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Ladder representation norms for hermitian symmetric groups

Mark G. Davidson and Ronald J. Stanke*

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Abstract. Let G be a connected noncompact simple Hermitian symmetric group with finite center. Let $\mathcal{H}(\lambda)$ denote the geometric realization of an irreducible unitary highest weight representation with highest weight λ. Then $\mathcal{H}(\lambda)$ consists of vector-valued holomorphic functions on G/K and the action of G on $\mathcal{H}(\lambda)$ is given in terms of a factor of automorphy. For highest weights λ corresponding to ladder representations, we obtain the G-invariant inner product on $\mathcal{H}(\lambda)$. This inner product arises as the pullback of an isometry $\Phi_\lambda : \mathcal{H}(\lambda) \rightarrow \mathcal{H}(\tilde{\lambda}) \otimes Y_\lambda$, where Y_λ is finite dimensional and the weight $\tilde{\lambda}$ corresponds to a scalar valued representation. In all but finitely many cases the G-invariant inner product on $\mathcal{H}(\tilde{\lambda})$ is known and is used to express the G-invariant inner product on $\mathcal{H}(\lambda)$. Explicit examples are given for families of ladder representations of $SU(p,q)$ and $SO^*(2n)$. Finally, inversion formulas for unitary intertwining operators between $\mathcal{H}(\lambda)$ and any equivalent realization are exhibited.

1. Introduction

The ladder representations are among the most reductive of the unitary highest weight representations. To be more specific, we refer to the Enright, Howe and Wallach classification [5] where the unitary highest weight modules associated to a K-type are parametrized by the union of a half-line and a set of isolated points. The endpoint of the half line and the isolated points are called reduction points because the corresponding Verma modules are reducible. The discrete series representations and the generalized limits of discrete series [8] lie within the half-line on the left and their unitary structures are well known. The ladder representations, on the other hand, appear among the right most isolated points and their unitary structures have been recently studied. [12], [13], [14].

In this article, we describe the invariant unitary norm of highest weight ladder representations, realized in the Harish-Chandra setting [6] where the relevant Hermitian symmetric space is a bounded domain. The unitary norm is obtained by constructing an equivariant isometry which maps from the ladder representation to a tensor product that has a uniquely determined scalar-valued unitary

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representation as one of it factors. The form of the unitary norm, being the pullback of this isometry, fundamentally depends on the norm of the corresponding scalar-valued representation. For all but finitely many ladder representations, the scalar-valued representation lies in the discrete series or limits thereof. The upshot of this construction is explicit formulas for norms of ladder representations. We now give a more detailed description of the results.

Let G be a connected noncompact simple Hermitian symmetric group with finite center. Let K denote a maximal compact subgroup of G. Highest weight representations of G are realized here as holomorphically induced multiplier representations. In this setting, the representation space $\mathcal{H}(\lambda)$ with highest weight λ consists of holomorphic, vector-valued functions on the Hermitian symmetric space G/K, viewed as a bounded domain. Ladder representations are characterized by the property that the K-types occur as a multiplicity-free chain.

Let $\Lambda_l(G)$ (resp. $\Lambda_0(G)$) denote the set of highest weights corresponding to unitary highest weight ladder (resp. scalar-valued) representations of G. Let $\mathcal{H}(\lambda)^K$ denote the K-finite vecotrs in $\mathcal{H}(\lambda)$. For $\lambda \in \Lambda_l(G)$ and $\mu \in \Lambda_0(G)$, we define in section five a K-map $\phi : \mathcal{H}(\lambda)^K \to \mathcal{H}(\mu) \otimes V_{\lambda-\mu}$, where $V_{\lambda-\mu}$ is K-irreducible with highest weight $\lambda - \mu$. For each $\lambda \in \Lambda_l(G)$, we uniquely determine a weight $\mu = \widetilde{\lambda}$ such that if $\widetilde{\lambda} \in \Lambda_0(G)$, the above map extends to a G-equivariant isometry (cf Theorem 6.1). The invariant inner product on $\mathcal{H}(\lambda)$ arises as the pullback of the inner product on $\mathcal{H}(\widetilde{\lambda}) \otimes V_{\lambda-\widetilde{\lambda}}$.

As the explicit examples for $G = SO^*(2n)$ in section seven show, it is not always the case that $\widetilde{\lambda} \in \Lambda_0(G)$. However, if one has $\widetilde{\lambda} \in \Lambda_0(\widetilde{G})$, where $\widetilde{G}$ is a finite covering group of G, then one can proceed as indicated above to obtain a $\widetilde{G}$-invariant, and thus G-invariant, inner product on $\mathcal{H}(\lambda)$.

For all but finitely many $\lambda \in \Lambda_l(G)$, the associated weight $\widetilde{\lambda}$ corresponds to a scalar-valued discrete series or generalized limit of discrete series representation. Inner products for such representations $\mathcal{H}(\widetilde{\lambda})$ are explicitly known and are given by integration over G/K or one of its boundary components. For such λ, the form of the inner product for $\mathcal{H}(\lambda)$ in turn inherits this feature (cf. Theorem 7.1).

In order to provide some concrete examples, we consider families of ladder representations of $SU(p,q)$ and $SO^*(2n)$ whose highest weights are found in Table 11.1 of [2]. For example, if $G = SU(p,q)$ then $\lambda = m\omega_{p-1} - (m+1)\omega_p \in \Lambda_l(G)$. If $m+1 \geq p+q$, then the inner product on $\mathcal{H}(\lambda)$ has the explicit form

$$(f,g)_\lambda = d_\lambda \int_D (f(z), g(z)) det(I - zz^*)^{m+1-p-q} dz,$$

where $D \subset \mathbb{C}^{p\times q}$ denotes the bounded realization of G/K, dz is Lebesgue measure and d_λ is a constant. The constraint $m+1 \geq p+q$ is precisely the condition for which $\mathcal{H}(\widetilde{\lambda})$ is a discrete series representation.

As an application of the explicit form of the invariant inner products given in section seven, we compute in section eight the form of the inverse of intertwining operators $\Xi : \mathbb{H}_\lambda \to \mathcal{H}(\lambda)$, where $\mathbb{H}_\lambda$ is any equivalent realization of $\mathcal{H}(\lambda)$. In the special case where $G = SU(p,q)$, $SO^*(2n)$ or $Sp(n,\mathbb{R})$ and $\mathbb{H}_\lambda$ is the harmonic realization of $\mathcal{H}(\lambda)$, we express the inverse for the unitary intertwining operator found in section eight of [3] as an integral over G/K. This expression lacks the differentiation and limits found in [12], [13], and [14], where $G = SU(p,q)$ is considered.

2. Preliminaries

Let G be a connected noncompact simple Lie group with finite center. Let K denote a maximal compact subgroup of G and assume that the space G/K is Hermitian symmetric. Choose a Cartan subgroup T in K. Let $\mathfrak{g}_o, \mathfrak{k}_o$ and $\mathfrak{t}_o$ denote the Lie algebras of G, K and T, respectively. Fix a Cartan decomposition $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$. By notational convention, the removal of the subscript $_0$ denotes complexification. Then there exists a decomposition $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$, where $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{p}$ satisfying

$$[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k} \text{ and } \overline{\mathfrak{p}}_\pm = \mathfrak{p}_\mp, \tag{2.1}$$

where the bar denotes conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_o$.

The subalgebra $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and we let Φ denote the roots corresponding to the pair $(\mathfrak{g}, \mathfrak{t})$. We let Φ_c and Φ_n denote the set of compact and noncompact roots, respectively. The root space corresponding to $\alpha \in \Phi$ is denoted by $\mathfrak{g}_\alpha$. We choose a positive system of roots Φ^+ so that, if $\Phi_n^+ = \Phi^+ \cap \Phi_n$ and $\Phi_n^- = (-\Phi^+) \cap \Phi_n$, then

$$\mathfrak{p}_+ = \oplus_{\alpha \in \Phi_n^+} \mathfrak{g}_\alpha \text{ and } \mathfrak{p}_- = \oplus_{\alpha \in \Phi_n^-} \mathfrak{g}_\alpha. \tag{2.2}$$

Let B denote the Killing form and θ denote Cartan involution on $\mathfrak{g}$. We choose root vectors $E_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Phi$, so that $B(E_\alpha, E_{-\alpha}) = 2/(\alpha, \alpha)$ and $\theta(\overline{E}_\alpha) = -E_{-\alpha}$, where $(\cdot, \cdot)$ is the standard inner product on the real space of linear functionals on $\mathfrak{t}$ taking purely imaginary values on $\mathfrak{t}_o$. If we set $H_\alpha = [E_\alpha, E_{-\alpha}]$, then

$$[H_\alpha, E_\alpha] = 2E_\alpha \text{ and } [H_\alpha, E_{-\alpha}] = -2E_{-\alpha}. \tag{2.3}$$

3. Holomorphically induced multiplier representations

In this article, we shall work exclusively with the geometric realization of unitary highest weight representations of G. In this realization, the representation space consists of vector-valued functions that are holomorphic on G/K. We adopt the Harish-Chandra realization [6] of G/K as a bounded domain of $\mathfrak{p}_+$. The representation of G is a multiplier representation that is expressed in terms of a factor of automorphy. We now discuss some basic facts of this realization. For additional details, see [6], [10].

Let (τ, V_τ) denote an irreducible unitary representation of K with representation space V_τ. Then τ extends to a holomorphic representation of $K_{\mathbb{C}}$ on V_τ. Let $j : G \times \overline{D} \to K_{\mathbb{C}}$ be the continuous map in Proposition (4.7) of [4]. Define the factor of automorphy J_τ by $J_\tau(g, z) = \tau(j(g, z))$ for $(g, z) \in G \times \overline{D}$. Let $\mathcal{O}(D, V_\tau)$ denote the space of V_τ-valued holomorphic functions on D. We define the associated multiplier representation $T = T_\tau$ of G on $\mathcal{O}(D, V_\tau)$ by

$$(T(g)F)(z) = J_\tau(g^{-1}, z)^{-1} F(g^{-1} \cdot z), \tag{3.1}$$

for $F \in \mathcal{O}(D, V_\tau)$, $g \in G$ and $z \in D$. Here $g \cdot z$ denotes the action of G on $\overline{D}$.

It is well-known that the K-finite vectors in $\mathcal{O}(D, V_\tau)$ are the polynomial functions (cf. Lemma 1.6 of [2]). Such functions have unique extensions to $\mathfrak{p}_+$

since D is open in $\mathfrak{p}_+$. Let $\mathbb{V}_\tau$ denote the $V_\tau-$valued polynomial functions on $\mathfrak{p}_+$.

Let $\mathbf{1}_v$ denote the constant function on $\mathfrak{p}_+$ whose value is $v \in V_\tau$ and let $\mathbb{V}_\tau^0$ denote the linear span of these constant functions. Let $L(\tau)$ denote the $\mathfrak{g}$-module generated by $\mathbb{V}_\tau^0$. It is known that $L(\tau)$ lies in each $\mathfrak{g}$-invariant submodule of $\mathbb{V}_\tau$. Consequently, $L(\tau)$ is irreducible. Moreover, if $v_\tau \in V_\tau$ is a highest weight vector for $\mathfrak{k}$, then $\mathbf{1}_{v_\tau}$ is a highest weight vector for $\mathfrak{g}$.

We conclude this section with an elementary fact regarding particular weight spaces of $\mathbb{V}_\tau$. Write Φ_n^+ as the set $\{\alpha_0, \alpha_1, \ldots, \alpha_s\}$, where α_0 is the simple noncompact root, and let $E_\alpha \in \mathfrak{g}_\alpha$ be chosen as in section two. Let $\gamma = (\gamma_0, \ldots, \gamma_s) \in \mathbb{N}_0^{s+1}$, where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, denote a multindex and $(\cdot, \cdot)$ denote the standard Hermitian inner product on $\mathfrak{g}$. By (6.3) of [4], for each weight vector v_μ of V_τ with weight μ, one observes that

$$f_{\gamma,\mu}(z) = (z, E_{\alpha_0})^{\gamma_0} \cdots (z, E_{\alpha_s})^{\gamma_s} v_\mu, \qquad z \in \mathfrak{p}_+,$$

is a nonzero weight vector with weight $-\sum_{i=0}^s \gamma_i \alpha_i + \mu$. Denote by W_ω the weight space in $\mathbb{V}_\tau$ with weight ω. Since $\mathbb{V}_\tau$ is spanned by the monomials $f_{\gamma,\mu}$, we have $\mathbb{V}_\tau = \oplus_\omega W_\omega$ where $W_\omega = \mathrm{span}_{\mathbb{C}}\{f_{\gamma,\mu} : \omega = -\sum_{i=0}^s \gamma_i \alpha_i + \mu\}$.

Proposition 3.1. *Let (τ, V_τ) be an irreducible unitary representation of K with highest weight λ. If α_0 denotes the simple noncompact root, then, for each integer $m \geq 0$, the weight space $W_{\lambda - m\alpha_0}$ is spanned by the vector $(\cdot, E_{\alpha_0})^m v_\lambda$, where v_λ is a unit highest weight vector for V_τ.*

Proof. By the above discussion, it suffices to show that if $-\sum_{i=0}^s \gamma_i \alpha_i + \mu = \lambda - m\alpha_0$, then $\mu = \lambda, \gamma_0 = m$ and $\gamma_1 = \ldots = \gamma_s = 0$. Recall that $\mu = \lambda - Q$ where Q is a sum of elements from Φ_c^+. If $-\sum_{i=0}^s \gamma_i \alpha_i + \mu = \lambda - m\alpha_0$, then $\sum_{i=0}^s \gamma_i \alpha_i + Q = m\alpha_0$. By writing each $\alpha_i, 1 \leq i \leq s$, and Q as the sum of simple roots, we conclude $\gamma_1 = \ldots = \gamma_s = 0$ and $Q = 0$. Consequently, we have $\mu = \lambda$ and $\gamma_0 = m$. ∎

4. Unitary highest weight representations

The purpose of this section is to give some general results on unitary highest weight representations of G. In particular, we point out that the results of [3] extend from the setting of a linear group G to the setting of a Hermitian, connected noncompact simple Lie group G with finite center. As a consequence of this extension, we employ a theorem of Kunze [11] which characterizes the unitary space of functions on G/K on which G acts irreducibly. The representation in this case is the representation T given in section three. However, the unitary norm in Kunze's result is given in an abstract way. The goal of the remaining sections is to make explicit this norm in specific cases.

Let λ denote the highest weight for the irreducible unitary representation (τ, V_τ). We change notation slightly and let $L(\lambda)$ denote the irreducible $\mathfrak{g}$-module defined in section three. Similarly, v_λ will replace v_τ and $\mathbf{1}_\lambda$ will replace $\mathbf{1}_{v_\tau}$. Let $\Lambda(\mathfrak{g})$ denote the set of highest weights such that the $\mathfrak{g}$-module $L(\lambda)$ is unitarizable. The set $\Lambda(\mathfrak{g})$ has been completely classified [5], [9]. We are interested in the subset $\Lambda(G) \subset \Lambda(\mathfrak{g})$ of highest weights that correspond to unitary highest weight representations of G and, in particular, their realizations as spaces of vector valued functions on G/K.

We begin with a main result in [3] on positive definite operator-valued kernel functions. If G is linear, $\lambda \in \Lambda(G)$, and J_λ is the factor of automorphy in section three, define $Q : D \times D \to Aut(V_\lambda)$ by

$$Q(w,z) = J(g_2, 0)J(g_2^{-1}, z)^{*-1}, \tag{4.1}$$

where $g_1, g_2 \in G$, $g_1 \cdot 0 = z$ and $g_2 \cdot 0 = w$. By Theorem 7.1 and Proposition 7.3 of [3], Q defines a positive definite operator-valued kernel function that is holomorphic in w and antiholomorphic in z.

There are only some minor details to check to see that the proofs in [3] extend to the setting of finite covering groups. For example, the calculations needed to establish the results through Proposition 6.2 are done at the level of the Lie algebra and so trivially extend. Theorem 6.1 extends by utilizing arguments similar to those found in the proof of Lemma (4.15) of [4]. One uses the connectedness of the subgroup A along with the finiteness of the center of G to establish the calculation on p. 26 of [3]. From this, both results from [3] follow by the arguments provided there.

Equipped with the positive definite operator Q in (4.1), one can construct a Hilbert space $\mathcal{H}(\lambda)$ of V_λ-valued functions on D for which the representation of G is given by the formula for T in (3.1). The details of the construction of $\mathcal{H}(\lambda)$ are found in [11]. We summarize the essential properties.

Proposition 4.1. *If $\lambda \in \Lambda(G)$, there exists a Hilbert space $\mathcal{H}(\lambda)$ of continuous functions $f : D \to V_\lambda$ with inner product $(\cdot,\cdot)_\lambda$ such that*

(1) for each $z \in D$, the map $e_z : \mathcal{H}(\lambda) \to V_\lambda$ defined by $e_z(f) = f(z)$ is continuous,

(2) $Q(w,z) = e_w e_z^*$ for all $w, z \in D$,

(3) the linear space $\mathrm{span}_{\mathbb{C}}\{w \to Q(w,z)v : z \in D, v \in V_\lambda\}$ is dense in $\mathcal{H}(\lambda)$,

(4) $(Q(\cdot,z)v, Q(\cdot,w)u)_\lambda = (Q(w,z)v,u)$ for all $z, w \in D$ and $v, u \in V_\lambda$,

(5) the formula (3.1) defines a strongly continuous unitary representation of G on $\mathcal{H}(\lambda)$.

Remark 4.2. *Since Q is holomorphic in the first variable, it follows that $\mathcal{H}(\lambda)$ consists of holomorphic functions on D. By (4.1), $Q(z,0)$ is the identity on V_λ. Then by proposition 4.1 (3) it follows that $\mathcal{H}(\lambda)$ contains the constant functions $\mathbf{1}_v$, $v \in V_\lambda$. If $\mathcal{H}(\lambda)^K$ denotes the K-finite vectors in $\mathcal{H}(\lambda)$, one concludes $\mathcal{H}(\lambda)^K = L(\lambda)$.*

For weights μ and α, we let the number $2(\mu,\alpha)/(\alpha,\alpha)$ be denoted by μ_α.

Proposition 4.3. *Let $\lambda \in \Lambda(G)$. Then for each integer $m \geq 0$, the weight vector $T(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)$ is a nonzero highest weight vector for K satisfying:*

(1) $\| T(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda) \|_\lambda^2 = (-1)^m m!\ \lambda_{\alpha_0}(\lambda_{\alpha_0} - 1) \cdots (\lambda_{\alpha_0} - m + 1)$,

(2) $T(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda) = c_\lambda(m)(\cdot, E_{\alpha_0})^m v_\lambda$,
where $c_\lambda(m) = \frac{(-1)^m}{(E_{\alpha_0}, E_{\alpha_0})^m}\lambda_{\alpha_0}(\lambda_{\alpha_0} - 1)\cdots(\lambda_{\alpha_0} - m + 1)$.

Proof. We suppress the use of the symbol T and denote the $\mathfrak{g}$-action by juxtaposition. To show (1), recall from Proposition 4.1 that $(\cdot,\cdot)_\lambda$ is the unitary inner product for $\mathcal{H}(\lambda)$, so we have $\| \overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) \|^2_\lambda = (-1)^m(E^m_{\alpha_0}\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda), \mathbf{1}_\lambda)_\lambda$. But by Lemma 4.2 (b) of [3], one knows

$$E^m_{\alpha_0}\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) = m!\lambda_{\alpha_0}(\lambda_{\alpha_0} - 1)\cdots(\lambda_{\alpha_0} - m + 1)\mathbf{1}_\lambda. \tag{4.2}$$

Since $(\mathbf{1}_\lambda, \mathbf{1}_\lambda)_\lambda = (v_\lambda, v_\lambda) = 1$, we have

$$\| \overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) \|^2_\lambda = (-1)^m m!\lambda_{\alpha_0}(\lambda_{\alpha_0} - 1)\cdots(\lambda_{\alpha_0} - m + 1),$$

which establishes (1).

To prove (2), we observe that the weight vector $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda)$ has weight $\lambda - m\alpha_0$. But $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) \in \mathcal{H}(\lambda)^K = L(\lambda) \subset \mathbb{V}_\lambda$, the space of K-finite vectors in $\mathcal{O}(D, V_\lambda)$. By Proposition 3.1, the weight space for the weight $\lambda - m\alpha_0$ in $\mathbb{V}_\lambda$ is spanned by the vector $(\cdot, E_{\alpha_0})^m v_\lambda$. Thus there exists a constant c such that $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) = c(\cdot, E_{\alpha_0})^m v_\lambda$. From (6.3) of [4], we see that $E^m_{\alpha_0}$ is the differential operator $(-1)^m\delta(E_{\alpha_0})^m$, where δ is the directional derivative. A straightforward computation shows that $\delta(E_{\alpha_0})^m((\cdot, E_{\alpha_0})^m v_\lambda) = m!\ (E_{\alpha_0}, E_{\alpha_0})^m v_\lambda$. Thus, by applying $E^m_{\alpha_0}$ to the equation $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) = c(\cdot, E_{\alpha_0})^m v_\lambda$, one concludes by (4.2) that $m!\ \lambda_{\alpha_0}(\lambda_{\alpha_0} - 1)\cdots(\lambda_{\alpha_0} - m + 1) = c(-1)^m m!\ (E_{\alpha_0}, E_{\alpha_0})^m$. This establishes (2).

Since α_0 is the simple noncompact root, it follows that for each $\alpha \in \Phi^+_c$, one has $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha_o}] = 0$. Consequently, for $\alpha \in \Phi^+_c, E_\alpha \cdot \overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) = \overline{E}^m_{\alpha_0}E_\alpha(\mathbf{1}_\lambda) = 0$ since $\mathbf{1}_\lambda$ is a highest weight vector. This shows that $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda)$ is a highest weight vector for K.

To show that $\overline{E}^m_{\alpha_0}(\mathbf{1}_\lambda) \neq 0$, it suffices by (1) to show that $\lambda_{\alpha_0} < 0$. For this, we refer to the proof of Theorem 5.1 of [3]. There it is shown that if μ is any weight of V_λ, γ_1 is the maximal root of $\mathfrak{p}_+$ and $\lambda \in \Lambda(G)$, then $\mu_{\gamma_1} < 0$. We observe that $-\alpha_0$ is the highest weight of $\mathfrak{p}_-$ and that $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are dual via the Killing form. Thus, if σ denotes the longest element of $\mathcal{W}_K$ (the Weyl group for K), then $-\sigma(\gamma_1) = -\alpha_0$. But then $\lambda_{\alpha_0} = 2(\lambda, \alpha_0)/(\alpha_0, \alpha_0) = 2(\sigma^{-1}(\lambda), \sigma^{-1}(\alpha_0))/(\sigma^{-1}(\alpha_0), \sigma^{-1}(\alpha_0)) = 2(\sigma^{-1}(\lambda), \gamma_1)/(\gamma_1, \gamma_1) = \sigma^{-1}(\lambda)_{\gamma_1}$. But $\mathcal{W}_K$ permutes the weights of V_λ. Putting $\mu = \sigma^{-1}(\lambda)$, we have $\lambda_{\alpha_0} = \mu_{\gamma_1} < 0$. ∎

5. Scalar-valued representations and isometries

Recall that $\Lambda(G)$ denotes the set of highest weights that correspond to irreducible, unitary highest weight representations of G. Let $\Lambda_0(G) \subset \Lambda(G)$ denote the subset of nonzero highest weights μ for which the corresponding representation (τ_μ, V_μ) of K satisfies $\dim(V_\mu) = 1$. We refer to the corresponding representation T_μ on $\mathcal{H}(\mu)$ as a scalar-valued representation of G. The purpose of this section is to associate to $\lambda \in \Lambda(G)$ a unique $\widetilde{\lambda} \in \Lambda_0(G)$ for which an isometry exists between a certain K-invariant subspace of $\mathcal{H}(\lambda)$ and a tensor product $\mathcal{H}(\widetilde{\lambda}) \otimes Y_\lambda$, where Y_λ is a finite dimensional Hilbert space.

For $\mu \in \Lambda_0(G)$, we let $\mathbb{C}_\mu$ denote the corresponding one dimensional K space and let $1_\mu \in \mathbb{C}_\mu$ denote a unit vector. The dual $\mathbb{C}_\mu^*$ of $\mathbb{C}_\mu$ is a K space with highest weight $-\sigma(\mu)$, where σ is the longest element of the Weyl group $\mathcal{W}_K$. Since $\dim(\mathbb{C}_\mu) = 1$, it follows that $\sigma(\mu) = \mu$. The dual of $\mathbb{C}_\mu$ is thus identified with $\mathbb{C}_{-\mu}$ and K acts on $\mathbb{C}_{-\mu}$ by multiplication by $\tau_\mu(k)^{-1}, k \in K$. For $\mu \in \Lambda_0(G)$ and $\lambda \in \Lambda(G)$, we form the K-space $\mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$ and observe that the natural representation of K on this space reduces to $1 \otimes \tau_\lambda \otimes 1$, where 1 denotes the trivial representation. It is routine to check that for each $\mu \in \Lambda_0(G)$ and $\lambda \in \Lambda(G)$, the mapping $\psi_{\mu,\lambda} : \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu} \to V_\lambda$ defined by

$$\psi_{\mu,\lambda}(\Sigma c_i \otimes v_i \otimes d_i) = \sum_i (c_i, 1_\mu)(d_i, 1_{-\mu})v_i$$

is a K-equivariant isometry.

For $\mu \in \Lambda_0(G)$, let $P(\mathfrak{p}_+, \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu})$ denote the K-space of $\mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$-valued polynomials on $\mathfrak{p}_+$. The K-action here is $(k \cdot f)(z) = (1 \otimes \tau_\lambda(k) \otimes 1)(f(k^{-1}.z))$ for $k \in K$ and $z \in \mathfrak{p}_+$. For $f \in P(\mathfrak{p}_+, \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu})$, define $R_{\mu,\lambda}(f)$ by

$$(R_{\mu,\lambda}(f))(z) = \psi_{\mu,\lambda}(f(z)), z \in \mathfrak{p}_+. \tag{5.1}$$

Then $R_{\mu,\lambda}$ defines a K-isomorphism of $P(\mathfrak{p}_+, \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu})$ onto $\mathbb{V}_\lambda$, the K-space of V_λ-valued polynomials on $\mathfrak{p}_+$.

For $\lambda \in \Lambda(G)$, we let $\{v_i\}$ denote an orthonormal basis of V_λ. For $\mu \in \Lambda_0(G)$, we define a map $\phi_{\mu,\lambda} : \mathcal{H}(\lambda)^K \to P(\mathfrak{p}_+, \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu})$ by the expression

$$\phi_{\mu,\lambda}(f)(z) = \sum_i (f(z), v_i) 1_\mu \otimes v_i \otimes 1_{-\mu}. \tag{5.2}$$

By a standard calculation, $\phi_{\mu,\lambda}$ is independent of the choice of orthonormal basis $\{v_i\}$. But then, for $k \in K$, one has

$$\begin{aligned}
\phi_{\mu,\lambda}(T(k)f)(z) &= \sum_i ((T(k)f)(z), v_i) 1_\mu \otimes v_i \otimes 1_{-\mu} \\
&= \sum_i (\tau_\lambda(k) f(k^{-1} \cdot z), v_i) 1_\mu \otimes v_i \otimes 1_{-\mu} \\
&= \sum_i (f(k^{-1} \cdot z), \tau_\lambda(k^{-1}) v_i) 1_\mu \otimes v_i \otimes 1_{-v} \\
&= (1 \otimes \tau_\lambda(k) \otimes 1)(\phi_{\mu,\lambda}(f)(k^{-1}.z)) \\
&= (k \cdot \phi_{\mu,\lambda} f)(z).
\end{aligned}$$

Thus $\phi_{\mu,\lambda}$ is a K-map. In addition, it is clear by the definition of $R_{\mu,\lambda}$ that

$$R_{\mu,\lambda} \circ \phi_{\mu,\lambda} = Id_{\mathcal{H}(\lambda)^K}. \tag{5.3}$$

In particular, $\phi_{\mu,\lambda}$ is injective.

For $\lambda \in \Lambda(G)$, let $V_{\lambda - m\alpha_0} \subset \mathcal{H}(\lambda)^K$ denote the K-irreducible space generated by the K-highest weight vector $T(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)$ in Proposition 4.3. If $\mu \in \Lambda_0(G)$, note that $\mathcal{H}(\mu)^K$ is a K-invariant space of $\mathbb{C}_\mu$-valued polynomials on $\mathfrak{p}_+$. Consequently, $\mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$ can be identified with a subspace of $P(\mathfrak{p}_+, \mathbb{C}_\mu \otimes V_\lambda \otimes \mathbb{C}_{-\mu})$.

Finally, note that $\mu \in \Lambda_0(G)$ is uniquely determined by the value of (μ, α_0). Indeed, $\mu \in \Lambda_0(G)$ is of the form $\mu = r\omega_0$ where $r \in \mathbb{R}$ and ω_0 denotes the fundamental weight corresponding to α_0.

Proposition 5.1. *Let $\lambda \in \Lambda(G)$, $\mu \in \Lambda_0(G)$ and let $B(\lambda)$ denote the K-space $\oplus_{m\geq 0} V_{\lambda - m\alpha_0}$.*

(1) For all integers $m \geq 0$, one has $\phi_{\mu,\lambda}(V_{\lambda-m\alpha_0}) \subset \mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$.

(2) The restricted map $\phi_{\mu,\lambda} : B(\lambda) \to \mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$ is norm-preserving if and only if μ satisfies $\mu_{\alpha_0} = \lambda_{\alpha_0}$.

Proof. To prove (1), we choose an orthonormal basis of V_λ by adjoining v_λ to an orthonormal basis of $(\mathbb{C}v_\lambda)^\perp$. Recall from Proposition 4.3 (2) that $T_\lambda(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda) = c_\lambda(m)(\cdot, E_{\alpha_0})^m v_\lambda$. Using the above basis in the definition of $\phi_{\mu,\lambda}$, one has

$$\phi_{\mu,\lambda}(T_\lambda(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)) = c_\lambda(m)(\cdot, E_{\alpha_0})^m 1_\mu \otimes v_\lambda \otimes 1_{-\mu}.$$

Again from Proposition 4.3 (2), one knows that $T_\mu(\overline{E}_{\alpha_0})^m(\mathbf{1}_\mu) = c_\mu(m)(\cdot, E_{\alpha_0})^m 1_\mu$. Consequently, since $T_\mu(\overline{E}_{\alpha_0})^m(\mathbf{1}_\mu) \in \mathcal{H}(\mu)^K$, one concludes that

$$\phi_{\mu,\lambda}(T(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)) \in \mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}.$$

But since $\phi_{\mu,\lambda}$ is K-equivariant and $\mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$ is K-invariant, one has $\phi_{\mu,\lambda}(V_{\lambda-m\alpha_0}) \subset \mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$ as well. This establishes (1).

To prove (2), first observe the center of K acts on $\mathfrak{p}_+$ by scalars. By (6.3) of [4] it follows that the K-finite vectors in $\mathcal{H}(\lambda)$ of different homogeneous degree must be orthogonal. It follows that $\oplus_{m\geq 0} V_{\lambda-m\alpha_0}$ is an orthogonal direct sum in $\mathcal{H}(\lambda)$ and, likewise, $\oplus_{m\geq 0}\phi_{\mu,\lambda}(V_{\lambda-m\alpha_0})$ is an orthogonal direct sum in $\mathcal{H}(\mu)^K \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$. Let $(\cdot,\cdot)_\otimes$ denote the canonical inner product on the Hilbert space $\mathcal{H}(\mu) \otimes V_\lambda \otimes \mathbb{C}_{-\mu}$. Since $V_{\lambda-m\alpha_0}$ is an irreducible K-space, there exists a constant $b_m > 0$ such that

$$(\phi_{\mu,\lambda}(f), \phi_{\mu,\lambda}(g))_\otimes = b_m (f,g)_\lambda, \quad \text{for all } \ f, g \in V_{\lambda-m\alpha_0}. \tag{5.4}$$

The above calculation shows that

$$\phi_{\mu,\lambda}(T_\lambda(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)) = \frac{c_\lambda(m)}{c_\mu(m)} T_\mu(\overline{E}_{\alpha_0})^m(\mathbf{1}_\mu) \otimes v_\lambda \otimes 1_{-\mu}.$$

Letting $f = g = T_\lambda(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda)$ in (5.4), we obtain

$$\begin{aligned}(\frac{c_\lambda(m)}{c_\mu(m)})^2 &(T_\mu(\overline{E_{\alpha_0}})^m(\mathbf{1}_\mu), T_\mu(\overline{E_{\alpha_0}})^m(\mathbf{1}_\mu))_\mu \\ &= b_m (T_\lambda(\overline{E_{\alpha_0}})^m(\mathbf{1}_\lambda), T_\lambda(\overline{E_{\alpha_0}})^m(\mathbf{1}_\lambda))_\lambda.\end{aligned} \tag{5.5}$$

By Proposition 4.3 (1), we know that

$$\| T_\lambda(\overline{E}_{\alpha_0})^m(\mathbf{1}_\lambda) \|_\lambda^2 = m!(E_{\alpha_0}, E_{\alpha_0})^m c_\lambda(m)$$

and

$$\| T_\mu(\overline{E}_{\alpha_0})^m(\mathbf{1}_\mu) \|_\mu^2 = m!(E_{\alpha_0}, E_{\alpha_0})^m c_\mu(m).$$

But then (5.5) reduces to $c_\lambda(m) = b_m c_\mu(m)$. Consequently $b_m = 1$ for all $m \geq 0$ if and only if $\lambda_{\alpha_0} = \mu_{\alpha_0}$. This completes the proof of (2). ∎

For $\lambda \in \Lambda(G)$, we define the weight $\widetilde{\lambda}$ by

$$\widetilde{\lambda} = (\lambda_{\alpha_0})\omega_0, \tag{5.6}$$

where ω_0 is the fundamental dominant weight corresponding to α_0.

Corollary 5.2. *If $\lambda, \widetilde{\lambda} \in \Lambda(G)$, then the restriction of $\phi_{\widetilde{\lambda},\lambda}$ to $B(\lambda)$ uniquely extends to an isometry $\Phi_\lambda : \overline{B(\lambda)} \to \mathcal{H}(\widetilde{\lambda}) \otimes V_\lambda \otimes \mathbb{C}_{-\widetilde{\lambda}}$ such that $\Phi_\lambda(f)(z) = \sum_i (f(z), v_i) 1_{\widetilde{\lambda}} \otimes v_i \otimes 1_{-\widetilde{\lambda}}$, where $\{v_i\}$ is an orthonormal basis of V_λ.*

Proof. The unique extension to an isometry Φ_λ on the closure of $B(\lambda)$ follows from Proposition 5.1 (2). To establish the formula for Φ_λ, recall from Proposition 4.1(1) that convergence in $\mathcal{H}(\lambda)$ implies pointwise convergence. Consequently, if $\{f_n\} \subset B(\lambda)$ converges to f in $\overline{B(\lambda)}$, then $(f_n(z), v)$ converges to $(f(z), v)$ for each $z \in D$ and $v \in V_\lambda$. Since both Φ_λ and point evaluation e_z are continuous, one has

$$\begin{aligned}
\Phi_\lambda f(z) &= e_z \Phi_\lambda(f) \\
&= \lim_{n\to\infty} e_z \Phi_\lambda(f_n) \\
&= \lim_{n\to\infty} \sum_i (f_n(z), v_i) 1_{\widetilde{\lambda}} \otimes v_i \otimes 1_{-\widetilde{\lambda}} \\
&= \sum_i (f(z), v_i) 1_{\widetilde{\lambda}} \otimes v_i \otimes 1_{-\widetilde{\lambda}}. \qquad \blacksquare
\end{aligned}$$

Remark 5.3. *In [13], certain orthogonal families of scalar-valued polynomials on D, $G = U(p,q)$, are studied. These polynomials are used to invert an intertwining operator which is subsequently used to define a G-invariant norm for ladder representations of G. However, these polynomials are easily seen to be an orthogonal basis of $B(\widetilde{\lambda})$. The proof of Proposition 5.1 (1) shows that the image of Φ_λ is contained in $B(\widetilde{\lambda}) \otimes V_\lambda \otimes \mathbb{C}_{-\widetilde{\lambda}}$. We can therefore see the relevance of these orthogonal families in describing the image of Φ_λ.*

6. Ladder Representations

In this section we restrict our attention to highest weights $\lambda \in \Lambda(G)$ that correspond to a ladder representation of G. We define ladder representations and show how the unitary inner product $(\cdot,\cdot)_\lambda$ of $\mathcal{H}(\lambda)$, discussed in section four, is completely determined by the unitary inner product of $\mathcal{H}(\widetilde{\lambda})$ (with $\widetilde{\lambda}$ defined as in (5.6)) and the isometry in Corollary 5.2.

Recall by Proposition 4.3 that for any $\lambda \in \Lambda(G)$, each of the K-types $V_{\lambda - m\alpha_0}$, $m \geq 0$, appears in $\mathcal{H}(\lambda)$. We define $(T_\lambda, \mathcal{H}(\lambda))$ to be a ladder representation of G if these are the only K-types that appear. Let $\Lambda_l(G)$ denote the subset of $\Lambda(G)$ such that $(T_\lambda, \mathcal{H}(\lambda))$ is a ladder representation. Then $\lambda \in \Lambda_l(G)$ if and only if

$$(\overline{\oplus_{m\geq 0} V_{\lambda - m\alpha_0}}) = \mathcal{H}(\lambda). \tag{6.1}$$

We define a representation S_λ of G on $\mathcal{O}(D, \mathbb{C}_{\widetilde{\lambda}} \otimes V_\lambda \otimes \mathbb{C}_{-\widetilde{\lambda}})$ by the formula

$$(S_\lambda(g)f)(z) = (1 \otimes J_\lambda(g^{-1}, z)^{-1} \otimes 1) f(g^{-1} \cdot z), \tag{6.2}$$

for $z \in D$. Here, 1 denotes the identity map on $\mathbb{C}_{\widetilde{\lambda}}$ as well as $\mathbb{C}_{-\widetilde{\lambda}}$.

Theorem 6.1. *Let $\lambda \in \Lambda_l(G)$. If $\widetilde{\lambda} \in \Lambda_0(G)$ then Φ_λ intertwines T_λ and S_λ. Moreover, the invariant unitary inner product $(\cdot,\cdot)_\lambda$ on $\mathcal{H}(\lambda)$ is given by the formula*

$$(f_1, f_2)_\lambda = \sum_i ((f_1(\cdot), v_i)1_{\widetilde{\lambda}}, (f_2(\cdot), v_i)1_{\widetilde{\lambda}})_{\widetilde{\lambda}},$$

where $f_1, f_2 \in \mathcal{H}(\lambda), \{v_i\}$ is an orthonormal basis of V_λ and $(\cdot,\cdot)_{\widetilde{\lambda}}$ denotes the invariant unitary inner product on $\mathcal{H}(\widetilde{\lambda})$.

Proof. The intertwining property is a consequence of the form of Φ_λ in Corollary 5.2. Now let $f_1, f_2 \in \mathcal{H}(\lambda)$ and let $(\cdot,\cdot)_\otimes$ denote the inner product on the Hilbert space $\mathcal{H}(\widetilde{\lambda}) \otimes V_\lambda \otimes \mathbb{C}_{-\widetilde{\lambda}}$. Since Φ_λ is an isometry, we have

$$\begin{aligned}
(f_1, f_2)_\lambda &= (\Phi_\lambda(f_1), \Phi_\lambda(f_2))_\otimes \\
&= (\sum_i (f_1(\cdot), v_i)1_{\widetilde{\lambda}} \otimes v_i \otimes 1_{-\widetilde{\lambda}}, \sum_j (f_2(\cdot), v_j)1_{\widetilde{\lambda}} \otimes v_j \otimes 1_{-\widetilde{\lambda}})_\otimes \\
&= \sum_{i,j} ((f_1(\cdot), v_i)1_{\widetilde{\lambda}}, (f_2(\cdot), v_j)1_{\widetilde{\lambda}})_{\widetilde{\lambda}}(v_i, v_j) \\
&= \sum_i ((f_1(\cdot), v_i)1_{\widetilde{\lambda}}, (f_2(\cdot), v_i)1_{\widetilde{\lambda}})_{\widetilde{\lambda}}. \qquad \blacksquare
\end{aligned}$$

Remark 6.2. *As seen in the next section, it can happen that $\widetilde{\lambda} \notin \Lambda_0(G)$ but $\widetilde{\lambda} \in \Lambda_0(\tilde{G})$ for some covering group $\tilde{G}$ of G. The representation $T = T_\lambda$ of G lifts to a unitary representation of $\tilde{G}$ on the same Hilbert space $\mathcal{H}(\lambda)$. Theorem 6.1 then applies and gives the unitary inner product on $\mathcal{H}(\lambda)$, which is $\tilde{G}$-invariant. However, this inner product is also G-invariant via the covering map $\tilde{G} \to G$.*

7. The role of scalar-valued representations

In all but finitely many cases, the associated scalar-valued representation $\mathcal{H}(\widetilde{\lambda})$, where $\widetilde{\lambda}$ is defined in (5.6), is a discrete series or a generalized limit of discrete series representation [8]. Since the unitary inner product for such scalar-valued representations is well-known, we can describe the unitary inner product on $\mathcal{H}(\lambda)$ using the formula for $(\cdot,\cdot)_\lambda$ found in Theorem 6.1. Explicit examples of the unitary inner product $(\cdot,\cdot)_\lambda$ for families of ladder representations of the groups $SU(p,q)$ and $SO^*(2n)$ appear at the end of this section.

We begin with some basic facts about the unitary inner product for $\mathcal{H}(\mu)$ where μ corresponds to a scalar-valued discrete series representation. We define the subset $\Lambda_0^d(G)$ of $\Lambda_0(G)$ to be the set of highest weights μ satisfying $(\mu+\rho,\beta) < 0$, where $\beta \in \Phi_n^+$ is the maximal noncompact root and ρ denotes half the sum of the roots in Φ^+. Then each $\mu \in \Lambda_0^d(G)$ corresponds to a discrete series representation of G.

Let $dm(z)$ denote the G-invariant measure on D and let M_μ, $\mu \in \Lambda_0^d(G)$, denote the function on D determined by the function $g \to |J_\mu(g,0)^{-1}|^2$. Denote by $d\mu(z)$ the measure $M_\mu(z)\,dm(z)$, normalized so that $\int_D d\mu(z) = 1$. Then the inner product on $\mathcal{H}(\mu)$ is given by

$$(f,h)_\mu = \int_D (f(z), h(z))_{\mathbb{C}_\mu}\, d\mu(z). \tag{7.1}$$

Put $r = \operatorname{rank}(D)$ and let $\mu_i \in \Lambda_0(G)$ satisfy $(\mu_i+\rho)_\beta = c(i-1)$, $1 \le i \le r$, where c is given on page 115 of [5]. These weights, $\Lambda_0^{\lim}(G)$, have corresponding unitary structures explicitly given in [8]. We summarize the results found there. Let $dm_i(z)$ denote the quasi-invariant measure on the i^{th} boundary component $\mathcal{B}_i$ of D. The $\mathbb{R}$-valued function on $K \times G_i$ given by $(k,g) \to |J_{\mu_i}(kgc_i, 0)^{-1}|^2$ (cf. (6.14) [8]) determines a function M_i on $\mathcal{B}_i$. Let $d\mu_i(z)$ denote the normalized measure on $\mathcal{B}_i$ corresponding to $M_i(z)\, dm_i(z)$. Then the (densely defined) inner product on $\mathcal{H}(\mu_i)$, $1 \le i \le r$, is given by

$$(f,h)_{\mu_i} = \int_{\mathcal{B}_i} (f(z), h(z))_{\mathbb{C}_\mu} d\mu_i(z), \qquad \text{for } f,h \in \mathcal{H}(\mu_i)^K. \tag{7.2}$$

Theorem 7.1. *Let $\lambda \in \Lambda_l(G)$.*

(1) If $\widetilde{\lambda} \in \Lambda_0^d(G)$ then

$$(f,h)_\lambda = \int_D (f(z), h(z)) d\mu(z), \qquad \textit{for} \quad f,g \in \mathcal{H}(\lambda).$$

(2) If $\widetilde{\lambda} \in \Lambda_0^{\lim}(G)$ then

$$(f,h)_\lambda = \int_{\mathcal{B}_i} (f(z), h(z)) d\mu_i(z), \qquad \textit{for} \ f,h \in \mathcal{H}(\lambda)^K.$$

Proof. To show (1), assume $\widetilde{\lambda} \in \Lambda_0^d(G)$. Let $\{v_i\}$ be an orthonormal basis of V_λ. Then by Theorem 6.1, we obtain

$$\begin{aligned}(f,h)_\lambda &= \sum_i ((f(\cdot), v_i)1_{\widetilde{\lambda}}, (h(\cdot), v_i)1_{\widetilde{\lambda}})_{\widetilde{\lambda}} \\ &= \sum_i \int_D (f(z), v_i)\overline{(h(z), v_i)} d\mu(z) \quad \text{by (7.1)} \\ &= \int_D (f(z), h(z)) d\mu(z).\end{aligned}$$

This proves (1). The proof of (2) follows in the same way by (7.2) and the fact that Φ_λ maps K-finite vectors to K-finite vectors. ∎

Explicit examples for $\mathbf{SU(p,q)}$

We now let $G = SU(p,q)$ and utilize Theorem 7.1 to explicitly describe the unitary norm for a family of ladder representations of G. This family includes all but finitely many of the ladder representations of $SU(p,q)$ discussed in section eleven of [2].

Let $m \ge 1$ be an integer. Let $\{\omega_i\}$ denote the set of standard fundamental dominant weights found in [2]. Let $\lambda^m = -(m+1)\omega_p + m\omega_{p+1}$ and $\lambda_m = mw_{p-1} - (m+1)w_p$. Since α_p is the simple noncompact root, we see from (5.6) that $\widetilde{\lambda}^m = \widetilde{\lambda}_m = -(m+1)\omega_p$. Note that the weight $-(m+1)\omega_p$ always lies in $\Lambda_0(G)$ and corresponds to a highest weight of a discrete series representation of $SU(p,q)$ if and only if $m+1 \geq p+q$. From [7], p. 84, the invariant measure $dm(z)$ on D is given (up to a positive multiple) by $\det(I - zz^*)^{-(p+q)}dz$ where $dz = \Pi_{r,s} dx_{rs} dy_{rs}$ with $z_{rs} = x_{rs} + iy_{rs}$. Moreover, by [1], one has $M_{\widetilde{\lambda}_m}(z) = M_{\widetilde{\lambda}^m}(z) = det(I - zz^*)^{m+1}$. If $m+1 \geq p+q$ and $f, h \in \mathcal{H}(\lambda_m)$, then we have by Theorem 7.1 the formula

$$(f,h)_{\lambda_m} = d_{\lambda_m} \int_D (f(z), h(z)) \det(I - zz^*)^{m+1-p-q} dz, \tag{7.3}$$

where $d_{\lambda_m}^{-1} = \int_D det(I - zz^*)^{m+1-p-q} dz$. By replacing the inner product on V_{λ_m} in (7.3) with the inner product on V_{λ^m}, one obtains the formula for $(\cdot,\cdot)_{\lambda^m}$ on $\mathcal{H}(\lambda^m)$.

Explicit examples for $\mathbf{SO^*(2n)}$

We now let $G = SO^*(2n)$ where $n \geq 4$. For each integer $m \geq 1$, the weight $\lambda = m\omega_{n-1} - (m+2)\omega_n$ corresponds to a ladder representation of $SO^*(2n)$. Following the notation in [2], since $\alpha_n = e_{n-1} + e_n$ is the simple noncompact root, we have $\widetilde{\lambda} = -(m+2)\omega_n$. Now by Section 7.22 of [2] the highest weights of the unitary scalar-valued representations of $SO^*(2n)$ are all of the form $-2k\omega_n$ where $k \geq 0$ is an integer. In particular, if m is odd, then $\widetilde{\lambda} \notin \Lambda_0(SO^*(2n))$. We shall return to the case where m is odd. For the moment, assume m is even. From [7], p. 85, the invariant measure on D is given (up to a multiple) by $det(I + z\overline{z})^{-n+1}dz$, where $dz = \Pi_{r<s} dx_{rs} dy_{rs}$ with $z_{rs} = x_{rs} + iy_{rs}$. The function $M_{\widetilde{\lambda}}$ in this case is $M_{\widetilde{\lambda}}(z) = det(I + z\overline{z})^{\frac{m}{2}+1}$. The weight $\widetilde{\lambda}$ corresponds to a highest weight for a discrete series representation when $\frac{m}{2} - n + 2 \geq 0$.

Then for $f, h \in \mathcal{H}(\lambda)$, we have by Theorem 7.1 that

$$(f,h)_{\lambda} = d_{\lambda} \int_D (f(z), h(z)) \det(I + z\overline{z})^{\frac{m}{2}-n+2} dz, \tag{7.4}$$

where $d_{\lambda}^{-1} = \int_D det(I + z\overline{z})^{\frac{m}{2}-n+2} dz$. We observed above that for odd m, the weight $\widetilde{\lambda} = -(m+2)\omega_n$ is not the highest weight for a unitary highest weight presentation of $SO^*(2n)$. However, for odd m such that the square integrable condition $\frac{m}{2} - n + 2 \geq 0$ holds, $\widetilde{\lambda}$ does correspond to a discrete series representation of the two-fold covering group $SO^*(2n)^\sim$ of $SO^*(2n)$. Remark 6.2 applies in this case and we conclude that the formula for $(\cdot,\cdot)_\lambda$ appearing in (7.4) remains valid for the m odd case.

8. Applications to Intertwining Operators

In Theorem 7.1 we observed that for $\lambda \in \Lambda_l(G)$, the unitary inner product $(\cdot,\cdot)_\lambda$ for $\mathcal{H}(\lambda)$ can inherit the intrinsic simplicity of the unitary norm of an associated discrete representation of G on $\mathcal{H}(\widetilde{\lambda})$. We now show that this simplicity is also manifested in the form of the inverse of a unitary intertwining operator $\Xi : \mathbb{H}_\lambda \to \mathcal{H}(\lambda)$, where $\mathbb{H}_\lambda$ denotes any equivalent realization of $\mathcal{H}(\lambda)$.

If $\lambda \in \Lambda(G)$, we let $(\omega_\lambda, \mathbb{H}_\lambda)$ denote an irreducible unitary highest weight representation of G with highest weight λ. Let v_λ denote a unit highest weight vector and V_λ its K-invariant span. For $z \in \mathfrak{p}_+$, define the operator $q_z : V_\lambda \to \mathbb{H}_\lambda$

by the formula

$$q_z(v) = \sum_{n=0}^{\infty} \frac{d\omega_\lambda(\overline{z})^n}{n!}(v), \tag{8.1}$$

where $v \in V_\lambda$ and $d\omega_\lambda$ denotes the derived action of ω_λ. By Theorem 5.1 of [3] the series (8.1) converges in $\mathbb{H}_\lambda$ if and only if $z \in D$. Moreover, by Theorem 7.2 of [3], the expression

$$(\Xi f)(z) = q_z^*(f), \quad f \in \mathbb{H}_\lambda, z \in D, \tag{8.2}$$

defines a unitary intertwining operator $\Xi : \mathbb{H}_\lambda \to \mathcal{H}(\lambda)$. Here, $q_z^* : \mathbb{H}_\lambda \to V_\lambda$ denotes the adjoint of the operator $q_z, z \in D$.

Proposition 8.1. *Let $\lambda \in \Lambda_l(G)$. If $\widetilde{\lambda} \in \Lambda_0^d(G)$, then the inverse Ξ^{-1} of Ξ is given by*

$$\Xi^{-1} F = \int_D q_z(F(z)) d\mu(z), \qquad \textit{for } F \in \mathcal{H}(\lambda).$$

Proof. Let $< \cdot, \cdot >$ denote the unitary inner product on $\mathbb{H}_\lambda$. Using the form of $(\cdot,\cdot)_\lambda$ in Theorem 7.1 (1), we have for $f \in \mathbb{H}_\lambda$ and $F \in \mathcal{H}(\lambda)$

$$(\Xi f, F)_\lambda = \int_D (q_z^*(f), F(z)) d\mu(z) = \int_D < f, q_z(F(z)) > d\mu(z).$$

Since Ξ is unitary, the proposition now follows. ∎

In order to give a concrete example of the inversion formula in Proposition 8.1, we turn to the harmonic realization of unitary highest weight representations of the linear groups $SU(p,q), SO^*(2n)$ and $Sp(n,\mathbb{R})$. We follow the notation found in section seven of [2]. Let M be the complex space of matrices in Table 7.2 of [2]. The relevant Hilbert space is the Fock space $\mathbb{F}$ consisting of $\mathbb{C}$-valued holomorphic functions on M which are square integrable with respect to the Gaussian measure on M. We let $\theta : M \to \mathfrak{p}_+$ denote the map on page 55 of [2] and let $\mathbb{I}$ denote the ideal generated by the matrix entries of θ and the constants. Let V_λ denote an irreducible K space of harmonic polynomials (cf. (7.8), [2]). Then the closure of $\mathbb{I}V_\lambda$, denoted by $\mathbb{H}_\lambda$, is an irreducible subspace for ω_λ, the harmonic representation of G. By (7.3) of [2], the action of $\mathfrak{p}_-$ on $\mathbb{H}_\lambda$ is given by $d\omega_\lambda(x) f(y) = (\theta(y), \overline{x}) f(y)$, for $x \in \mathfrak{p}_-$, $y \in M$ and $f \in \mathbb{H}_\lambda$. Consequently, one finds by (8.1) that for $z \in D$

$$q_z h(y) = e^{(\theta(y),z)} h(y), \qquad h \in V_\lambda, y \in M. \tag{8.3}$$

For $\lambda \in \Lambda(G)$ $(G = SU(p,q), SO^*(2n)$ or $Sp(n,\mathbb{R}))$, we have by (8.3) of [3] that the intertwining operator Ξ has the form

$$(\Xi f)(z,y) = \int_M f(w) e^{(z,\theta(w))} K_\lambda(y,w) d\mu(w),$$

where K_λ denotes the reproducing kernel of V_λ.

If $\lambda \in \Lambda_l(G)$ and $\widetilde{\lambda} \in \Lambda_0^d(G)$, then we have by Proposition 8.1 and (8.3) that Ξ^{-1} has the explicit form

$$(\Xi^{-1} F)(y) = \int_D e^{(\theta(y),z)} F(z,y) d\mu(z), \tag{8.4}$$

where $F \in \mathcal{H}(\lambda)$ and $y \in M$.

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Mathematics Department
LSU
Baton Rouge, LA 70809
davidson@math.lsu.edu

Mathematics Department
Baylor University
Waco, TX 76798
Ronald_Stanke@Baylor.edu

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