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An Explicit Formula for Sums of Powers of Integers in Terms of Stirling Numbers

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Abstract

In this note we present an explicit formula for the sum of powers of the first n terms of a general arithmetic sequence in terms of Stirling numbers. We then provide an algorithm for calculating the sum of consecutive powers of integers.

1 Introduction

This paper is concerned with sums of pth powers of the first n terms of a general arithmetic sequence

$$S_{p,(a,d)}(n) = a^p + (a+d)^p + \cdots + (a+(n-1)d)^p, \ n \geq 1$$

where p is a nonnegative integer, and a, d are complex numbers with $d \neq 0$. In particular, we have

$$S_{p,(1,1)}(n) = 1^p + 2^p + 3^p + \cdots + n^p,$$

which has been studied extensively by many authors.

The properties of $S_{p,(a,d)}(n)$ were obtained by Howard [4] via the following generating function

$$\mathcal{B}_0(z) := \sum_{p\geq 0} S_{p,(a,d)}(n) \frac{z^p}{p!} = \sum_{k=0}^{n-1} e^{(a+kd)z}. \tag{1}$$

For recent articles on this subject, see [1, 6, 8].

In this note we establish a generalization of the well-known formula [3, 5, 9]

$$S_{p,(1,1)}(n) = \sum_{k=0}^{p} k! \left\{ {p \atop k} \right\} \binom{n+1}{k+1}, \ p \geq 1.$$

for the sums $S_{p,(a,d)}(n)$. Also, we provide an algorithm for calculating the sum of the pth powers of the first n terms of a general arithmetic sequence.

First we present some definitions and notations and some results that will be useful in the rest of the paper. For $x \in \mathbb{C}$, the falling factorial $(x)_n$ is defined by $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1)$ for $n > 0$. The generalized binomial coefficient is defined as follows:

$$\binom{x}{n} = \begin{cases} \frac{(x)_n}{n!}, & \text{if } n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k$$

and satisfy the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k) \ \text{ for } 1 \leq k \leq n. \tag{2}$$

The Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k .$$

These numbers count the number of ways to partition a set of n elements into exactly k nonempty subsets.

For any positive integer r, the r-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r$ are obviously generalizations of Stirling numbers of the second kind. These numbers count the number of partitions of a set of n objects into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. The exponential generating function is given by

$$\sum_{n\geq k}\left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!} = \frac{1}{k!}e^{rz}\left(e^z-1\right)^k.$$

For more details about these numbers see [2, 3].

2 Main results

First, we obtain the following result via the generalized Stirling transform method.

Theorem 1. *For all $n \geq 1$, non-negative integer p and complex numbers a and d ($d \neq 0$), we have*

$$S_{p,(a,d)}\left(n\right) = d^p \sum_{k=0}^{p} k! \left\{ {p \atop k} \right\} \left(\binom{n+\frac{a}{d}}{k+1} - \binom{\frac{a}{d}}{k+1} \right). \tag{3}$$

Proof. Since

$$S_{p,(a,d)}\left(n\right) = d^p S_{p,\left(\frac{a}{d},1\right)}\left(n\right),$$

we can investigate the sums $S_{p,\left(\frac{a}{d},1\right)}\left(n\right)$. It follows from (1) and [7, Theorem 4] that

$$\begin{aligned}
\mathcal{A}_0\left(z\right) &= \sum_{m\geq 0} a_{0,m}\frac{z^m}{m!} \\
&= \mathcal{B}_0\left(\ln\left(1+z\right)\right) \\
&= \sum_{k=0}^{n-1}\left(1+z\right)^{\frac{a}{d}+k} \\
&= \sum_{m\geq 0}\frac{z^m}{m!}\sum_{k=0}^{n-1}\left(\frac{a}{d}+k\right)_m.
\end{aligned}$$

Since

$$\sum_{k=0}^{n}\left(x+k\right)_m := \frac{\left(x+n+1\right)_{m+1}-\left(x\right)_{m+1}}{m+1},$$

we can easily verify that

$$\begin{aligned}
a_{0,m} &= \left[z^m\right]\mathcal{A}_0\left(z\right) \quad (4)\\
&= m!\left(\binom{n+\frac{a}{d}}{m+1} - \binom{\frac{a}{d}}{m+1} \right) \quad (5)
\end{aligned}$$

where $[z^n]f(z) = f_n$ denotes the operation of extracting the coefficient of z^n in the formal power series $f(z) = \sum_n f_n \frac{z^n}{n!}$.

Now, from [7, Corollary 1] we get

$$\sum_{k=0}^{m} \frac{1}{d^{p+k}} s(m,k)\, S_{p+k,(a,d)}(n) = \sum_{k=0}^{p} \left\{ {p+m \atop k+m} \right\}_m (m+k)! \left(\binom{n+\frac{a}{d}}{m+k+1} - \binom{\frac{a}{d}}{m+k+1} \right). \quad (6)$$

and the proof is completed by letting $m = 0$ in the above identity. □

Setting $p = 0$ in (6), one obtains the following recursive formula for the sum of the pth powers of the first n terms of a general arithmetic sequence involving the Stirling numbers of the first kind:

Corollary 2. *We have*

$$\sum_{k=0}^{m} \frac{1}{d^k} s(m,k)\, S_{k,(a,d)}(n) = m! \left(\binom{n+\frac{a}{d}}{m+1} - \binom{\frac{a}{d}}{m+1} \right).$$

Thus, for example, when $m = 0, 1, 2, 3$, we obtain

$$\begin{aligned} S_{0,(a,d)}(n) &= n, \\ S_{1,(a,d)}(n) &= d \left(\binom{n+\frac{a}{d}}{2} - \binom{\frac{a}{d}}{2} \right), \\ -\frac{1}{d} S_{1,(a,d)}(n) + \frac{1}{d^2} S_{2,(a,d)}(n) &= 2 \left(\binom{n+\frac{a}{d}}{3} - \binom{\frac{a}{d}}{3} \right), \\ \frac{2}{d} S_{1,(a,d)}(n) - \frac{3}{d^2} S_{2,(a,d)}(n) + \frac{1}{d^3} S_{3,(a,d)}(n) &= 6 \left(\binom{n+\frac{a}{d}}{4} - \binom{\frac{a}{d}}{4} \right). \end{aligned}$$

In the next paragraph, we propose an algorithm based on a three-term recurrence relation for calculating the pth powers of the first n terms of a general arithmetic sequence $S_{p,(a,d)}(n)$. It is convenient to introduce the following sequence $A_{p,m}^{(a,d)}(n)$ with two indices by

$$A_{p,m} := A_{p,m}^{(a,d)}(n) = \frac{S_{0,(a,d)}(n)}{a_{0,m}} \sum_{k=0}^{m} \frac{1}{d^k} s(m,k)\, S_{p+k,(a,d)}(n), \quad (7)$$

with $A_{0,m} = n$ and $A_{p,0} = S_{p,(a,d)}(n)$.

Theorem 3. *The sequence $A_{p,m}^{(a,d)}(n)$ satisfies the following three-term recurrence relation*

$$A_{p+1,m} = d \frac{a_{0,m+1}}{a_{0,m}} A_{p,m+1} + dm A_{p,m}, \quad (8)$$

with the initial sequence $A_{0,m} = n$, and $a_{0,m}$ is defined by (5).

Proof. From (7) and (2) , we have

$$A_{p,m+1}=\frac{n+1}{a_{0,m+1}}\sum_{k=0}^{m+1}\frac{1}{d^k}s\left(m+1,k\right)S_{p+k,(a,d)}\left(n\right)$$
$$=\frac{n+1}{a_{0,m+1}}\sum_{k=1}^{m+1}\frac{1}{d^k}\left(s\left(m,k-1\right)-ms\left(m,k\right)\right)S_{p+k,(a,d)}\left(n\right).$$

After some rearrangement, we get

$$A_{p,m+1}=\frac{a_{0,m}}{da_{0,m+1}}A_{p+1,m}-\frac{a_{0,m}m}{a_{0,m+1}}A_{p,m}.$$

This completes the proof. $\square$

As an immediate application of (8) we have the following algorithm for evaluating Faulhaber's formula $S_{p,(1,1)}\left(n\right)$. Starting with the sequence $R_{0,m}:=n$, as the first row of the matrix $\left(R_{p,m}\right)_{p,m\geq 0}$, each entry is determined recursively by

$$R_{p+1,m}=mR_{p,m}+\frac{\left(m+1\right)\binom{n+1}{m+2}}{\binom{n+1}{m+1}-\delta_{0,m}}R_{p,m+1},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

Then $R_{p,0}:=S_{p,(1,1)}\left(n\right)$ is Faulhaber's formula.

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References

[1] A. Bazsó and I. Mező, On the coefficients of power sums of arithmetic progressions. *J. Number Theory* **153** (2015), 117–123.

[2] A. Z. Broder, The r-Stirling numbers. *Discrete Math.* **49** (1984), 241–259.

[3] R. L. Graham and D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, 1994.

[4] F. T. Howard, Sums of powers of integers via generating functions. *Fibonacci Quart.* **34** (1996), 244–256.

[5] J. Quaintance and H. W. Gould. *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H. W. Gould*, World Scientific Publishing, 2016.

[6] M. Merca, An alternative to Faulhaber's formula. *Amer. Math. Monthly* **122** (2015), 599–601.

[7] M. Rahmani, Generalized Stirling transform. *Miskolc Math. Notes* **15** (2014), 677–690.

[8] R. Schumacher, An extended version of Faulhaber's formula, *J. Integer Sequences* **19** (2016), Article 16.4.2.

[9] R. Wituła, K. Kaczmarek, P. Lorenc, E. Hetmaniok and M. Pleszczynski, Jordan numbers, Stirling numbers and sums of powers. *Discuss. Math. Gen. Algebra Appl.* **34** (2014), 155–166.

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