d\omega_{\mathbb{A}}^M(m) d\omega_{\mathbb{A}}^{U^Q}(u)$$

for $f \in C_c(M_Q(\mathbb{A}))$ defines a Haar measure on $M_Q(\mathbb{A})$ and is denoted by $\nu_{M_Q(\mathbb{A})}$. We also define the Haar measure $\nu_{M_Q(\mathbb{A})^1}$ on $M_Q(\mathbb{A})^1$ by $\nu_{M_Q(\mathbb{A})^1} = \nu_{M_Q(\mathbb{A})}/d_Q \mu_{Z_Q^+}$. There exists a positive constant C_Q such that

$$\omega_{\mathbb{A}}^{M_Q} = C_Q \nu_{M_Q(\mathbb{A})}. \tag{1.2}$$

We define left $G(\mathbb{A})^1$-invariant measures on the homogeneous space $Y_Q = G(\mathbb{A})^1/Q(\mathbb{A})^1$. Since $Q(\mathbb{A})^1 = M(\mathbb{A})^1 U_Q(\mathbb{A})$, $\nu_{M(\mathbb{A})^1} \omega_{\mathbb{A}}^{U_Q}$ defines a Haar measure, say $\nu_{Q(\mathbb{A})^1}$, of $Q(\mathbb{A})^1$. We note that both $G(\mathbb{A})^1$ and $Q(\mathbb{A})^1$ are unimodular. Therefore, both $\omega_{Y_Q} = \omega_{G(\mathbb{A})^1}/\omega_{Q(\mathbb{A})^1}$ and $\nu_{Y_Q} = \nu_{G(\mathbb{A})^1}/\nu_{Q(\mathbb{A})^1}$ yield $G(\mathbb{A})^1$-invariant measures on Y_Q. We identify Y_Q with $G(\mathbb{A})/Z_G^+ Q(\mathbb{A})^1$ and set $K^Q = K^G \cap Q(\mathbb{A})$. Then the mapping

$$\iota_Q : K^G/K^Q \times Z_Q^+/Z_G^+ \to Y_Q : (hK^Q, zZ_G^+) \mapsto hzZ_G^+ Q(\mathbb{A})^1$$

is an isomorphism. Set $\mu_{Z_Q^+/Z_G^+} = \mu_{Z_Q^+}/\mu_{Z_G^+}$.

Lemma 1. *For any left K^G-invariant function $f \in C_0(Y_Q)$, one has*

$$\int_{Y_Q} f(y) d\nu_{Y_Q}(y) = \frac{d_Q}{d_G} \int_{Z_Q^+/Z_G^+} f(\iota_Q(K^Q, zZ_G^+)) \delta_Q(z) d\mu_{Z_Q^+/Z_G^+}(zZ_G^+),$$

where δ_Q^{-1} *is the modular character of* $Q(\mathbb{A})$.

Proof. Let $\iota_Q^*\nu_{Y_Q}$ be the measure on $K^G/K^Q \times Z_Q^+/Z_G^+$ obtained by the pull-back of ν_{Y_Q}. We define the measure $\widehat{\mu}_{Z_Q^+/Z_G^+}$ on Z_Q^+/Z_G^+ by the mapping

$$\varphi \mapsto \int_{K^G/K^Q\times Z_Q^+/Z_G^+} \varphi(zZ_G^+)d\iota_Q^*\nu_{Y_Q}(hK^Q, zZ_G^+), \qquad (\varphi \in C_0(Z_Q^+/Z_G^+))$$

Then we have

$$\int_{Y_Q} f(y)d\nu_{Y_Q}(y) = \int_{Z_Q^+/Z_G^+} f(\iota_Q(K^Q, zZ_G^+))d\widehat{\mu}_{Z_Q^+/Z_G^+}(zZ_G^+)$$

for any left K^G-invariant function $f \in C_0(Y_Q)$. We determine $\widehat{\mu}_{Z_Q^+/Z_G^+}$. Let $\psi \in C_0(G(\mathbb{A})^1)$ be a left K^G-invariant function on $G(\mathbb{A})^1$. On the one hand, we have

$$\begin{aligned}
&\int_{G(\mathbb{A})^1} \psi(g)d\nu_{G(\mathbb{A})^1}(g) \\
&= \int_{K^G}\int_{M(\mathbb{A})/Z_G^+}\int_{U(\mathbb{A})} \psi(hmu)\delta_P(m)d\nu_{K^G}(h)d(\omega_{\mathbb{A}}^M/d_G\mu_{Z_G^+})(m)d\omega_{\mathbb{A}}^U(u) \\
&= \int_{Z_P^+/Z_G^+}\int_{M(\mathbb{A})^1}\int_{U(\mathbb{A})} \psi(zm_1u)\delta_P(z)d(d_P\mu_{Z_P^+}/d_G\mu_{Z_G^+})(z)d\omega_{M(\mathbb{A})^1}(m_1)d\omega_{\mathbb{A}}^U(u) \\
&= \frac{d_Q}{d_G}\int_{Z_P^+/Z_Q^+}\int_{Z_Q^+/Z_G^+}\int_{M(\mathbb{A})^1}\int_{U(\mathbb{A})} \psi(z_1z_2m_1u)\delta_{PQ}(z_1)\delta_Q(z_2) \\
&\qquad \times d(d_P\mu_{Z_P^+}/d_Q\mu_{Z_Q^+})(z_1)d\mu_{Z_Q^+/Z_G^+}(z_2)d\omega_{M(\mathbb{A})^1}(m_1)d\omega_{\mathbb{A}}^U(u).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{G(\mathbb{A})^1} \psi(g)d\nu_{G(\mathbb{A})^1}(g) \\
&= \int_{Y_Q}\left\{\int_{Q(\mathbb{A})^1} \psi(gq)d\nu_{Q(\mathbb{A})^1}(q)\right\} d\nu_{Y_Q}(gQ(\mathbb{A})^1) \\
&= \int_{K^G/K^Q\times Z_Q^+/Z_G^+}\{\int_{K^{M_Q}}\int_{M(\mathbb{A})/Z_Q^+}\int_{U^Q(\mathbb{A})}\int_{U_Q(\mathbb{A})} \psi(\iota_Q(h_1K^Q, zZ_G^+)h_2mu_1u_2) \\
&\qquad \times \delta_{PQ}(m)d\nu_{K^{M_Q}}(h_2)d(\omega_{\mathbb{A}}^M/d_Q\mu_{Z_Q^+})(m)d\omega_{\mathbb{A}}^{U^Q}(u_1)d\omega_{\mathbb{A}}^{U_Q}(u_2)\} \\
&\qquad \times d\iota_Q^*\nu_{Y_Q}(h_1K_Q, zZ_G^+) \\
&= \int_{Z_Q^+/Z_G^+}\{\int_{Z_P^+/Z_Q^+}\int_{M(\mathbb{A})^1}\int_{U(\mathbb{A})} \psi(z_2z_1m_1u)\delta_{PQ}(z_1)d(d_P\mu_{Z_P^+}/d_Q\mu_{Z_Q^+})(z_1) \\
&\qquad \times d\omega_{M(\mathbb{A})^1}(m_1)d\omega_{\mathbb{A}}^U(u)\}d\widehat{\mu}_{Z_Q^+/Z_G^+}(z_2) \\
&= \int_{Z_P^+/Z_Q^+}\int_{Z_Q^+/Z_G^+}\int_{M(\mathbb{A})^1}\int_{U(\mathbb{A})} \psi(z_1z_2m_1u)\delta_{PQ}(z_1)d(d_P\mu_{Z_P^+}/d_Q\mu_{Z_Q^+})(z_1) \\
&\qquad \times d\widehat{\mu}_{Z_Q^+/Z_G^+}(z_2)d\omega_{M(\mathbb{A})^1}(m_1)d\omega_{\mathbb{A}}^U(u).
\end{aligned}$$

Therefore, we obtain $\widehat{\mu}_{Z_Q^+/Z_G^+} = d_Qd_G^{-1}\delta_Q(z_2)\mu_{Z_Q^+/Z_G^+}$. ∎

Corollary . *For any left K^G-invariant function $f \in C_0(Y_Q)$, one has*

$$\int_{Y_Q} f(y)d\omega_{Y_Q}(y) = \frac{C_G d_Q}{C_Q d_G} \int_{Z_Q^+/Z_G^+} f(\iota_Q(K^Q, zZ_G^+))\delta_Q(z)d\mu_{Z_Q^+/Z_G^+}(zZ_G^+). \quad \blacksquare$$

2. An analog of Hermite's constant

We fix a strongly k-rational $\xi \in \mathfrak{R}_k^s(G)$ and a k-rational representation $\pi \in \xi$. We also fix a free $\mathfrak{O}$-lattice L in the k-vector space $V_\pi(k)$ and a $\mathfrak{O}$-basis $e_1, \cdots, e_n$ of L, where $n = \dim \pi$. For every $v \in \mathfrak{V}$, we define the norm $||\cdot||_{L,v}$ of the k_v-vector space $V_\pi(k_v)$ as follows;

$$(2.1)\quad ||x_1e_1 + \cdots + x_ne_n||_{L,v} = \begin{cases} \sup(|x_1|_v, \cdots, |x_n|_v) & (v \in \mathfrak{V}_f) \\ (|x_1|_v^2 + \cdots + |x_n|_v^2)^{1/2} & (v \text{ is real}) \\ |x_1|_v + \cdots + |x_n|_v & (v \text{ is imaginary}). \end{cases}$$

We define height functions on $GL_n(\mathbb{A})V_\pi(k)$. Let $||\cdot||_v$, $v \in \mathfrak{V}$ be a norm on $V_\pi(k_v)$ compatible with $|\cdot|_v$. Assume that $||\cdot||_v = ||\cdot||_{L,v}$ holds for almost all $v \in \mathfrak{V}$. Then we set

$$||x||_{\mathbb{A}} = \prod_{v \in \mathfrak{V}} ||x_v||_v, \qquad (x \in GL_n(\mathbb{A})V_\pi(k)).$$

Such $||\cdot||_{\mathbb{A}}$ is called a height function on $GL_n(\mathbb{A})V_\pi(k)$ (cf.[8]). The following property is obvious by its definition.

$$(2.2)\qquad ||ax||_{\mathbb{A}} = |a|_{\mathbb{A}}||x||_{\mathbb{A}} \quad \text{for any } a \in \mathbb{A}^\times \text{ and } x \in GL_n(\mathbb{A})V_\pi(k).$$

A height function $||\cdot||_{\mathbb{A}}$ is said to be K^G-invariant if $||\pi(h)x||_{\mathbb{A}} = ||x||_{\mathbb{A}}$ holds for all $h \in K^G$ and $x \in GL_n(\mathbb{A})V_\pi(k)$. Furthermore, $||\cdot||_{\mathbb{A}}$ is said to be normalized if $||x_0||_{\mathbb{A}} = 1$ for a highest weight vector $x_0 \in D_\pi(k)\backslash\{0\}$. Then one has $||x||_{\mathbb{A}} = 1$ for any highest weight vector $x \in D_\pi(k)\backslash\{0\}$ because of the one-dimensionality of $D_\pi(k)$ and (2.2). It is obvious that K^G-invariant and normalized height functions on $GL_n(\mathbb{A})V_\pi(k)$ exist.

In the following, we fix a K^G-invariant normalized height function $||\cdot||_{\mathbb{A}}$ on $GL_n(\mathbb{A})V_\pi(k)$. Let $x_0 \in D_\pi(k)\backslash\{0\}$ be a highest weight vector of π. For each $g \in G(\mathbb{A})^1$, we define the function $\Phi_g : G(k) \to \mathbb{R}_+^\times$ by $\Phi_g(\gamma) = ||\pi(g\gamma)x_0||_{\mathbb{A}}$. From $\pi(\gamma)x_0 = \lambda_\pi(\gamma)x_0$, $(\gamma \in Q_\pi(k))$ and (2.2), it follows that Φ_g is right $Q_\pi(k)$-invariant and is independent of the choice of x_0. By the reduction theory (cf.[1, §16],[8]), it is known that Φ_g attains its minimum value at a point in the intersection of $G(k)$ and a Siegel set of $G(\mathbb{A})$. We set $\Psi(g) = \min_{\gamma \in G(k)/Q_\pi(k)} \Phi_g(\gamma)$. Then it is easy to see that Ψ is a continuous function on $K^G\backslash G(\mathbb{A})^1/G(k)$ and is bounded from above (cf.[1, 16.10]).

Proposition 2. *Ψ has the maximum value.*

Proof. Set $H = (\operatorname{Ker}\pi)^0$, $P_H = P \cap H$ and $S_H = S \cap H$. Let Δ_k^H be the set of simple k-roots of H with respect to S_H and P_H. We also set $\Delta_k^{G/H} = \Delta_k^G \backslash \Delta_k^H$

and $S_{G/H} = (\cap_{\alpha \in \Delta_k^H} \operatorname{Ker}\alpha)^0$. Then $Z_P^+ = S_{G/H}^+ S_H^+$. For $\kappa > 0$ and every standard k-parabolic subgroup Q, define a subset $Z_Q^+(\kappa)$ of Z_Q^+ by

$$Z_Q^+(\kappa) = \{z \in Z_Q^+ \ : \ |\alpha(z)|_{\mathbb{A}} < \kappa \text{ for all } \alpha \in \Delta_k\}.$$

By the reduction theory, we can choose $\kappa > 1$ and a compact subset Ω in $P(\mathbb{A})$ so that $G(\mathbb{A}) = K^G \Omega Z_P^+(\kappa) G(k)$. Since π is trivial on $H(\mathbb{A})$, it is enough to prove that Ψ attains its maximum value at a point in $\Omega Z_P^+(\kappa) S_H^+ / Z_G^+ S_H^+ = \Omega S_{G/H}^+(\kappa)/Z_G^+$. By compactness of Ω, there is a constant C such that

$$\Psi(\omega z) \leq ||\pi(\omega z)x_0||_{\mathbb{A}} \leq C||\pi(z)x_0||_{\mathbb{A}} = C|\lambda_\pi(z)|_{\mathbb{A}} \tag{2.3}$$

holds for all $\omega \in \Omega$ and $z \in Z_P^+$. Since $\rho_k(\lambda_\pi) \in \mathbf{X}^*(S/S_H) \subset \mathbf{X}^*(S_{G/H}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\rho_k(\lambda_\pi)$ is represented by a $\mathbb{Q}$-linear combination of $\alpha \in \Delta_k^{G/H}$ modulo $\mathbf{X}^*(Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e., we have

$$\rho_k(\lambda_\pi) \equiv \sum_{\alpha \in \Delta_k^{G/H}} c_\alpha \alpha \mod \mathbf{X}^*(Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.4}$$

For the moment, we assume that the following condition is satisfied;

$$c_\alpha > 0 \text{ for all } \alpha \in \Delta_k^{G/H}. \tag{2.5}$$

Let Q_β be the standard maximal k-parabolic subgroup of G correspoinding to $\Delta_k \backslash \{\beta\}$ for $\beta \in \Delta_k^{G/H}$. By (2.3) and (2.4), one has for $\omega \in \Omega$, $z \in Z_{Q_\beta}^+(\kappa_1)$ and $\kappa_1 > 0$

$$\Psi(\omega z) \leq C|\lambda_\pi(z)|_{\mathbb{A}} \leq C\kappa_1^{c_\beta}.$$

Under the condition (2.5), this concludes that if κ_1 is sufficiently small, then $\Psi(\omega z)$ is also small for all $\omega \in \Omega$ and $z \in S_{Q_\beta}^+(\kappa_1)$. In other words, if we define a subset $S_{G/H}^+(\kappa, \kappa_1)$ of $S_{G/H}^+(\kappa)$ by

$$S_{G/H}^+(\kappa, \kappa_1) = S_{G/H}^+(\kappa) \backslash \bigcup_{\beta \in \Delta_k^{G/H}} Z_{Q_\beta}^+(\kappa_1),$$

then we have

$$\sup_{g \in K^G \backslash G(\mathbb{A})^1 / G(k)} \Psi(g) = \sup_{g \in \Omega S_{G/H}^+(\kappa, \kappa_1)/Z_G^+} \Psi(g).$$

Therefore, $\Psi(g)$ has the maximum value since $\Omega S_{G/H}^+(\kappa, \kappa_1)/Z_G^+$ is relatively compact in $G(\mathbb{A})^1$. To complete the proof, we have to show (2.5). Since it is an assetion concerning root systems, we may assume that $H = 1$. Furthermore, it is reduced to the cases that G is almost k-simple. Thus $\rho_k(\lambda_\pi)$ is non-zero and is represented as a non-negative integral linear combination of the fundamental k-weights m_α, $\alpha \in \Delta_k$. Let

$$m_\alpha = \sum_{\beta \in \Delta_k} d_{\alpha,\beta} \beta, \qquad (d_{\alpha,\beta} \in \mathbb{Q}).$$

It is sufficient to show

$$d_{\alpha,\beta} > 0 \text{ for all } \alpha, \beta \in \Delta_k .$$

For an absolute fundamental weight ℓ_a, let

$$\ell_a = \sum_{b \in \Delta} d'_{a,b} b, \qquad (d'_{a,b} \in \mathbb{Q}).$$

Then we have

$$d_{\alpha,\beta} = \sum_{a \in \rho_k^{-1}(\alpha)} \sum_{b \in \rho_k^{-1}(\beta)} d'_{a,b} .$$

Therefore, we need to show that $d'_{a,b} \geq 0$ for all a, b and there are $a \in \rho_k^{-1}(\alpha)$ and $b \in \rho_k^{-1}(\beta)$ such that $d'_{a,b} > 0$. Since G is almost k-simple, there are a finite field extention L/k and an absolutely almost simple group G_1 defined over L so that G is k-isomorphic to the scalar restriction $R_{L/k}(G_1)$ of G_1. Let Φ^{G_1} be the absolute root system of G_1 and Δ^{G_1} a set of simple roots in Φ^{G_1}. Then Φ is an orthogonal direct sum of $[L:k]$ copies of Φ^{G_1}, and hence Δ is of the form

$$\Delta = \bigsqcup_{i=1}^{[L:k]} \Delta^i , \qquad \Delta^i = \Delta^{G_1} .$$

We have $d'_{a,b} = 0$ if $a \in \Delta^i$, $b \in \Delta^j$ and $i \neq j$. Since $\rho_k(\Delta^i) = \Delta_k \cup \{0\}$ for each i, it is enough to prove $d'_{a,b} > 0$ for $a, b \in \Delta^i$. This follows from a classification table of the simple root systems ([4]). ■

Let $\gamma'_\pi = \max_{g \in K^G \backslash G(\mathbb{A})^1 / G(k)} \Psi(g)$. Then we call $\gamma_\pi = (\gamma'_\pi)^{2/[k:\mathbb{Q}]}$ the Hermite constant attached to π and $||\cdot||_{\mathbb{A}}$. We write $\gamma_\pi(||\cdot||_{\mathbb{A}})$ for γ_π if we need to emphasize the dependence of $||\cdot||_{\mathbb{A}}$. In an example below, we shall clarify a relation of γ_π and the original Hermite's constant.

Example 1. Let V be an n-dimensional vector space defined over k. We fix a free $\mathfrak{O}$-lattice L in V and its $\mathfrak{O}$-basis $e_1, \cdots, e_n$. We identify $G = GL(V)$ with GL_n with respect to the basis $e_1, \cdots, e_n$. Let $||\cdot||_{L,\mathbb{A}} = \prod_{v \in \mathfrak{V}} ||\cdot||_{L,v}$ be the height function on $GL_n(\mathbb{A})V(k)$, where $||\cdot||_{L,v}$ is defined similary as (2.1). Let S be the subgroup of diagonal matirces in G and P the subgroup of upper triangular matrices in G. For $g \in S$, $\epsilon_i(g)$ denotes the i-th diagonal element of g for $1 \leq i \leq n$. Thus ϵ_i is a rational character of S. As usual, the root system $\Phi = \Phi_k$ is given by $\{\epsilon_i - \epsilon_j \ : \ 1 \leq i, j \leq n, \quad i \neq j\}$. We have $\Delta = \Delta_k = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \ : \ 1 \leq i \leq n-1\}$. The fundamental weights ℓ_d are given by

$$\ell_d = \epsilon_1 + \cdots + \epsilon_d .$$

Then

$$\mathfrak{L}_+ = \mathfrak{L}_+^{\Gamma_k} = \{\sum_{d=1}^{n} c_d \ell_d \ : \ 0 \leq c_d \in \mathbb{Z} \text{ for all } d\} .$$

We take the maximal compact subgroup K^G as follows;

$$K^G = \prod_{v\in\mathfrak{V}} K_v^G, \qquad K_v^G = \{g \in G(k_v) \ : \ ||gx||_{L,v} = ||x||_{L,v} \text{ for all } x \in V(k_v)\}.$$

We consider the classes $\xi_d \in \mathfrak{R}(G)$ corresponding to the fundamental weights ℓ_d. Let $\pi_d \in \xi_d$ be the d-th exterior representation of G, i.e. $V_{\pi_d} = \bigwedge^d V$. Then $v_d = e_1 \wedge \cdots \wedge e_d$ is a highest weight vector of π_d. We denote by Q_d the stabilizer of the highest weight space D_{π_d} of π_d in G. The Levi-subgroup M_{Q_d} is isomorphic to $GL_d \times GL_{n-d}$. Let $L_d = \bigwedge^d L$ be a lattice in V_{π_d}. We define the height function $||\cdot||_{L_d,\mathbb{A}}$ on $GL(V_{\pi_d},\mathbb{A})V_{\pi_d}(k)$ similary as above. By definition,

$$\gamma_{\pi_d}(||\cdot||_{L_d,\mathbb{A}}) = (\max_{g\in G(\mathbb{A})^1/G(k)} \min_{\gamma\in G(k)/Q_d(k)} ||\pi_d(g\gamma)v_d||_{L_d,\mathbb{A}})^{2/[k:\mathbb{Q}]}.$$

This γ_{π_d} is equal to the constant $\gamma_{n,d}(k)$ defined by Thunder, and further, if $k = \mathbb{Q}$, this is equal to the constsnat defined by Rankin.

3. A lower bound of γ_π

We fix $\xi \in \mathfrak{R}_k^s(G)$, a k-rational representation $\pi \in \xi$ and a K^G-invariant normalized height function $||\cdot||_{\mathbb{A}}$ on $GL_n(\mathbb{A})V_\pi(k)$, where $n = \dim \pi$. Let $x_0 \in D_\pi(k)$. We define the function $f_\pi : G(\mathbb{A}) \to \mathbb{R}_+^\times$ by $f_\pi(g) = ||\pi(g)x_0||_{\mathbb{A}}$. We have

$$f_\pi(gh) = |\lambda_\pi(h)|_{\mathbb{A}} f_\pi(g) \qquad \text{for all } h \in Q_\pi(\mathbb{A}) \text{ and } g \in G(\mathbb{A}).$$

Therefore, f_π is regarded as a function on $Y_{Q_\pi} = G(\mathbb{A})^1/Q_\pi(\mathbb{A})^1$. In the following, we assume that ξ is maximal, i.e., $\rho_k(\lambda_\xi)$ is a positive integer multiple of one fundamental k-weight m_α. Then Q_π is the standard maximal k-parabolic subgroup associated to $\Delta_k\backslash\{\alpha\}$. For simplicity, we omit the subscript π and write Q for Q_π. Let n_Q be the positive integer such that $n_Q^{-1}\alpha|_{Z_Q}$ is a $\mathbb{Z}$-base of $\mathbf{X}^*(Z_Q/Z_G)$. We set $\alpha_Q = n_Q^{-1}\alpha|_{Z_Q}$. Then the Haar measuer $\mu_{Z_Q^+/Z_G^+}$ equals the pull-back of the measure dx/x by the isomorphism $|\alpha_Q|_{\mathbb{A}} : Z_Q^+/Z_G^+ \to \mathbb{R}_+^\times$. Note that $\dim U_Q \alpha|_{Z_Q}$ is the restriction of the k-rational character $m \mapsto \det(\mathrm{Ad}(m)|_{\mathrm{Lie}(U_Q)})$ of M_Q to Z_Q, where by $\mathrm{Ad}(m)|_{\mathrm{Lie}(U_Q)}$ we mean the adjoing action of m on the Lie algebra of U_Q. Since $\delta_Q(m) = |\det(\mathrm{Ad}(m)|_{\mathrm{Lie}(U_Q)})|_{\mathbb{A}}$, we have

$$\delta_Q(z) = |\alpha_Q(z)|_{\mathbb{A}}^{e_Q}, \qquad (z \in Z_Q(\mathbb{A})) \tag{3.1}$$

where $e_Q = n_Q \dim U_Q$. The quotient morphism $Z_Q \to Z_Q/Z_G$ induces an isomorphism $\mathbf{X}^*(Z_Q/Z_G) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbf{X}^*(Z_Q \cap G^{ss}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where G^{ss} denotes the derived group of G. Under the identification $\mathbf{X}^*(Z_Q \cap G^{ss}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbf{X}^*(Z_Q/Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}$, there exists the positive rational number e_ξ such that

$$\rho_k(\lambda_\xi)|_{Z_Q\cap G^{ss}} = e_\xi \alpha_Q. \tag{3.2}$$

Then $f_\pi(\iota_Q(K^Q, zZ_G^+)) = |\alpha_Q(z)|_{\mathbb{A}}^{e_\xi}$ holds for any $zZ_G^+ \in Z_Q^+/Z_G^+$.

Lemma 2. *For any measurable function φ on $\mathbb{R}$, one has*

$$\int_{Y_Q} \varphi(f_\pi(g))d\omega_{Y_Q}(g) = \frac{C_G d_Q}{C_Q d_G e_\xi} \int_{\mathbb{R}_+^\times} \varphi(x) x^{\frac{e_Q}{e_\xi}} \frac{dx}{x}.$$

Proof. By Corollary to Lemma 1,

$$\begin{aligned}
&\int_{Y_Q} \varphi(f_\pi(g))d\omega_{Y_Q}(g) \\
&= \frac{C_G d_Q}{C_Q d_G} \int_{Z_Q^+/Z_G^+} \varphi(f_\pi(\iota_Q(K^Q, zZ_G^+)))\delta_Q(z) d\mu_{Z_Q^+/Z_G^+}(zZ_G^+) \\
&= \frac{C_G d_Q}{C_Q d_G} \int_{Z_Q^+/Z_G^+} \varphi(|\alpha_Q(z)|_{\mathbb{A}}^{e_\xi})|\alpha_Q(z)|_{\mathbb{A}}^{e_Q} d\mu_{Z_Q^+/Z_G^+}(zZ_G^+) \\
&= \frac{C_G d_Q}{C_Q d_G} \int_{\mathbb{R}_+^\times} \varphi(x^{e_\xi}) x^{e_Q} \frac{dx}{x}.
\end{aligned}$$

By change of variables, we obtain the assertion. ∎

Theorem 1. *One has*

$$\gamma_\pi(||\cdot||_{\mathbb{A}}) \geq \left(\frac{C_Q d_G e_Q \tau(G)}{C_G d_Q \tau(Q)}\right)^{\frac{2e_\xi}{[k:\mathbb{Q}]e_Q}}.$$

As is known from the proof, this estimate is independent of the choice of a K^G-invariant normalized height function $||\cdot||_{\mathbb{A}}$.

Proof. Fix a positive real number t such that

$$\frac{C_G d_Q}{C_Q d_G e_Q} t^{e_Q/e_\xi} < \frac{\tau(G)}{\tau(Q)}.$$

Let ε be a sufficiently small positive real number satisfying

$$\frac{C_G d_Q}{C_Q d_G e_Q}(t^{e_Q/e_\xi} + \varepsilon) < \frac{\tau(G)}{\tau(Q)}.$$

We define the function $\varphi(x)$ by

$$\varphi(x) = \begin{cases} 1 & (0 \leq x \leq t) \\ \left(\dfrac{x}{t}\right)^{\frac{e_Q}{e_\xi}(-1-\varepsilon^{-1}t^{e_Q/e_\xi})} & (t < x). \end{cases}$$

Then, by Lemma 2,

$$\begin{aligned}
\int_{Y_Q} \varphi(f_\pi(g))d\omega_{Y_Q}(g) &= \frac{C_G d_Q}{C_Q d_G e_\xi} \int_{\mathbb{R}_+^\times} \varphi(x) x^{e_Q/e_\xi} \frac{dx}{x} \\
&= \frac{C_G d_Q}{C_Q d_G e_Q}(t^{e_Q/e_\xi} + \varepsilon) < \frac{\tau(G)}{\tau(Q)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\int_{Y_Q}\varphi(f_\pi(g))d\omega_{Y_Q}(g)\\&=\frac{1}{\tau(Q)}\int_{G(\mathbb{A})^1/G(k)}\sum_{\gamma\in G(k)/Q(k)}\varphi(f_\pi(g\gamma))d(\omega_{G(\mathbb{A})^1}/\omega_{G(k)})(g)<\frac{\tau(G)}{\tau(Q)}.\end{aligned}$$

Therefore there exists $g\in G(\mathbb{A})^1$ such that $\sum_{\gamma\in G(k)/Q(k)}\varphi(f_\pi(g\gamma))<1$. This implies $\varphi(f_\pi(g\gamma))<1$ for all $\gamma\in G(k)/Q(k)$. By definition of φ, we have $f_\pi(g\gamma)>t$ for all $\gamma\in G(k)/Q(k)$. Namely

$$\min_{\gamma\in G(k)/Q(k)}f_\pi(g\gamma)>t\,.$$

Hence

$$\begin{aligned}\gamma'_\pi=\max_{g\in G(\mathbb{A})^1/G(k)}\min_{\gamma\in G(k)/Q(k)}f_\pi(g\gamma)&\geq\sup\{t>0\;:\;\frac{C_Gd_Q}{C_Qd_Ge_Q}t^{e_Q/e_\xi}<\frac{\tau(G)}{\tau(Q)}\}\\&=\left(\frac{C_Qd_Ge_Q\tau(G)}{C_Gd_Q\tau(Q)}\right)^{\frac{e_\xi}{e_Q}}.\end{aligned}$$

■

4. Examples

First, we compute C_G. We use the same notations as in Sections 1 and 3. We state an integration formula. Let w_0 be an element in the normalizer of S in G which represents the longest element in the relative Weyl group of G with respect to S. We set $U^-=w_0^{-1}Uw_0$. Then the homogeneous space $U^-\backslash G$ has the right G-invariant gauge form $\omega^{U^-}\backslash\omega^G$ defined over k matching with ω^{U^-} and ω^G. By the Bruhat decomposition, P is regarded as a Zariski open subset of $U^-\backslash G$. Then $(\omega^{U^-}\backslash\omega^G)|_P$ yields a right invariant gauge form on P defined over k. Since a right invariant gauge form on P defined over k is unique up to scalar multiplications, we can choose the gauge form ω^M so that $(\omega^{U^-}\backslash\omega^G)|_P=\omega^U\omega^M$. Then, for each $v\in\mathfrak{V}$ and $\varphi\in C_0(G(k_v))$, we have

$$\begin{aligned}&\int_{G(k_v)}\varphi(g)d\omega_v^G\\ (4.1)\qquad&=\int_{U^-(k_v)}\int_{U(k_v)}\int_{M(k_v)}\varphi(u^-um)d\omega_v^{U^-}(u^-)d\omega_v^U(u)d\omega_v^M(m)\\&=\int_{U^-(k_v)}\int_{M(k_v)}\int_{U(k_v)}\varphi(u^-mu)\delta_{P,v}(m)d\omega_v^{U^-}(u^-)d\omega_v^M(m)d\omega_v^U(u)\end{aligned}$$

where $\delta_{P,v}^{-1}$ denotes the modular character of $P(k_v)$. We define some notations. We fix $v\in\mathfrak{V}$ for the moment. By the Iwasawa decompositon, $g\in U^-(k_v)$ is decomposed to $h_v(g)m_v(g)u_v(g)$ with $h_v(g)\in K_v^G$, $m_v(g)\in M(k_v)$, $u_v(g)\in$

$U(k_v)$. Then the function η_v of $U^-(k_v)$ defined by $\eta_v(g) = \delta_{P,v}(m_v(g))^{-1}$ is well-defined. It is known that the integral

$$I_v(G,P) = \int_{U^-(k_v)} \eta_v(g)\,d\omega_v^{U^-}(g)$$

converges. We set $I_\infty(G,P) = \prod_{v\in\mathfrak{V}_\infty} I_v(G,P)$. Let K_∞^G and K_f^G denote the infinite and the finite part of K^G, respectively. Similarly, K_∞^P and K_f^P are defined.

Lemma 3. *One has*

$$C_G = |D_k|^{(\dim P - \dim G)/2}\frac{\omega_f^G(K_f^G)}{\omega_f^P(K_f^P)} I_\infty(G,P)\,.$$

Proof. The proof is essentially same as [12]. See also [11, Proposition 5.2]. For a given $\varphi \in C_0(K_\infty^P\backslash P(k_\infty))$, let Φ be a left K^G-invariant function on $G(\mathbb{A})$ such that the support of Φ is contained in $G(k_\infty)K_f^G$ and $\Phi(h_\infty p_\infty h_f) = \varphi(p_\infty)$ for $h_\infty \in K_\infty^G$, $p_\infty \in P(k_\infty)$, $h_f \in K_f^G$. On the one hand, we have

$$\begin{aligned}
&\int_{G(\mathbb{A})} \Phi(g)\,d\omega_{\mathbb{A}}^G(g)\\
&= C_G \int_{G(\mathbb{A})} \Phi(g)\,d\nu_{G(\mathbb{A})}(g)\\
&= C_G \int_{M(\mathbb{A})}\int_{U(\mathbb{A})} \Phi(mu)\delta_P(m)\,d\omega_{\mathbb{A}}^M(m)\,d\omega_{\mathbb{A}}^U(u)\\
&= C_G |D_k|^{-\dim P/2}\omega_f^P(K_f^P)\\
&\qquad\times \int_{M(k_\infty)}\int_{U(k_\infty)} \varphi(m_\infty u_\infty)\delta_{P,\infty}(m_\infty)\,d\omega_\infty^M(m_\infty)\,d\omega_\infty^U(u_\infty)\,,
\end{aligned}$$

where $\delta_{P,\infty} = \prod_{v\in\mathfrak{V}_\infty}\delta_{P,v}$. On the other hand, by (4.1), we have

$$\begin{aligned}
&\int_{G(\mathbb{A})} \Phi(g)\,d\omega_{\mathbb{A}}^G(g)\\
&= |D_k|^{-\dim G/2}\omega_f^G(K_f^G)\int_{G(k_\infty)} \Phi(g)\,d\omega_\infty^G(g)\\
&= |D_k|^{-\dim G/2}\omega_f^G(K_f^G)\int_{U^-(k_\infty)}\int_{M(k_\infty)}\int_{U(k_\infty)} \Phi(u^- mu)\delta_{P,\infty}(m)\\
&\qquad\times d\omega_\infty^{U^-}(u^-)\,d\omega_\infty^M(m)\,d\omega_\infty^U(u)\\
&= |D_k|^{-\dim G/2}\omega_f^G(K_f^G)\int_{U^-(k_\infty)}\int_{M(k_\infty)}\int_{U(k_\infty)} \varphi(mu)\eta_\infty(u^-)\delta_{P,\infty}(m)\\
&\qquad\times d\omega_\infty^{U^-}(u^-)\,d\omega_\infty^M(m)\,d\omega_\infty^U(u)\\
&= |D_k|^{-\dim G/2}\omega_f^G(K_f^G) I_\infty(G,P)\\
&\qquad\times \int_{M(k_\infty)}\int_{U(k_\infty)} \varphi(mu)\delta_{P,\infty}(m)\,d\omega_\infty^M(m)\,d\omega_\infty^U(u)\,.
\end{aligned}$$

These imply the assertion. ∎

Corollary . *If Q is a standard k-parabolic subgroup of G, then*

$$\frac{C_Q}{C_G} = |D_k|^{\dim U_Q/2} \frac{\omega_f^Q(K_f^Q) I_\infty(M_Q, P^Q)}{\omega_f^G(K_f^G) I_\infty(G, P)} .$$

Proof. It follows from the following two equations;

$$\begin{aligned} \frac{\omega_f^P(K_f^P)}{\omega_f^G(K_f^G)} \cdot \frac{\omega_f^{M_Q}(K_f^{M_Q})}{\omega_f^{P^Q}(K_f^G \cap P^Q(\mathbb{A}_f))} &= \frac{\omega_f^U(K_f^G \cap U(\mathbb{A}_f))\omega_f^{M_Q}(K_f^{M_Q})}{\omega_f^G(K_f^G)\omega_f^{U^Q}(K_f^G \cap U^Q(\mathbb{A}_f))} \\ &= \frac{\omega_f^{U_Q}(K_f^G \cap U_Q(\mathbb{A}_f))\omega_f^{M_Q}(K_f^{M_Q})}{\omega_f^G(K_f^G)} . \end{aligned}$$

$$\frac{|D_k|^{(\dim P^Q - \dim M_Q)/2}}{|D_k|^{(\dim P - \dim G)/2}} = \frac{|D_k|^{-\dim U^Q/2}}{|D_k|^{-\dim U/2}} = |D_k|^{\dim U_Q/2} . \qquad \blacksquare$$

From now on, we assume that G is split over k. Then G has a model over $\mathfrak{O}$, i.e., there is a smooth affine group scheme $\mathfrak{G}$ of finite type over $\mathfrak{O}$ such that the generic fiber $\mathfrak{G} \times_{\mathfrak{O}} k$ is isomorphic to G and the special fiber $\mathfrak{G} \times_{\mathfrak{O}} \mathfrak{O}_\mathfrak{p}/\mathfrak{p}$ is a split reductive algebraic group over $\mathfrak{O}_\mathfrak{p}/\mathfrak{p}$ for all finite primes $\mathfrak{p} \in \mathfrak{V}_f$. In this case, we choose $K_\mathfrak{p}^G$ as $\mathfrak{G}(\mathfrak{O}_\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{V}_f$. Let $r_k(G)$ be the k-rank of the derived group G^{ss} of G and let $a_1^G, \cdots, a_{r_k(G)}^G$ be positive integers such that the polynomial in X

$$\prod_{i=1}^{r_k(G)} (X^{2a_i^G - 1} + 1)$$

is the Poincaré polynomial of the compact form of $G^{ss}(\mathbb{C})$. Then we have

$$\omega_f^G(K_f^G) = \sigma_k(G)^{-1} \prod_{i=1}^{r_k(G)} \zeta_k(a_i^G)^{-1} ,$$

where $\zeta_k(\cdot)$ denotes the Dedekind zeta function of k (cf.[15]). In a similar fashion, one has

$$\omega_f^Q(K_f^Q) = \omega_f^{M_Q}(K_f^{M_Q}) = \sigma_k(M_Q)^{-1} \prod_{i=1}^{r_k(M_Q)} \zeta_k(a_i^{M_Q})^{-1}.$$

Since the representations σ_G and σ_{M_Q} are trivial, $\sigma_k(G)/\sigma_k(M_Q)$ is equal to

$$(\mathrm{Res}_{s=1}\zeta_k(s))^{\dim Z_G - \dim Z_Q} .$$

Furthermore, it is known that $I_\infty(G, P)$ is described as

$$I_\infty(G,P) = \left\{ \prod_{\substack{\alpha \in \Phi_k \\ \alpha > 0}} \sqrt{\pi} \frac{\Gamma(c_\alpha/2)}{\Gamma((c_\alpha+1)/2)} \right\}^{r_1} \left\{ \prod_{\substack{\alpha \in \Phi_k \\ \alpha > 0}} 2\pi \frac{\Gamma(c_\alpha)}{\Gamma(c_\alpha+1)} \right\}^{r_2} , \tag{4.2}$$

where r_1 and r_2 denote the numbers of real and imaginary places of k, respectively, $\Gamma(\cdot)$ is the Gamma function and the constant c_α is defined by

$$c_\alpha = (\frac{1}{2} \sum_{\substack{\beta \in \Phi_k \\ \beta > 0}} \beta, \alpha^\vee)$$

for each positive root $\alpha \in \Phi_k$ (cf.[6],[12]). Summing up, we obtain the following estimate of γ_π;

Theorem 2. *Let G and K_f^G be as above, $\xi \in \mathfrak{R}_k^s(G) = \mathfrak{R}(G)$ be a maximal class, $\pi \in \xi$ be a k-rational representation of G and $Q = Q_\pi$ be the stabilizer of the highest weight space D_π in G. Then, for any K^G-invariant normalized height $||\cdot||_{\mathbb{A}}$ on $GL(V_\pi, \mathbb{A})V_\pi(k)$, one has*

$$\gamma_\pi(||\cdot||_{\mathbb{A}}) \geq \left\{ \frac{|D_k|^{\dim U_Q/2}}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{i=1}^{r_k(G)} \zeta_k(a_i^G)}{\prod_{j=1}^{r_k(M_Q)} \zeta_k(a_j^{M_Q})} \frac{I_\infty(M_Q, P^Q)}{I_\infty(G, P)} \frac{d_G e_Q \tau(G)}{d_Q \tau(Q)} \right\}^{\frac{2e_\xi}{[k:\mathbb{Q}]e_Q}}.$$

The factors $I_\infty(G,P)$ and $I_\infty(M_Q, P^Q)$ are calculated by the formula (4.2). ∎

We shall apply this theorem to classical split groups.

Example 2. The case of $G = GL(n)$. We use the same notations as in Example 1. We fix $1 \leq d \leq n-1$ and set $Q = Q_d$. For a positive integer q, let $\xi_d^q \in \mathfrak{R}(G)$ be the class whose highest weight is equal to $q\ell_d$ and let $\pi_d^q \in \xi_d^q$ be a k-rational representation. It follows from easy calculations that

$$d_G = n, \quad d_Q = d(n-d), \quad e_Q = d(n-d), \quad e_{\xi_d^q} = q\frac{d(n-d)}{n}$$

$$\frac{\omega_f^G(K_f^G)}{\omega_f^{M_Q}(K_f^{M_Q})} = \frac{1}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{i=n-d+1}^{n} \zeta_k(i)}{\prod_{j=2}^{d} \zeta_k(j)}$$

$$\frac{I_\infty(M_Q, P^Q)}{I_\infty(G,P)} = \left(\frac{\prod_{i=n-d+1}^{n} \pi^{-i/2}\Gamma(i/2)}{\prod_{j=2}^{d} \pi^{-j/2}\Gamma(j/2)} \right)^{r_1} \left(\frac{\prod_{i=n-d+1}^{n} (2\pi)^{1-i}\Gamma(i)}{\prod_{j=2}^{n} (2\pi)^{1-j}\Gamma(j)} \right)^{r_2}$$

We note that

$$\tau(G) = \tau(Q) = 1 \ \text{ and } \ \mathrm{Res}_{s=1}\zeta_k(s) = |D_k|^{-1/2} \frac{2^{r_1}(2\pi)^{r_2} h_k R_k}{w_k},$$

where h_k is the class number of k, R_k the regulator of k and w_k the number of roots of 1 in k. If we put $Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$, we obtain

$$\gamma_{\pi_d^q} \geq \left(\frac{|D_k|^{d(n-d)/2} n}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{i=n-d+1}^{n} Z_k(i)}{\prod_{j=2}^{d} Z_k(j)} \right)^{2q/([k:\mathbb{Q}]n)}.$$

When $q = 1$, this estimate is the same as Thunder's one (0.2).

Example 3. The case of $G = SO(n,n+1), Sp(2n)$ or $SO(n,n)$. Let V be an m-dimensional vector space over k equipped with a non-degenerate ε-symmetric bilinear form Ψ with Witt index n. We consider the following three cases;

$$\begin{aligned} &(B_n)\ m = 2n+1 \text{ and } \varepsilon = 1. \\ &(C_n)\ m = 2n \text{ and } \varepsilon = -1. \\ &(D_n)\ m = 2n \text{ and } \varepsilon = 1. \end{aligned}$$

We choose a k-basis $e_1, \cdots, e_m$ of $V(k)$ by which Ψ is represented as follows;

$$\Psi = \begin{cases} \begin{pmatrix} 0 & 0 & J_n \\ 0 & 2 & 0 \\ J_n & 0 & 0 \end{pmatrix} & (B_n) \\ \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} & (C_n) \\ \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} & (D_n) \end{cases} \qquad \text{where } J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let L be the free $\mathfrak{O}$-lattice generated by $e_1, \cdots, e_m$. We set

$$\begin{aligned} G &= \{g \in SL(V) \ : \ {}^t g \Psi g = \Psi\}, \\ K_v^G &= \{g \in G(k_v) \ : \ ||gx||_{L,v} = ||x||_{L,v} \text{ for all } x \in V(k_v)\}, \quad (v \in \mathfrak{V}) \end{aligned}$$

Let S be the group of diagonal matrices in G and P the group of upper triangular matrices in G. For $g \in S$, $\epsilon_i(g)$ denotes the i-th diagonal element of g. Then $\epsilon_1, \cdots, \epsilon_n$ is a $\mathbb{Z}$-basis of $\mathbf{X}^*(S)$. The simple roots α_i and the fundamental weights ℓ_i are given as follows;

$$\alpha_i = \epsilon_i - \epsilon_{i+1},\ (1 \le i \le n-1), \quad \alpha_n = \begin{cases} \epsilon_n & (B_n) \\ 2\epsilon_n & (C_n) \\ \epsilon_{n-1} + \epsilon_n & (D_n) \end{cases}$$

$$\ell_i = \begin{cases} \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-2} + \epsilon_{n-1} - \epsilon_n) & (D_n \text{ and } i = n-1) \\ \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) & (B_n \text{ and } i = n \text{ or } D_n \text{ and } i = n) \\ \epsilon_1 + \cdots + \epsilon_i & (\text{otherwise}) \end{cases}$$

We fix $1 \le d \le n$ and a positive integer q so that $q\ell_d \in \mathbf{X}^*(S)$. Let $\xi_d^q \in \mathfrak{R}(G)$ be the class whose highest weight is equal to $q\ell_d$ and let $\pi_d^q \in \xi_d^q$ be a k-rational representation of G. Then $Q_d = Q_{\pi_d^q}$ be the standard maximal parabolic subgroup of G associated with $\Delta \backslash \{\alpha_d\}$. We denote by U_d the unipotent radical of Q_d and by M_d the Levi subgroup of Q_d containing S. For convenience, we define the constant $\delta(D_n)$ by

$$\delta(D_n) = \begin{cases} 1 & (D_n \text{ and } d = n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

It is easy to check the following

$$\dim U_d = \begin{cases} \frac{1}{2}d(d+1) + 2d(n-d) & (B_n \text{ or } C_n) \\ \frac{1}{2}(d+\delta(D_n))(d+\delta(D_n)-1) + 2d(n-d-\delta(D_n)) & (D_n) \end{cases}$$

and

$$\begin{aligned} & d_G = 1, \quad d_{Q_d} = d + \delta(D_n), \\ & e_{Q_d} = \begin{cases} 2\dim U_d & (C_n \text{ and } d = n \text{ or } D_n \text{ and } d \geq n-1) \\ \dim U_d & (\text{otherwise}) \end{cases} \\ (4.3) \qquad & \frac{\tau(G)}{\tau(Q_d)} = \begin{cases} 2 & (B_n \text{ and } d = n \text{ or } D_n \text{ and } d \geq n-1) \\ 1 & (\text{otherwise}) \end{cases} \\ & e_{\xi_d^q} = \begin{cases} qn/2 & (B_n \text{ and } d = n \text{ or } D_n \text{ and } d \geq n-1) \\ qd & (\text{otherwise}) \end{cases} \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \prod_{i=1}^{r_k(G)} \zeta_k(a_i^G) \prod_{j=1}^{r_k(M_d)} \zeta_k(a_j^{M_d})^{-1} \\ (4.4) \qquad & = \frac{1}{\prod_{j=2}^{d+\delta(D_n)} \zeta_k(j)} \times \begin{cases} \prod_{i=n-d+1}^{n} \zeta_k(2i) & (B_n \text{ or } C_n) \\ \frac{\zeta_k(n)}{\zeta_k(n-d)} \prod_{i=n-d}^{n-1} \zeta_k(2i) & (D_n \text{ and } d \leq n-2) \\ \zeta_k(n) \prod_{i=1}^{n-1} \zeta_k(2i) & (D_n \text{ and } d \geq n-1) \end{cases} \end{aligned}$$

We define the function $\Lambda_k(s)$ by

$$\Lambda_k(s) = \left\{\pi^{-s/2}\Gamma(s/2)\right\}^{r_1} \left\{(2\pi)^{1-s}\Gamma(s)\right\}^{r_2}.$$

For $x \in \mathbb{R}$, we write $[x]$ for the largest integer not exceeding x, and define $c_{n,d}$ by

$$(B_n) \quad \prod_{h=n-d+1}^{n} \frac{\Lambda_k(2h)}{\Lambda_k(2h-1)} \prod_{i=1}^{[(n-d)/2]} \frac{\Lambda_k(2i)}{\Lambda_k(2n-2d-2i+1)} \prod_{j=1}^{[n/2]} \frac{\Lambda_k(2n-2j+1)}{\Lambda_k(2j)},$$

$$(C_n) \quad \frac{\Lambda_k(n+1)}{\Lambda_k(n-d+1)} \prod_{i=1}^{[(n-d)/2]} \frac{\Lambda_k(2i+1)}{\Lambda_k(2n-2d-2i+2)} \prod_{j=1}^{[n/2]} \frac{\Lambda_k(2n-2j+2)}{\Lambda_k(2j+1)},$$

$$(D_n) \quad \prod_{i=1}^{[(n-d)/2]} \frac{\Lambda_k(2i-1)}{\Lambda_k(2n-2d-2i)} \prod_{j=1}^{[n/2]} \frac{\Lambda_k(2n-2j)}{\Lambda_k(2j-1)}.$$

Then we have

$$\frac{I_\infty(M_d, P^{Q_d})}{I_\infty(G,P)} = c_{n,d} \prod_{i=1}^{d+\delta(D_n)} \frac{\Lambda_k(n-d+i)}{\Lambda_k(i)}. \tag{4.5}$$

(4.3)–(4.5) yield an explicit description of a lower bound of $\gamma_{\pi_d^q}$. For instance, when $d = q = 1$, then $\pi_1 : G \to GL(V)$ is the natural representation and

$$\gamma_{\pi_1} \geq \begin{cases} \left\{ \dfrac{|D_k|^{n-1/2}(2n-1)}{\mathrm{Res}_{s=1}\zeta_k(s)} Z_k(2n) \right\}^{2/((2n-1)[k:\mathbb{Q}])} & (B_n \text{ or } C_n) \\ \left\{ \dfrac{|D_k|^{n-1}(2n-2)}{\mathrm{Res}_{s=1}\zeta_k(s)} \dfrac{Z_k(2(n-1))Z_k(n)}{Z_k(n-1)} \right\}^{1/((n-1)[k:\mathbb{Q}])} & (D_n) \end{cases}$$

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560-0043
Japan
watanabe@math.wani.osaka-u.ac.jp

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