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Extending a Recent Result on Hyper m-ary Partition Sequences

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Abstract

A hyper m-ary partition of an integer n is defined to be a partition of n where each part is a power of m and each distinct power of m occurs at most m times. Let $h_m(n)$ denote the number of hyper m-ary partitions of n and consider the resulting sequence. We show that the hyper m_1-ary partition sequence is a subsequence of the hyper m_2-ary partition sequence, for $2 \leq m_1 < m_2$.

1 Introduction

In 2004, Courtright and Sellers [2] defined a hyper m-ary partition of an integer n to be a partition of n for which each part is a power of m and each power of m occurs at most m

times. They denote the number of hyper m-ary partitions of n as $h_m(n)$ and showed that they satisfy the following recurrence relation:

$$h_m(mq) = h_m(q) + h_m(q-1), \tag{1}$$
$$h_m(mq+s) = h_m(q), \text{ for } 1 \le s \le m-1. \tag{2}$$

Several of these hyper m-ary partition sequences can be found in the On-line Encyclopedia of Integer Sequences [7]. In particular, h_2 is A002487, h_3 is A054390, h_4 is A277872, and h_5 is A277873.

The sequence h_2 A002487, the hyperbinary partition sequence, is well known. It is commonly known as the Stern sequence based on Stern's work [8]. Northshield [5] gives an extensive summary of the many uses and applications of A002487. Calkin and Wilf [1] also studied h_2, outlining a connection between this sequence and a sequence of fractions they defined and used to give an enumeration of the rationals. Since then, several authors have studied similar restricted binary and m-ary partition functions; see [3, 4, 6] for additional examples.

In this paper, we will be analyzing hyper m-ary partitions of n while also considering the base m representation of n. Thus, it will be convenient to have clear and distinct notation. In particular, for $m \ge 2$, let $(n_r, n_{r-1}, \dots, n_1, n_0)_m$ be the base m representation of positive integer n where $0 \le n_i < m$, $n_r \ne 0$, and $n = \sum_{i=0}^{r} n_i m^i$. Also, for $2 \le m_1 < m_2$ and $n = (n_r, n_{r-1}, \dots, n_1, n_0)_{m_1}$, we define a change of base function, $F_{m_1,m_2}(n) = (n_r, n_{r-1}, \dots, n_1, n_0)_{m_2}$.

Next, we write a hyper m-ary partition of n as $[x_r, x_{r-1}, \dots, x_1, x_0]_m$ where $0 \le x_i \le m$ and $n = \sum_{i=0}^{r} x_i m^i$. Here, we may allow any of the x_i to be 0 so that each hyper m-ary partition of n is the same length r as the base m representation of n. Furthermore, let $H_m(n)$ be the set of all distinct hyper m-ary partitions of n. Observe that $h_m(n)$ is the cardinality of this set.

Recently, the authors gave an identity relating h_2 to h_3 and then generalized this identity to show that h_2 is a subsequence of h_m for any m [4]. This result involved giving a bijection between $H_2(\ell)$ and $H_m(k)$, where $k = F_{2,m}(\ell)$. In this note, the authors will follow a similar process to show that h_{m_1} is a subsequence of h_{m_2}, for $2 \le m_1 < m_2$.

2 A preliminary example

Consider the integer $37 = (1,1,0,1)_3$ and use the change of base function to find the integer with the same digits in base 4. In particular, $F_{3,4}(37) = (1,1,0,1)_4 = 81$. Now consider the hyper 3-ary partitions of 37 and the hyper 4-ary partitions of 81.

$$\begin{aligned} 37 &= 1 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0 \\ &= 1 \cdot 3^3 + 3 \cdot 3^1 + 1 \cdot 3^0 \\ &= 3 \cdot 3^2 + 3 \cdot 3^1 + 1 \cdot 3^0 \end{aligned} \qquad \begin{aligned} 81 &= 1 \cdot 4^3 + 1 \cdot 4^2 + 1 \cdot 4^0 \\ &= 1 \cdot 4^3 + 4 \cdot 4^1 + 1 \cdot 4^0 \\ &= 4 \cdot 4^2 + 4 \cdot 4^1 + 1 \cdot 4^0 \end{aligned}$$

Adopting the notation for hyper m-ary partitions and the sets of these partitions, rewrite these partitions in the following way:

$$\begin{aligned} H_3(37) &= \{\,[1,1,0,1]_3,\ [1,0,3,1]_3,\ [0,3,3,1]_3\,\}\,; \\ H_4(81) &= \{\,[1,1,0,1]_4,\ [1,0,4,1]_4,\ [0,4,4,1]_4\,\}\,. \end{aligned}$$

Note that the number of hyper 3-ary partitions of 37 is the same as the number of hyper 4-ary partitions of 81. In other words,

$$h_3(37) = h_4\left(F_{3,4}(37)\right) = h_4(81)\,.$$

We also observe that the coefficients of the partitions are similar, indicating that there is a relationship between the partitions in each set. This relationship will be further explored in the next section.

3 Bijections between hyper m-ary partitions and hyper $(m+1)$-ary partitions

We now verify the result suggested by the example in the prior section by considering hyper m-ary partitions of an integer ℓ and the hyper $(m+1)$-ary partitions of $k = F_{m,m+1}(\ell)$.

Lemma 1. *For a positive integer ℓ, let $k = F_{m,m+1}(\ell)$. Define $g_m : H_{m+1}(k) \to H_m(\ell)$ by mapping*

$$[c_r, c_{r-1}, \ldots, c_2, c_1, c_0]_{m+1} \mapsto [b_r, b_{r-1}, \cdots, b_2, b_1, b_0]_m$$

according to the following rules:

$$\begin{aligned} c_i = 0 &\longrightarrow b_i = 0 \\ c_i = 1 &\longrightarrow b_i = 1 \\ &\vdots \\ c_i = m-2 &\longrightarrow b_i = m-2 \\ c_i = m-1 &\longrightarrow b_i = m-1 \\ c_i = m &\longrightarrow b_i = m-1 \\ c_i = m+1 &\longrightarrow b_i = m. \end{aligned}$$

Then g_m is a bijection.

Proof. It is clear from the definition that g_m is a function. So we first show that g_m is one-to-one. Suppose $x = [x_r, x_{r-1}, \ldots x_2, x_1, x_0]_{m+1}$ and $y = [y_r, y_{r-1}, \ldots y_2, y_1, y_0]_{m+1}$ are two hyper $(m+1)$-ary partitions of k such that $x \neq y$. Then there must be at least one digit that doesn't match. Let $J = \{j_1, j_2, \ldots, j_n\}$ be the set of indices such that $x_j \neq y_j$. Then we have two cases.

First suppose without loss of generality that there is an index j such that $x_j \notin \{m-1, m\}$. Then the j^{th} digit of $g_m(x)$ will be different than the j^{th} digit of $g_m(y)$. Thus $g_m(x) \neq g_m(y)$.

Now suppose that $x_j \in \{m-1, m\}$ and $y_j \in \{m-1, m\}$ for all $j \in J$. Let $J_1 = \{j \in J : x_j = m-1\}$ and $J_2 = \{j \in J : x_j = m\}$. Note that $y_j = m$ for all $j \in J_1$ and $y_j = m-1$ for all $j \in J_2$. Also observe that

$$\begin{aligned} x &= \sum_{j \notin J} x_j m^j + \sum_{j \in J_1} (m-1)m^j + \sum_{j \in J_2} m \cdot m^j \\ y &= \sum_{j \notin J} y_j m^j + \sum_{j \in J_1} m \cdot m^j + \sum_{j \in J_2} (m-1)m^j. \end{aligned}$$

Since $x_j = y_j$ for all $j \notin J$,

$$\begin{aligned} x - y &= \sum_{j \in J_1} (m-1-m)m^j + \sum_{j \in J_2} (m-m+1)m^j \\ &= \sum_{j \in J_2} m^j - \sum_{j \in J_1} m^j. \end{aligned}$$

Observe that $x - y = 0$ since x and y are two different hyper $(m+1)$-ary partitions of the same number k, implying

$$\sum_{j \in J_2} m^j - \sum_{j \in J_1} m^j = 0.$$

However, since J_1 and J_2 are disjoint, this is impossible. Thus it must be the case that when $x \neq y$, one of x_j or y_j must be outside of $\{m-1, m\}$ so that $g_m(x) \neq g_m(y)$ as seen above. Thus g_m is one-to-one.

To show that g_m is onto, consider $b = [b_r, b_{r-1}, \cdots, b_2, b_1, b_0]_m \in H_m(\ell)$. We then define $c = [c_r, c_{r-1}, \cdots, c_2, c_1, c_0]_{m+1}$ in the following way. If $b_i \in \{0, 1, 2, \ldots, m-3, m-2\}$, then set $c_i = b_i$ and if $b_i = m$, set $c_i = m+1$. Now suppose $b_i = m-1$. Let v be the minimal index with $v < i$ such that $b_v \neq m-1$. If v does not exist, then set $c_i = m-1$. If v does exist with $b_v = m$, then set $c_i = m$. If v exists with $b_v \in \{0, 1, 2, \ldots, m-2\}$, then set $c_i = m-1$. Notice that we may verify that $c \in H_{m+1}(k)$ by converting c into the base $m+1$ representation of k. Therefore b is the image of c under g_m and thus g_m is onto. $\square$

This bijection implies that the number of m-ary partitions of any integer ℓ is the same as the number of $(m+1)$-ary partitions of $F_{m,m+1}(\ell)$.

4 Hyper m_1-ary partitions and hyper m_2-ary partitions

In this section, we use the result of Lemma 1 to define a more general bijection between $H_{m_1}(n)$ and $H_{m_2}(F_{m_1,m_2}(n))$ for $m_2 > m_1 + 1$. To do this, we need the following lemma about hyper m_2-ary partitions of an integer n.

In the following proof, observe that multiplying a partition $[x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_m$ by m corresponds to shifting the coefficients to the left one place and adding an additional 0 as the last coefficient.

Lemma 2. *Let $m_2 > m_1 + 1$. If the base m_2 representation of an integer n contains only digits from the set $\{0, 1, 2, \ldots, m_1 - 1\}$, then there are no hyper m_2-ary partitions of n which use any of the coefficients $m_1, m_1 + 1, \ldots, m_2 - 2$.*

Proof. We will prove this by induction on n. Assume that for all $q < n$, when the base m_2 representation of q contains only digits from the set $\{0, 1, 2, \ldots, m_1 - 1\}$, then there are no hyper m_2-ary partitions of n which use any of the coefficients $m_1, m_1 + 1, \ldots, m_2 - 2$.

First, consider when $n = m_2 q$ and suppose that in base m_2 the digits of n come from the set $\{0, 1, 2, \ldots, m_1 - 1\}$. This means the digits in the base m_2 representation of q also come only from this set. Now, apply the recurrence (1) to write $h_{m_2}(m_2 q) = h_{m_2}(q) + h_{m_2}(q - 1)$. This implies that every hyper m_2-ary partition of n is obtained from either a hyper m_2-ary partition of q or a hyper m_2-ary partition of $q - 1$.

Observe that a hyper m_2-ary partition of n obtained from a hyper m_2-ary partition of q is found by multiplying the latter partition by m_2, thereby shifting the coefficients of q and appending a 0 at the end. This results in hyper m_2-ary partitions of n whose coefficients are the same as the coefficients of hyper m_2-ary partitions of q, along with an additional 0. Similarly, a hyper m_2-ary partition of n that is obtained from a hyper m_2-ary partition of $q - 1$ is found by shifting the digits of the latter partition and appending an m_2 to the end. This means we may write

$$\begin{aligned} H_{m_2}(n) = &\{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0, 0]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q)\} \\ &\cup \{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0, m_2]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q-1)\}. \end{aligned}$$

Since $q - 1$ and q are less than n, by the induction hypothesis we know the coefficients of all hyper m_2-ary partitions of $q - 1$ and q are from the set $\{0, 1, 2, \ldots, m_1 - 1, m_2 - 1, m_2\}$. Thus, the coefficients of any hyper m_2-ary partition of n are also from this set.

Now assume that $n = m_2 q + s$, where $1 \leq s \leq m_2 - 1$. Observe that since the base m_2 representation of n contains only digits from the set $\{0, 1, 2, \ldots, m_1 - 1\}$, then we must have $1 \leq s \leq m_1 - 1$. Furthermore, when $n = m_2 q + s$, apply the recurrence (2) to conclude that a hyper m_2-ary partition of n is obtained from a hyper m_2-ary partition of q by multiplying the latter partition by m_2 and appending s to the end, where $1 \leq s \leq m_1 - 1$. So,

$$H_{m_2}(n) = \{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0, s]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q)\}.$$

Since $q < n$, the coefficients of all hyper m_2-ary partitions of q are in the set $\{0, 1, 2, \ldots, m_1 - 1, m_2 - 1, m_2\}$. Since s is an element of this set, we conclude that the coefficients of hyper m_2-ary partitions of n come from the same set.

Therefore, in all cases, the hyper m_2-ary partitions of n never contain any of the coefficients $m_1, m_1 + 1, \ldots, m_2 - 2$. $\square$

Now we are ready to prove there is a bijection between hyper m_1-ary partitions of an integer ℓ and hyper m_2-ary partitions of $k = F_{m_1,m_2}(\ell)$.

Lemma 3. *Let ℓ be a positive integer and set $k = F_{m_1,m_2}(\ell)$. Define $\phi : H_{m_2}(k) \to H_{m_1}(\ell)$ by mapping*

$$[c_r, c_{r-1}, \ldots, c_2, c_1, c_0]_{m_2} \mapsto [b_r, b_{r-1}, \cdots, b_2, b_1, b_0]_{m_1}$$

according to the following rules:

$$\begin{aligned} c_i = 0 &\longrightarrow b_i = 0 \\ c_i = 1 &\longrightarrow b_i = 1 \\ &\vdots \\ c_i = m_1 - 1 &\longrightarrow b_i = m_1 - 1 \\ c_i = m_2 - 1 &\longrightarrow b_i = m_1 - 1 \\ c_i = m_2 &\longrightarrow b_i = m_1. \end{aligned}$$

Then, ϕ is a bijection.

Proof. If $m_2 = m_1 + 1$, then the result follows immediately from Lemma 1. So, we assume that $m_2 > m_1 + 1$. From the definition of k, we know the base m_2 representation of k includes only digits less than or equal to $m_1 - 1$. So, we apply Lemma 2 to conclude that none of the hyper m_2-ary partitions in $H_{m_2}(k)$ have any coefficients between m_1 and $m_2 - 2$, inclusive. Thus, ϕ need only specify how to map coefficients from the set $\{0, 1, \ldots, m_1 - 1, m_2 - 1, m_2\}$.

Now, using the bijection g_m given in Lemma 1, define a new function $G : H_{m_2}(k) \to H_{m_1}(\ell)$ as follows:

$$G = g_{m_1} \circ g_{m_1+1} \circ g_{m_1+2} \circ \cdots \circ g_{m_2-2} \circ g_{m_2-1} \ .$$

It is clear from Lemma 1 that when we apply G to any m_2-ary partition coefficient which is less than or equal to $m_1 - 1$, the coefficient maps to itself. When we apply G to a partition coefficient of $m_2 - 1$, we see that

$$m_2 - 1 \xmapsto{g_{m_2-1}} m_2 - 2 \xmapsto{g_{m_2-2}} m_2 - 3 \xmapsto{g_{m_2-3}} \cdots \xmapsto{g_{m_1}} m_1 - 1 \ .$$

Finally, when we apply G to a partition coefficient of m_2, we see that

$$m_2 \xmapsto{g_{m_2-1}} m_2 - 1 \xmapsto{g_{m_2-2}} m_2 - 2 \xmapsto{g_{m_2-3}} \cdots \xmapsto{g_{m_1}} m_1 \ .$$

Thus, $G = \phi$.

We have ϕ equal to a finite composition of bijective functions. Therefore, ϕ is a bijection. □

Lemma 3 leads to the following identity between values of h_{m_1} and h_{m_2}.

Theorem 4. *Let $2 \le m_1 < m_2$. For positive integer ℓ, set $k = F_{m_1,m_2}(\ell)$. Then*

$$h_{m_2}(k) = h_{m_1}(\ell).$$

Proof. The values ℓ and k given here match Lemma 3 and we know that $h_{m_1}(\ell) = |H_{m_1}(\ell)|$ and $h_{m_2}(k) = |H_{m_2}(k)|$. Lemma 3 gives a bijection between these finite sets. Therefore, we conclude that the sets must have the same cardinality. □

As an immediate corollary, we now state a final result regarding the relationships between hyper m-ary partition sequences for different values of m.

Corollary 5. *Let $2 \le m_1 \le m_2$. Then h_{m_1} is a subsequence of h_{m_2}.*

These theorems extend the results in [4], ultimately showing that the subsequence identity holds for any hyper m_1-ary and hyper m_2-ary partition sequences.

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