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On the Largest Product of Primes with Bounded Sum

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Abstract

Let $h(n)$ denote the largest product of primes whose sum is $\leqslant n$, and $g(n)$ denote the Landau function, which is the largest product of powers of primes whose sum is $\leqslant n$. In this article, several properties of $h(n)$ are given and compared to similar properties of $g(n)$. Special attention is paid to the behavior of the largest prime factor of $h(n)$.

1 Introduction

If $n \geqslant 2$ is an integer, let us define $h(n)$ as the greatest product of a family of primes $q_1 < q_2 < \cdots < q_j$ the sum of which does not exceed n. Let ℓ be the additive function such that $\ell(p^\alpha) = p^\alpha$ for p prime and $\alpha \geqslant 1$. In other words, if the standard factorization of M into primes is $M = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_j^{\alpha_j}$ we have $\ell(M) = q_1^{\alpha_1} + q_2^{\alpha_2} + \cdots + q_j^{\alpha_j}$ and $\ell(1) = 0$. If μ denotes the Möbius function, $h(n)$ can also be defined by

$$h(n) = \max_{\substack{\ell(M) \leqslant n \\ \mu(M) \neq 0}} M. \tag{1}$$

Note that

$$\ell(h(n)) \leqslant n. \tag{2}$$

Landau [11] introduced the function $g(n)$ as the maximal order of an element in the symmetric group S_n; he showed that

$$g(n) = \max_{\ell(M) \leqslant n} M. \tag{3}$$

Sequences $(h(n))_{n\geqslant 1}$ and $(g(n))_{n\geqslant 1}$ are sequences A159685 and A000793 in the OEIS (*On-line Encyclopedia of Integer Sequences*).

From Eqs. (1) and (3), it follows that

$$h(n) \leqslant g(n), \quad (n \geqslant 0). \tag{4}$$

In [5] we gave some properties of $h(n)$ and an algorithm to compute $h(n)$ for large values of n.

In Section 2 below, these properties of $h(n)$ are recalled and compared to similar properties of $g(n)$. We also explain how the algorithm given in [5] can be adapted to calculate $h(n)$ for all n up to 10^{10}.

In Section 3, we recall various results about the distribution of primes.

Section 4 is devoted to effective and asymptotic estimates for $\log h(n)$, $\omega(h(n))$ and the differences $\log g(n) - \log h(n)$ and $\omega(h(n)) - \omega(g(n))$.

The last section, Section 6, studies the largest prime factor $P^+(h(n))$ of $h(n)$. This study uses the same tool (the so-called G-sequences) introduced by Grantham [9], and developed in [5] to estimate $P^+(g(n))$. The G-sequences are described in Section 5. The last result of the paper is a comparison between $P^+(h(n))$ and $\log h(n)$.

1.1 Notation

1. p denotes a generic prime. For $i \geqslant 1$, p_i is the i^{th} prime.
2. $\pi(x) = \sum_{p \leqslant x} 1$ is the number of primes $\leqslant x$.
3. $\theta(x)$ is the Chebyshev function

$$\theta(x) = \sum_{p \leqslant x} \log p. \tag{5}$$

4. Θ is the least upper bound of the real parts of the zeros of the Riemann ζ function. Under the Riemann hypothesis $\Theta = 1/2$.
5. $\log_2 x$ represents the iterated logarithm $\log\log x$.
6. $\mathrm{Li}(x)$, the integral logarithm of x, is defined for $x > 1$ by

$$\mathrm{Li}(x) = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right) = \gamma + \log_2 x + \sum_{n=1}^{+\infty} \frac{(\log x)^n}{n \cdot n!},$$

where $\gamma = 0.577\cdots$ is Euler's constant.

7. For each integer $N > 1$, $P^+(N)$ is the largest prime factor of N and $\omega(N) = \sum_{p|N} 1$ is the number of prime factors of N.

Let us write $\sigma_0 = 0$, $N_0 = 1$, and, for $j \geqslant 1$,

$$N_j = p_1 p_2 \cdots p_j \quad \text{and} \quad \sigma_j = p_1 + p_2 + \cdots + p_j = \ell(N_j). \tag{6}$$

For $n \geqslant 0$, let $k = k(n)$ denote the integer $k \geqslant 0$ such that

$$\sigma_k = p_1 + p_2 + \cdots + p_k \leqslant n < p_1 + p_2 + \cdots + p_{k+1} = \sigma_{k+1}. \tag{7}$$

2 General properties of $h(n)$

2.1 Theoretical properties of $h(n)$

In this section we recall the properties of $h(n)$ that we will use [5]. First of all (cf. [5, Prop 3.1]) for each nonnegative integer j,

$$h(\sigma_j) = N_j. \tag{8}$$

Proposition 1. *Let n be a nonnegative integer and $k = k(n)$. Then*

$$\log N_k = \theta(p_k) \leqslant \log h(n) \leqslant \theta(p_{k+1}) = \log N_{k+1}. \tag{9}$$

Proof. From the definition of $k(n)$ we have $\sigma_k \leqslant n < \sigma_{k+1}$. Using Eq. (8) and the fact that h is nondecreasing, this gives $N_k \leqslant h(n) \leqslant N_{k+1}$. Taking logarithms, we get Eq. (9). $\square$

From Eq. (9) we have $h(n) \geqslant N_k$ and, since $h(n)$ is squarefree,

$$P^+(h(n)) \geqslant p_{k(n)}. \tag{10}$$

We also have $h(n) = \prod_{p|h(n)} p \leqslant \prod_{p \leqslant P^+(h(n))} p$ and thus

$$\log h(n) \leqslant \theta(P^+(h(n))). \tag{11}$$

In [5, (1.9)], for $n \geqslant 0$, $h_j(n)$ is defined for each integer j satisfying $0 \leqslant j \leqslant k(n)$ by:

$$h_j(n) = \max_{\substack{\ell(M) \leq n \\ \mu(M) \neq 0,\ \omega(M) = j}} M, \tag{12}$$

where $\omega(M)$ is the number of prime factors of M. The result [5, Theorem 6.1] implies that, for each n, the sequence $(h_j(n))_{0 \leqslant j \leqslant k(n)}$ is increasing, and therefore,

$$h(n) = h_{k(n)}(n).$$

This result that could appear quite obvious at first sight, is not so easy to prove. It depends strongly on the distribution of prime numbers. It has an obvious consequence (cf. [5, corollary (6.1)]):

Theorem 2. *The number of prime factors of $h(n)$, $\omega(h(n))$ is given by*

$$\omega(h(n)) = k(n), \tag{13}$$

where $k(n)$ is defined in Eq. (7).

Theorem 3. *Let j be a nonnegative integer.*

(i) *We have $h(\sigma_{j+1} - 1) = h(\sigma_{j+1} - 2) = N_{j+1}/2$.*

(ii) *If $q \leqslant p_{j+1}$ is a prime, then $h(\sigma_{j+1} - q) = N_{j+1}/q$.*

(iii) *If $j \geqslant 1$ and if a is an even number satisfying $4 \leqslant a < p_{j+1}$, we have*

$$h(\sigma_{j+1} - a) = h(\sigma_{j+1} - a - 1). \tag{14}$$

(iv) *The number $h(n)$ is odd when $n = \sigma_{j+1} - 1$ or $n = \sigma_{j+1} - 2$. It is even for $\sigma_j \leqslant n \leqslant \sigma_{j+1} - 3$.*

(v) *For $n \geqslant 1$ the inequality $h(n) \leqslant 2h(n-1)$ holds.*

(vi) *We have*

$$\liminf_{n\to\infty} \frac{h(n)}{h(n-1)} = 1 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{h(n)}{h(n-1)} = 2.$$

Proof. *(i)* is from [5, Proposition 5.3].

(ii) follows from [5, Eqs. (8.13) and (8.6)].

(iii) is [5, Proposition 5.1].

(iv) The first part is implied by (i). Now assume that $\sigma_j \leqslant n \leqslant \sigma_{j+1} - 3$. From Theorem 2 we know that $\omega(h(n)) = j$. If 2 does not divide $h(n)$ we would have

$$\ell(h(n)) \geqslant 3 + 5 + \cdots + p_{j+1} = \sigma_{j+1} - 2,$$

in contradiction with the inequality (2), which proves (iv).

(v) First let us consider the case $h(n)$ even; we then have $\ell(h(n)/2) = \ell(h(n)) - 2 \leqslant n-2$, so that, by Eq. (1), $h(n-1) \geqslant h(n)/2$ holds. If $h(n)$ is odd, from (iv), we have $n = \sigma_{j+1} - 1$ or $n = \sigma_{j+1} - 2$ for some $j \geqslant 1$ and, from (i) and (ii), $h(\sigma_{j+1} - 1) = h(\sigma_{j+1} - 2) = N_{j+1}/2$ and $h(\sigma_{j+1} - 3) = N_{j+1}/3$, which completes the proof of (v). Note that $h(n) = 2h(n-1)$ implies $h(n-1)$ odd and $h(n)$ even. From (iv), this occurs if and only if $n = \sigma_j$ for some j.

(vi) From the fact that h is nondecreasing we get $h(n)/h(n-1) \geqslant 1$ while Eq. (14) shows that $h(n) = h(n-1)$ for infinitely many n, which gives the value of the lim inf. From (v) we have $h(n)/h(n-1) \leqslant 2$ for all $n \geqslant 1$. Moreover, from Eq. (8) we have $h(\sigma_{j+1}) = N_{j+1}$, which, together with (i), shows that $h(\sigma_{j+1})/h(\sigma_{j+1}-1) = 2$ holds for infinitely many j's and the proof of (vi) is completed.

$\square$

Remark 4. Properties *(iv), (v) and (vi) of $h(n)$ are analogous to the following properties of $g(n)$.*

(iv) *$g(n)$ is odd only for $n \in \{3, 8, 15\}$ (cf. [16, p. 142]).*

(v) *For $n \geqslant 1$, we have $g(n) \leqslant 2g(n-1)$ (cf. [16, p. 143]).*

(vi) $\lim_{n\to\infty} g(n)/g(n-1) = 1$ *is proved in [17].*

Proposition 5. *For $n \geqslant 1$, let $\gamma_h(n)$ denote the cardinality of the set $\{h(m); 1 \leqslant m \leqslant n\}$. Then, with $k = k(n)$ defined by Eq. (7), we have*

$$\frac{(k+2)(k-1)}{2} \leqslant \gamma_h(n) \leqslant n - \frac{\sigma_k - k}{2}, \tag{15}$$

and, when $n \to \infty$,

$$\frac{2n}{\log n} \lesssim \gamma_h(n) \lesssim \frac{n}{2}. \tag{16}$$

Proof. First, we observe that $\gamma_h(n)$ is the number of $m \leqslant n$ such that $h(m) > h(m-1)$ holds. From Theorem 3 (ii), for $1 \leqslant i \leqslant j$ and $m = \sigma_j - p_i$ we have $\ell(h(m)) = m$ and thus $h(m) > h(m-1)$ holds, so that

$$\gamma_h(n) \geqslant \sum_{j=2}^{k} (\gamma_h(\sigma_j - 1) - \gamma_h(\sigma_{j-1} - 1)) \geqslant \sum_{j=2}^{k} j = \frac{(k+2)(k-1)}{2}.$$

To prove the upper bound in Eq. (15), we note that $n - \gamma_h(n)$ is the number of $m \leqslant n$ such that $h(m) = h(m-1)$. From Theorem 3 (i) and (iii), for $j \geqslant 2$ and $\sigma_j \leqslant m < \sigma_{j+1}$, we have $h(m) = h(m-1)$ for $\sigma_{j+1} - m \in \{1, 4, 6, \ldots, p_{j+1} - 1\}$, so that, as $h(0) = h(1) = 1 < h(2) = 2 < h(3) = 3 = h(4)$ we get

$$n - \gamma_h(n) \geqslant 2 + \sum_{j=2}^{k-1} \frac{p_{j+1} - 1}{2} = 2 + \frac{\sigma_k - 5}{2} - \frac{k-2}{2} \geqslant \frac{\sigma_k - k}{2}.$$

Finally, to prove (16), we use Lemma 9 below. We get $\sigma_k = \sum_{p \leqslant p_k} p \sim \dfrac{p_k^2}{2\log p_k}$. But, from the prime number theorem, we have $p_k \sim k \log k$ so that $\log p_k \sim \log k$,

$$\sigma_k \sim \frac{k^2 \log k}{2} \sim \sigma_{k+1}, \quad n \sim \frac{k^2 \log k}{2}, \quad \log k \sim \frac{\log n}{2} \quad \text{and} \quad k \sim 2\sqrt{\frac{n}{\log n}}$$

which, together with (15), proves (16). $\square$

Remark 6. The estimates for $\gamma_g(n)$ are weaker (cf. [16, p. 162–164] and [18, p. 218]). However it seems difficult to show $\gamma_h(n) \sim n/2$ which is probably true.

Proposition 7. *Let $k \geqslant 1$ and $n \geqslant 2$ satisfy $k(n) = k$ (defined by Eq. (7)), so that $\sigma_k \leqslant n < \sigma_{k+1}$ holds.*

(i) If $\sigma_k \leqslant n < \sigma_k + p_{k+1} - p_k$, then we have $h(n) = h(\sigma_k) = N_k$ and $P^+(h(n)) = p_k$.

(ii) If $\sigma_k + p_{k+1} - p_k \leqslant n < \sigma_{k+1}$, then we have $P^+(h(n)) \geqslant p_{k+1}$.

Proof. From Theorem 3 (ii), we have

$$h(\sigma_k + p_{k+1} - p_k) = N_{k+1}/p_k > N_k. \tag{17}$$

From Eq. (13), we have $\omega(h(n)) = k$, and from (10) we get $P^+(h(n)) \geqslant p_k$.

- If $P^+(h(n)) = p_k$, then we have $h(n) = N_k$, which, from Eqs. (17) and (1), implies $n < \sigma_k + p_{k+1} - p_k$.

- If $P^+(h(n)) \geqslant p_{k+1}$, then, from Eq. (2), we have

$$n \geqslant \ell(h(n)) \geqslant P^+(h(n)) + p_1 + \cdots + p_{k-1} \geqslant \sigma_k + p_{k+1} - p_k.$$

□

Corollary 8. *There exist arbitrary long intervals on which $h(n)$ is constant.*

Proof. Since the difference $p_{k+1} - p_k$ is not bounded, this follows from Proposition 7 (i) above. Nicolas proved a similar result for $g(n)$ in [16, p. 158]. □

2.2 Computation of $h(n)$

The algorithm given in [5] was used to compute $h(10^k)$ for $1 \leqslant k \leqslant 35$. Let us recall a few facts.

Let $k = k(n)$ be defined above by Eq. (7). The value $h(n)$ may be written as the product of two terms:

$$h(n) = N_k \cdot G(p_k, n - \sigma_k),$$

where $G(p, m)$ is defined by

$$G(p, m) = \max \frac{Q_1 Q_2 \cdots Q_s}{q_1 q_2 \cdots q_s},$$

the maximum being taken over the primes $Q_1, Q_2, \ldots, Q_s, q_1, q_2, \ldots, q_s$, $s \geqslant 0$, satisfying

$$2 \leqslant q_s < q_{s-1} < \cdots < q_1 \leqslant p < Q_1 < Q_2 < \cdots < Q_s \quad \text{and} \quad \sum_{i=1}^{s} (Q_i - q_i) \leqslant m.$$

The algorithm given in [5] for computing an isolated value $h(n)$ is composed of two steps.

(i) The first step is the computation of $k = k(n)$, p_k and σ_k.

(ii) The second step is the computation of the fraction $h(n)/N_k = G(p_k, n - \sigma_k)$,

When computing $h(n)$ for an n larger than, say 10^5, most of the computation time is devoted to the first step.

In this article we needed to compute the values of $h(n)$ for all n up to x, for some values x, the largest of them being $x = 10^{10}$ (this computation took about 100 hours of one processor on an AMD Shanghai computer with 8 processors). In this case, the computation of p_k and σ_k is done once and for all for the n belonging to the same $[\sigma_k, \sigma_{k+1})$. So, working by slices on the successive $[\sigma_k, \sigma_{k+1})$, the time of computation is mostly devoted to the computation of $G(p_k, n - \sigma_k)$.

3 About the distribution of primes

3.1 Some lemmas

Lemma 9. *Let us write $S(x) = \sum_{p \leqslant x} p$. When x tends to infinity,*

$$S(x) \sim \frac{x^2}{2 \log x}. \tag{18}$$

Proof. Massias et al. [12, Lemme B] proved that $S(x) = \mathrm{Li}(x^2) + O\left(x^2 e^{-a\sqrt{\log x}}\right)$ for some $a > 0$, which implies (18), since $\mathrm{Li}(t) \sim t/\log t$ when $t \to \infty$. □

Lemma 10 below is [14, Lemma 2].

Lemma 10. *Let a be a nonnegative real number and $\Phi = \Phi_a$ the function defined by*

$$\Phi(x) = \sqrt{x \log x}\left(1 + \frac{\log_2 x - a}{2 \log x}\right).$$

Then Φ is increasing and concave for $x > 1$.

Proposition 11 is from [14, Point (ii), Proposition 1, p. 672].

Proposition 11. *For $k \geqslant 4398$, that is $p_k \geqslant 42061$, and $b = 1.16$, we have*

$$\theta(p_k) \geqslant \Phi_b(\sigma_{k+1}) = \sqrt{\sigma_{k+1} \log \sigma_{k+1}}\left(1 + \frac{\log_2 \sigma_{k+1} - b}{2 \log \sigma_{k+1}}\right). \tag{19}$$

3.2 The error term in the prime number theorem

Let θ be the Chebyshev function defined in (5).

(i) There is some $a > 0$ such that

$$\theta(x) = x + O\Big(xe^{-a\sqrt{\log x}}\Big). \tag{20}$$

(ii) If $\Theta < 1$, we have

$$\theta(x) = x + O\big(x^{\Theta} \log^2 x\big). \tag{21}$$

(iii) Under the Riemann hypothesis, we have

$$|\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x, \qquad x \geqslant 599. \tag{22}$$

Points (i) and (ii) may be found in [10] or [8], for instance. Point (iii) is proved in [21, p. 337].

3.3 Effective bounds

We shall use the following results of P. Dusart:

$$\theta(x) < x \text{ for } x \leqslant 8 \cdot 10^{11} \quad (\text{cf. [7, Table 6.6]}) \tag{23}$$

$$\theta(x) < x + \frac{x}{36\,260} \leqslant 1.000\,028\, x \quad (x > 0)\ (\text{cf. [7, Proposition 5.1]}) \tag{24}$$

$$|\theta(x) - x| \leqslant \frac{0.05\, x}{\log^2 x} \quad (x \geqslant 122\,568\,683)\ (\text{cf. [7, Theorem 5.2]}) \tag{25}$$

3.4 Distances between primes

(i) Dusart [7, Proposition 6.8] has proved that, for $x \geqslant 396\,738$, the interval

$$\left[x,\, x + \frac{x}{25\log^2 x}\right] \tag{26}$$

contains a prime number. This implies, for $p_i \geqslant 396\,833 = p_{33\,609}$,

$$p_{i+1} \leqslant p_i + \frac{p_i}{25\, \log^2 p_i}. \tag{27}$$

(ii) Under the Riemann hypothesis, Formula (22) implies that, for $x \geqslant 599$, the interval $\big[x - \sqrt{x}\log^2 x/(4\pi),\, x\big]$ contains a prime number.

Still under the Riemann hypothesis, Cramér [2] proved that there exists b such that the interval $[x,\, x+b\sqrt{x}\log x]$ contains a prime. Ramaré et al. [19, Th. 1] have made effective this result by proving that, for $x \geqslant 2$, the interval

$$\left[x-\frac{8}{5}\sqrt{x}\log x,\; x\right] \tag{28}$$

contains a prime number, which implies that, for $p_i \geqslant 3$,

$$p_{i-1} \geqslant p_i - \frac{8}{5}\sqrt{p_i}\log p_i. \tag{29}$$

In [2], the "Cramér Conjecture" is stated as

$$p_{i+1}-p_i = O\big(\log^2 p_i\big)\,. \tag{30}$$

This conjecture is supported by numerical computations (cf., for example, [15]).

3.5 The η_k functions

Let $k \geqslant 1$ be integer. By the prime number theorem, the quotient p_{i-k}/p_i tends to 1 when $i \to +\infty$. Thus, for $i_0 \geqslant k+1$ there is at most a finite number of i's such that $\dfrac{p_{i-k}}{p_i} \leqslant \dfrac{p_{i_0-k}}{p_{i_0}} < 1$, and the following definition makes sense.

Definition 12. We define η_k on the interval $[p_k,\, +\infty)$ by

$$\eta_k(x) = \min\left\{\frac{p_{i-k}}{p_i} \mid p_i > x\right\}. \tag{31}$$

From (31) we see that η_k is a nondecreasing and right-continuous step function whose discontinuity points are primes called *η_k–champion numbers*. By convention, p_k is considered as an η_k–champion. The following lemma is proved in [4, §2.4]

Lemma 13.

(i) Let x be $\geqslant p_k$. For all $y \geqslant x$, the interval $(\eta_k(x)y,\, y]$ contains at least k prime numbers, and $\eta_k(x)$ is the largest real number λ such that $(\lambda y,\, y]$ contains at least k prime numbers for all $y \geqslant x$.

(ii) Let p be an η_k–champion. Then for all $x \geqslant p$, $\eta_k(x) \geqslant \eta_k(p)$, in particular, the interval $(\eta_k(p)x,\, x]$ contains at least k prime numbers.

Tables of the first few η_k–champion numbers for $k = 1, 2, 3$, may be found in [4, Tables 3,4,5], or on [3] for more values. Proposition 14 (cf. [4, Proposition 2.1] for more details) recalls some results that we shall use later.

Proposition 14.

(i) From [1, p. 562] there exists $a > 0$ such that, for $x \geqslant p_k$, we have

$$\eta_k(x) \geqslant 1 - k\frac{a}{x^{0.475}}\,.$$

(ii) Let i_0 be defined by $i_0 = 33\,609$. For i close to i_0 the values of p_i are

$i =$	33 608	33 609	33 610	33 611	33 612
$p_i =$	396 733	396 833	396 871	396 881	396 883

For $x \geqslant p_{i_0+k-1}$ we have from Eq. (26) that

$$\eta_k(x) \geqslant 1 - \frac{k}{25 \log^2 x}. \tag{32}$$

(iii) Under the Riemann hypothesis, for $x \geqslant \max(p_k, e^2)$, we have from (28)

$$\eta_k(x) \geqslant 1 - \frac{8k}{5}\frac{\log x}{\sqrt{x}}. \tag{33}$$

(iv) Under Cramér's conjecture (30), there exists $a > 0$ such that, for $x \geqslant \max(p_k, e^2)$

$$\eta_k(x) \geqslant 1 - ka\frac{\log^2 x}{x}. \tag{34}$$

3.6 The $\theta_{\min}$, θ_d, and δ_3 functions

In this subsection we introduce three functions, $\theta_{\min}$, θ_d, δ_3, defined on the real interval $[1, +\infty)$. More information about them can be found in [4, §2].

Definition 15. The function $\theta_{\min}$ is the nondecreasing right-continuous step function defined by

$$\theta_{\min}(y) = \inf_{x \geqslant y} \frac{\theta(x)}{x} = \inf_{p_i > y} \frac{\theta(p_{i-1})}{p_i}. \tag{35}$$

A table of the first few $\theta_{\min}$–champion numbers p and their rounded-down records $\theta_{\min}(p)$ may be found in [4, Table 1] or on the web pages of the first author [3, Tabulation de thetamin]. If $p < q$ are two consecutive $\theta_{\min}$–champion numbers, then $\theta_{\min}$ is constant on $[p, q)$ and equal to $\theta_{\min}(p) = \theta(q^-)/q$ where q^- is the prime preceding q. If $x \geqslant p$, then $\theta(x)/x \geqslant \theta_{\min}(p)$ holds.

Definition 16. Let us define $\theta_d(y)$ for $y \geqslant 1$ by

$$\theta_d(y) = \sup_{x \geqslant y} \left| \frac{\theta(x)}{x} - 1 \right| \log^2 x,$$

so that, for $x \geqslant y$, we have

$$\left| \frac{\theta(x)}{x} - 1 \right| \leqslant \frac{\theta_d(y)}{\log^2 x}. \tag{36}$$

A table of the first few θ_d–champion numbers and records is given in [4, Table 2]. A more extensive table may be found in [3, Tabulation de thetad]. If $p < q$ are two consecutive θ_d–champion numbers, we have $\theta_d(p) = (1 - \theta(q^-)/q) \log^2 q$ where q^- is the prime preceding q. For $p \leqslant x < q$, $\theta_d(x) = \theta_d(p)$ and, for $x \neq 1$, $1 - \theta_d(p)/\log^2(x) \leqslant \theta(x)/x$.

Definition 17. Let us define the function δ_3, for $y \geqslant p_3 = 5$, by

$$\delta_3(y) = \sup_{x \geqslant y} (1 - \eta_3(x)) \log^2 x.$$

For $x \geqslant y$, we have

$$1 - \eta_3(x) \leqslant \frac{\delta_3(y)}{\log^2(x)}. \tag{37}$$

A table of the first few δ_3–champion numbers is given in [4, Table 6]. A more extensive table may be found in [3, Tabulation de delta3]. If $p < q$ are two consecutive δ_3–champion numbers, we have $\delta_3(p) = (1 - \eta_3(q^-)) \log^2(q)$ where q^- is the prime preceding q. For $p \leqslant x < q$, $\delta_3(x) = \delta_3(p)$ and, for $x \neq 1$, $1 - \delta_3(p)/\log^2(x) \leqslant \eta_3(x) \leqslant 1$.

4 Estimates for $\log h(n)$ and $\omega(h(n))$

4.1 An asymptotic equivalent of $\log h(n)$

Theorem 18. *When $n \to +\infty$, $\log h(n) \sim \sqrt{n \log n}$.*

Proof. Let be $k = k(n)$ defined by (7). From (9) it is sufficient to prove that, when n tends to infinity, $\log N_k \sim \sqrt{n \log n}$. When $k \to \infty$, $p_{k-1} \sim p_k$, and, from (18), $\sigma_{k-1} \sim \sigma_k \sim \dfrac{p_k^2}{2 \log p_k}$. With (7) this gives $n \sim \dfrac{p_k^2}{2 \log p_k}$, from which we infer

$$p_k \sim \sqrt{n \log n},$$

and, by the prime number theorem, $\log N_k = \theta(p_k) \sim p_k \sim \sqrt{n \log n}$. □

4.2 Effective estimates of $\log h(n)$

Theorem 19. *The assertion*

$$n \geqslant n(b) \implies \log h(n) > \sqrt{n \log n}\left(1 + \frac{\log_2 n - b}{2 \log n}\right) \tag{38}$$

is true for the following pairs $(b, n(b))$ (where each $n(b)$ is optimal):

b	2.0	1.8	1.6	1.4	1.20	1.18	1.16
$n(b)$	19 491	57 458	201 460	1 303 470	29 696 383	44 689 942	77 615 268

Proof. First suppose that $b = 1.16$ and $n \geqslant 87\,179\,593 = \sigma_{4398}$. Let $k = k(n)$ defined by (7) so that $\sigma_k \leqslant n < \sigma_{k+1}$ holds. From (9), Inequality (19) and Lemma 10, we may write

$$\log h(n) \geqslant \theta(p_k) \geqslant \Phi_b(\sigma_{k+1}) \geqslant \Phi_b(n).$$

This proves that (38) is true for $b = 1.16$ with $n(b) = 87\,179\,593$.

The computation, by the method described in §2.1, of all the values $h(n)$ for $n \leqslant 87\,179\,592$ gives, for each value of b the smallest value $n(b)$ such that Eq. (38) holds. ☐

Corollary 20. *We have*

$$\log h(n) > \sqrt{n \log n} \qquad \textit{for } n \geqslant 7\,387. \tag{39}$$

Proof. By using (38) with $b = 2$, if $n \geqslant 19\,491$ we may write

$$\log h(n) > \sqrt{n \log n}\left(1 + \frac{\log_2 n - 2}{2 \log n}\right) \geqslant \sqrt{n \log n}$$

since $\log_2 n \geqslant 2$. By computing $h(n)$ for $2 \leqslant n \leqslant 19\,490$, we prove that $7\,386$ is the largest integer such that $\log h(n) \leqslant \sqrt{n \log n}$. ☐

Theorem 21. *The following inequality is satisfied*

$$\log h(n) \leqslant \sqrt{n \log n}\left(1 + \frac{\log_2 n - 0.975}{2 \log n}\right), \quad n \geqslant 3. \tag{40}$$

Proof. This results from (4) and [14, Theorem 2]. ☐

Corollary 22. *For each $n \geqslant 2$,*

$$\frac{\log h(n)}{\sqrt{n \log n}} \leqslant 1.0482016\cdots, \tag{41}$$

the upper bound being attained for $n = 38343860 = \sigma_{2989} = \ell(N_{2989})$.

Proof. For $n \geqslant n_1 = 3.6 \cdot 10^9$, we have $\dfrac{\log_2 n - 0.975}{2\log n} \leqslant 0.048$. Thus, by (40), Inequality (41) is satisfied for $n \geqslant n_1$. The computation of the values $h(n)$ for n up to n_1 shows that the maximum value is attained on $n = \sigma_{2989}$. $\square$

It is possible to shorten the computation by using the trick described in [13, §4]. Let us recall briefly how it works.

From Theorem 18 and Corollary 20, we know that there exists $\lambda > 1$ such that

$$\forall n \geqslant 1, \quad \log h(n) \leqslant \lambda\sqrt{n\log n}, \tag{42}$$

with equality for some $n = n_0 \geqslant 2$. Moreover, we have

$$\ell(h(n_0)) = n_0 \tag{43}$$

since, if we should have $\ell(h(n_0)) \leqslant n_0 - 1$, we would have $h(n_0 - 1) = h(n_0)$ and $\log(h(n_0 - 1)/\sqrt{(n_0-1)\log(n_0-1)} > \lambda$, contradicting (42).

Now, let $M \geqslant 1$ be a squarefree integer. From (1.1) and (42), we have

$$\log M \leqslant \log h(\ell(M)) \leqslant \lambda\sqrt{\ell(M)\log\ell(M)} \tag{44}$$

and, by setting $M_0 = h(n_0)$, we get from (43)

$$\log M_0 = \lambda\sqrt{n_0\log n_0} = \lambda\sqrt{\ell(M_0)\log\ell(M_0)}. \tag{45}$$

The function $\phi : t \longrightarrow t\log t$ is a bijection from $(1, +\infty)$ to $(0, +\infty)$. Let us call ϕ^{-1} its inverse function. We set

$$u = u(M) = \phi^{-1}\left(\left(\frac{\log M}{\lambda}\right)^2\right), \quad u_0 = u(M_0), \quad \rho = \frac{2\sqrt{u_0\log u_0}}{\lambda(1+\log u_0)}.$$

We have

$$\log M = \lambda\sqrt{\phi(u)} = \lambda\sqrt{u\log u}, \qquad \log M_0 = \lambda\sqrt{u_0\log u_0}$$

so that (45) yields

$$u_0 = \ell(M_0).$$

Now, we set

$$f_1 = \ell(M) - u(M) \quad \text{ and } \quad f_2 = u(M) - \rho\log M. \tag{46}$$

As ϕ is increasing, $f_1 \geqslant 0$ is equivalent to $\phi(\ell(M)) \geqslant \phi(u(M))$ which follows from (44). Therefore, we have

$$f_1 = \ell(M) - u(M) \geqslant 0 = \ell(M_0) - u(M_0). \tag{47}$$

We have $f_2 = u - \lambda\rho\sqrt{u\log u}$ which is convex on $u > 1$ and whose derivative vanishes for $u = u_0$, due to the choice of ρ. Therefore, we have

$$f_2 \geqslant u_0 - \lambda\rho\sqrt{u_0\log u_0} = \ell(M_0) - \rho\log M_0. \tag{48}$$

By adding (47) to (48), we deduce from (46)

$$\forall \text{ squarefree } M \geqslant 1, \qquad \ell(M) - \rho\log M \geqslant \ell(M_0) - \rho\log M_0. \tag{49}$$

Let p be the largest prime factor of M_0 and assume that a prime $p' < p$ does not divide M_0; then applying (49) for $M = M_0/p$ and for $M = M_0 p'$ yields respectively $\rho \geqslant p/\log p$ and $\rho \leqslant p'/\log p'$ whence $p/\log p < p'/\log p'$. The function $t \mapsto t/\log t$ being minimal for $t = \exp(1)$, this is only possible if $p = 3$, $p' = 2$ and $M_0 = 3$, which is impossible since $\log 3 < \sqrt{3\log 3}$ holds. Therefore, there exists $k \geqslant 1$ such that $M_0 = N_k$, $\ell(M_0) = \sigma_k$ and

$$\lambda = \max_{j\geqslant 1,\ \sigma_j \leqslant 3.6\ 10^9} \frac{\log N_j}{\sqrt{\sigma_j \log\sigma_j}}.$$

4.3 An upper bound for $\log g(n) - \log h(n)$

Theorem 23. *There exists $C > 0$ such that, for $n \geqslant 1$,*

$$\log g(n) - C(n\log n)^{1/4} \leqslant \log h(n) \leqslant \log g(n). \tag{50}$$

We may choose $C = 5.68$.

Proof. Given (4), it is sufficient to prove the left inequality of (50). Let $P = P^+(g(n))$ be the largest prime factor of $g(n)$. Then, by [4, Theorem 5.1], $P \leqslant 1.27\sqrt{n\log n}$. By Bertrand's postulate, there exists a prime q such that

$$P < q \leqslant 2.54\sqrt{n\log n}. \tag{51}$$

Let us write

$$g(n) = \prod_{p\leqslant P} p^{a_p} \quad\text{ where }\quad a_p = v_p(g(n)),$$

and

$$g(n) = A\cdot B \quad\text{ with }\quad A = \prod_{p\leqslant P,\ a_p\geqslant 2} p^{a_p}, \quad B = \prod_{p\leqslant P,\ a_p = 1} p^{a_p}.$$

Then B divides $g(n)$, so that $\ell(B) \leqslant \ell(g(n))$ and, from (3), $\ell(g(n)) \leqslant n$, which implies $\ell(B) \leqslant n$. As B is squarefree, it follows from (3) that $h(n) \geqslant B = g(n)/A$ which implies

$$\log h(n) \geqslant \log g(n) - \log A. \tag{52}$$

Let us find an upper bound for A. If $a_p \geqslant 2$, we have $p^{a_p} \leqslant q + p^{a_p - 1}$ (otherwise $N' = qg(n)/p$ would be such that $N' > g(n)$ and $\ell(N') = \ell(g(n)) + p^{a_p-1} + q - p^{a_p} < \ell(g(n) \leqslant n$, contradicting the definition of $g(n)$). This gives $p^{a_p} \leqslant q/(1-1/p) \leqslant 2q$ and $p \leqslant (2q)^{1/a_p} \leqslant \sqrt{2q}$. From this we deduce $A \leqslant (2q)^{\pi(\sqrt{2q})}$ and, by using the inequality $\pi(t) \leqslant 1.26\, t/\log t$ (cf. [20, (3.6)]) and (51), we get

$$\log A \leqslant 1.26\frac{\sqrt{2q}}{\log\sqrt{2q}}\log(2q) = 2.52\sqrt{2q} \leqslant 5.68(n\log n)^{1/4},$$

which, with (52), ends the proof. $\square$

4.4 Study of $\omega(h(n)) - \omega(g(n))$

Theorem 24. *There exists a constant C such that, for $n \geqslant 2$,*

$$0 \leqslant \omega(h(n)) - \omega(g(n)) \leqslant C \frac{n^{1/4}}{(\log n)^{3/4}}. \tag{53}$$

Proof.
The lower bound: Let n be a positive integer, and define $k = k(n)$ by (7). Therefore we have

$$\sigma_k \leqslant n < \sigma_{k+1} \tag{54}$$

and, by Theorem 2, $\omega(h(n)) = k$. Further, from (3) and (54), we have

$$\sigma_{\omega(g(n))} = p_1 + p_2 + \cdots + p_{\omega(g(n))} \leqslant \sum_{p \mid g(n)} p \leqslant \ell(g(n)) \leqslant n < \sigma_{k+1}$$

whence $\omega(g(n)) \leqslant k = \omega(h(n))$ and the lower bound of (53) is proved.

The upper bound. Let $x_1 > 4$ be a real number and, for $i \geqslant 2$, x_i be the unique number such that $\dfrac{x_i^i - x_i^{i-1}}{\log x_i} = \dfrac{x_1}{\log x_1}$. The sequence (x_i) is decreasing and satisfies

$$x_i < x_1^{1/i} \tag{55}$$

so that $x_i < 2$ for $i > I = \left\lfloor \frac{\log x_1}{\log 2} \right\rfloor$. We set

$$N = N(x_1) = \prod_{i=1}^{I} \prod_{p \leqslant x_i} p. \tag{56}$$

Such an integer N is called in [6, §4] an ℓ–superchampion number associated with x_1. Let G denote the set of ℓ–superchampion numbers. If $N \in G$, the set $\{x_1; N(x_1) = N\}$ is an interval of positive length. Thus we can choose a particular value of x_1 such that, for $i \geqslant 1$, x_i is never prime (cf. [6, Lemma 4, 3.]). With such a value of x_1 (56) becomes

$$N = N(x_1) = \prod_{i=1}^{I} \prod_{p < x_i} p. \tag{57}$$

Now, let us define k_1 and $k_2 \leqslant k_1$ by

$$p_{k_1} < x_1 < p_{k_1+1} \quad \text{and} \quad p_{k_2} < \sqrt{x_1} < p_{k_2+1}. \tag{58}$$

It follows from (57) and (55) that, for every prime p, we have

$$p^{v_p(N)} < x_1 \quad \text{and} \quad v_p(N) = 1 \quad \text{for} \quad p_{k_2} < p \leqslant p_{k_1}. \tag{59}$$

Therefore, we get from (57) and (58)

$$\sigma_{k_1} = \sum_{p<x_1} p \leqslant \ell(N) \leqslant k_2 x_1 + \sigma_{k_1} \leqslant \sigma_{k_1+k_2}. \tag{60}$$

Let n be an integer, $n \geqslant 12$, and $N = N(x_1) < N' = N'(x'_1)$ two consecutive ℓ–superchampion numbers such that

$$\ell(N) \leqslant n < \ell(N').$$

Since N'/N is a prime p' (cf. [6, Lemma 4, 4.(i)]), from (57), either $p' = p_{k+1}$ or $x'_1 < p_{k+1}$, and, from (59), $(p')^{v_{p'}(N')} < x'_1 < p_{k+1}$. In both cases, $\ell(N') \leqslant \ell(N) + p_{k+1}$. Therefore, from (60), we get

$$\sigma_{k_1} \leqslant \ell(N) \leqslant n < \ell(N') \leqslant \ell(N) + p_{k+1} \leqslant \sigma_{k_1+k_2+1}, \tag{61}$$

and (13) from Theorem 2 yields

$$k_1 \leqslant \omega(h(n)) \leqslant k_1 + k_2.$$

On the other hand, [12, Lemma E] gives

$$\omega(g(n)) = \omega(N) + O\left(\frac{\sqrt{x_1}}{\log x_1}\right).$$

From (57) and (58), we get $\omega(N) = k_1$, and from (58) and the prime number theorem, we have

$$k_1 = \pi(x_1) \sim \frac{x_1}{\log x_1} \quad \text{and} \quad k_2 = \pi(\sqrt{x_1}) \sim \frac{2\sqrt{x_1}}{\log x_1}. \tag{62}$$

Thus, we get

$$\omega(h(n)) - \omega(g(n)) \leqslant k_1 + k_2 - \omega(N) + O\left(\frac{\sqrt{x_1}}{\log x_1}\right) \tag{63}$$

$$= k_2 + O\left(\frac{\sqrt{x_1}}{\log x_1}\right) = O\left(\frac{\sqrt{x_1}}{\log x_1}\right). \tag{64}$$

Finally, from (62), we have $k_1 + k_2 \sim k_1 \sim O\left(\dfrac{x_1}{\log x_1}\right)$, which, with (61) and Lemma 9, yields $n \sim \sum_{p\leqslant x_1} p \sim \dfrac{x_1^2}{2\log x_1}$. Therefore, $x_1 \sim \sqrt{n \log n}$ and $\log x_1 \sim (1/2)\log n$ holds, which completes the proof of Theorem 24. □

4.5 Asymptotic estimates for $\log h(n)$ and $\omega(h(n))$

Theorem 25. *There exists $a > 0$ such that, when $n \to +\infty$,*

$$\log h(n) = \sqrt{\mathrm{Li}^{-1}(n)} + O\left(\sqrt{n}e^{-a\sqrt{\log n}}\right). \tag{65}$$

If $\Theta > 1/2$ and $\xi < \Theta$, then

$$\log h(n) = \sqrt{\mathrm{Li}^{-1}(n)} + \Omega_{\pm}\left((n \log n)^{\xi/2}\right). \tag{66}$$

Under the Riemann hypothesis, i.e., $\Theta = 1/2$, we have

$$\log h(n) = \sqrt{\mathrm{Li}^{-1}(n)} + O\big(((n \log n)^{1/4})\big). \tag{67}$$

$$\log h(n) < \sqrt{\mathrm{Li}^{-1}(n)} \quad \textit{for } n \textit{ large enough.} \tag{68}$$

Proof. The above theorem is proved in [12, Theorem 1] with $g(n)$ instead of $h(n)$, so that its proof follows from Theorem 23. ☐

Theorem 26. *There exists $a > 0$ such that, when $n \to +\infty$,*

$$\omega(h(n)) = \mathrm{Li}\left(\sqrt{\mathrm{Li}^{-1}(n)}\right) + O\Big(\sqrt{n} e^{-a\sqrt{\log n}}\Big). \tag{69}$$

If $\Theta > 1/2$ and $\xi < \Theta$, then

$$\omega(h(n)) = \mathrm{Li}\left(\sqrt{\mathrm{Li}^{-1}(n)}\right) + \Omega_{\pm}\left((n \log n)^{\xi/2}\right). \tag{70}$$

Under the Riemann hypothesis, i.e., $\Theta = 1/2$, we have

$$\omega(h(n)) = \mathrm{Li}\left(\sqrt{\mathrm{Li}^{-1}(n)}\right) + O\big(n^{1/4} (\log n)^{-3/4}\big).$$

Proof. The proof follows from [12, Théorème 2] and from Theorem 24. ☐

Note that the corollary of [12, p. 225] is also valid with $g(n)$ replaced by $h(n)$ to give an asymptotic expansion of $\log h(n)$ according to the powers of $1/\log n$.

We hope to make more precise the behavior of $h(n)$ and $\omega((h(n))$ under the Riemann hypothesis in another paper.

5 G-sequences, a tool for bounding above $P^+(h(n))$

Lemma 27 is [4, Proposition 3.3]. Lemma 28 below is an obvious corollary.

Lemma 27. *Let $n \geqslant 2$ be an integer and p, p' be distinct primes such that $p+p' \leqslant P^+(h(n))$. Then, at least one of p, p' divides $h(n)$.*

Lemma 28. *If q is a prime divisor of $h(n)$, there exists at most one prime $\leqslant q/2$ that does not divide $h(n)$.*

Definition 29. Let γ, γ' be such that $0 < \gamma < \gamma' < 1$ and $\gamma' < \dfrac{1+\gamma^2}{2}$. We define

$$\alpha = 2\gamma' - 1 \quad \text{and} \quad \beta = \gamma^2. \tag{71}$$

Then $\alpha < \beta$ and the pair of intervals (I, J) defined by

$$I(\gamma, \gamma') = (\alpha,\, \beta] \text{ and } J(\gamma, \gamma') = (\gamma,\, \gamma'] .$$

is called the G-pair associated with (γ, γ').

Lemma 30 can be found in [9, Lemma 2]. (In [9], this result is enunciated in the case of the Landau function $g(n)$.)

Let I be an interval and λ be a real number. Let us write $\lambda I = \{\lambda x; x \in I\}$.

Lemma 30. *Let (I, J) be a G-pair, $n \geqslant 2$ and q a prime factor of $h(n)$. If qI contains* at least *one prime divisor of $h(n)$, then* at most *one prime in qJ fails to divide $h(n)$.*

Proof. Let us remark that $\gamma = \sqrt{\beta}$ and $\gamma' = (1+\alpha)/2$. By contradiction, let us suppose that p, p' in qJ are distinct primes that do not divide $h(n)$. Let q' be a prime in qI dividing $h(n)$. Let $M = \dfrac{pp'}{qq'} h(n)$. Then

$$\ell(M) = p + p' - q - q' + \ell(h(n)) \leqslant 2\gamma' q - q - \alpha q + \ell(h(n)) = \ell(h(n)).$$

But $pp' - qq' > (\sqrt{\beta} q)^2 - q(\beta q) = 0$, so $M > h(n)$, giving a contradiction. $\square$

Definition 31. A *G-sequence of length ℓ* is a finite sequence $(\gamma_j)_{0 \leqslant j \leqslant \ell+1}$ satisfying $\gamma_0 = 0$, $\gamma_1 = \dfrac{1}{2}$ and, for $1 \leqslant j \leqslant \ell$

$$0 < \gamma_j < 1 \quad \text{and} \quad \gamma_j < \gamma_{j+1} < \frac{1+\gamma_j^2}{2}. \tag{72}$$

Let us define $I_0 = \left(0,\, \frac{1}{4}\right]$, $J_0 = \left(0,\, \frac{1}{2}\right]$ and, for $1 \leqslant j \leqslant \ell$, I_j and J_j by

$$\alpha_j = 2\gamma_{j+1} - 1,\ \beta_j = \gamma_j^2,\ I_j = (\alpha_j,\, \beta_j] \text{ and } J_j = (\gamma_j,\, \gamma_{j+1}] . \tag{73}$$

From (73) we have that, for $j \geqslant 1$, the pair $((\alpha_j,\, \beta_j] , (\gamma_j,\, \gamma_{j+1}])$ is a G-pair.

In Section 5.2 we study the *uniform G-sequences* such that the quotient α_j/β_j is a constant.

Definition 32. Let $\Gamma = (\gamma_j)_{0 \leqslant j \leqslant \ell+1}$ be a G-sequence and $y \geqslant 12$. For $1 \leqslant i \leqslant \ell$ we let m_i denote the cardinality of the set of indices $j \in \{0, 1, \ldots, \ell\}$ such that $I_i \cap J_j \neq \emptyset$. The G-sequence Γ is *y-admissible* if, for every $\lambda \geqslant y$ and each i, $1 \leqslant i \leqslant \ell$, the interval λI_i contains at least $m_i + 1$ primes.

Remark 33. For every G-sequence, by (73), $I_1 = (\alpha_1,\, 1/4] \subset J_0 = (0,\, 1/2]$; thus $m_1 = 1$. Therefore, if $(\gamma_j)_{0\leqslant j\leqslant \ell+1}$ is admissible, the interval $yI_1 = (y\alpha_1,\, y/4]$ contains at least $m_1+1 = 2$ primes. This needs $y/4 \geqslant 3$, i.e., $y \geqslant 12$.

Remark 34. By Lemma 13 *(i), the G-sequence $(\gamma_j)_{0\leqslant j\leqslant \ell+1}$ is y–admissible if and only, for $1 \leqslant j \leqslant \ell$, we have the inequality $\alpha_j/\beta_j \leqslant \eta_{m_j+1}(y\beta_j)$.*

Lemma 35. *Let $\Gamma = (\gamma_j)_{0\leqslant j\leqslant \ell+1}$ be a G-sequence of length ℓ and m an integer such that $m \geqslant m_j + 1$ for $1 \leqslant j \leqslant \ell$. Let us suppose that y is a real number such that,*

$$\alpha_j/\beta_j \leqslant \eta_m(y\beta_j) \quad \textit{for} \quad 1 \leqslant j \leqslant \ell. \tag{74}$$

Then Γ is y–admissible.

Proof. Since $(r, x) \mapsto \eta_r(x)$ is nonincreasing in r, we may write

$$\alpha_j/\beta_j \leqslant \eta_m(y\beta_j) \leqslant \eta_{m_j+1}(y\beta_j)$$

and, by using the previous Remark 34, Γ is y–admissible. □

Proposition 36. *Let $\Gamma = (\gamma_j)_{0\leqslant j\leqslant \ell+1}$ be a G-sequence of length ℓ, and q a prime factor of $h(n)$. If Γ is q-admissible then, for all j, $0 \leqslant j \leqslant \ell$ the interval $qJ_j = q\,(\gamma_j,\, \gamma_{j+1}]$ contains at most one prime which does not divide $h(n)$. Moreover*

$$\theta(q\gamma_{\ell+1}) - \ell \log q < \theta(q\gamma_{\ell+1}) - \ell \log q - \sum_{j=1}^{\ell+1} \log \gamma_j \leqslant \log h(n). \tag{75}$$

Proof. Let $\mathcal{P}(j)$ be the property that *there exists at most one prime number in qJ_j that does not divide $h(n)$.*

If $j = 0$, by Lemma 28, $qJ_0 = (0,\, q/2]$ contains at most one prime which does not divide $h(n)$. So $\mathcal{P}(0)$ is true.

By Remark 33, $m_1 = 1$. Thus, by Definition 32, qI_1 contains at least two primes, p, p'. Since the upper bound of qI_1 is $q/4$, by Lemma 27 we have that p or p' divides $h(n)$, and, by Lemma 30, there is at most one prime in qJ_1 which does not divide $h(n)$. Thus $\mathcal{P}(1)$ is true.

Let $j \in [2,\, \ell]$ such that $\mathcal{P}(r)$ is true for $r < j$. The upper bound of I_j is $\beta_j = \gamma_j^2 < \gamma_j$. We thus have $qI_j \subset (0,\, q\gamma_j] = \bigcup_{r=0}^{j-1} qJ_r$. By the induction hypothesis, each of the intervals qJ_r, $(0 \leqslant r \leqslant j-1)$, contains at most one prime number which does not divide $h(n)$. Since qI_j intersects m_j of these intervals, it contains at most m_j primes not dividing $h(n)$. But, by hypothesis, qI_j contains at least $m_j + 1$ primes. Thus, one of them divides $h(n)$ and, by Lemma 30, this implies that qJ_j contains at most one prime which does not divide $h(n)$. This is to say that $\mathcal{P}(j)$ is true.

So we have just proved that $h(n)$ is divisible by all the primes in $(0,\, q\gamma_{\ell+1}]$, but at most one prime $q_j \in q\,(\gamma_j,\, \gamma_{j+1}]$ for each $j = 0, 1, 2, \ldots, \ell$. Since q divides $h(n)$ we have

$$h(n) \geqslant\; q\,\frac{\prod_{p\leqslant q\gamma_{\ell+1}} p}{\prod_{j=0}^{\ell} q_j} \;\geqslant\; q\,\frac{\prod_{p\leqslant q\gamma_{\ell+1}} p}{\prod_{j=0}^{\ell} q\gamma_{j+1}}.$$

Applying log, this gives the second inequality in (75). The first inequality comes from $\displaystyle\sum_{j=1}^{\ell+1} \log\gamma_j < 0$. ☐

Proposition 37. *Let n_0, y, a be positive real numbers such that*

$$12 \leqslant y \leqslant a\sqrt{n_0 \log n_0}$$

and $\ell \geqslant 1$, integer. We assume that $\Gamma = (\gamma_j)_{0\leqslant j\leqslant \ell+1}$, is a G-sequence y-admissible of length ℓ. Let us define

$$D_\ell = \gamma_{\ell+1}\theta_{\min}(y\gamma_{\ell+1}) - \frac{\ell\log y + \sum_{j=1}^{\ell+1}\log\gamma_j}{y}, \tag{76}$$

and let us suppose that $D_\ell > 0$. Then, for $n \geqslant n_0$, we have

$$P^+(h(n)) \leqslant \max(a, b)\sqrt{n\log n} \quad \textit{with} \quad b = \frac{1.0482017}{D_\ell}. \tag{77}$$

$$P^+(h(n)) \leqslant \max(a, b')\sqrt{n\log n} \quad \textit{with} \quad b' = \frac{1}{D_\ell}\left(1 + \frac{\log_2 n_0 - 0.975}{2\log n_0}\right). \tag{78}$$

The second upper bound is better than the first one for $n \geqslant 3\,259\,922\,785$.

Proof. Let n be an integer, $n \geqslant n_0$. Let us set, for simplification, $q = P^+(h(n))$.

(i) If $q < y$, then, by hypothesis, we have

$$q < y \leqslant a\sqrt{n_0\log n_0} \leqslant a\sqrt{n\log n}. \tag{79}$$

(ii) If $q \geqslant y$, then $q\gamma_{\ell+1} \geqslant y\gamma_{\ell+1}$. With Equation (35) this gives $\theta(q\gamma_{\ell+1}) \geqslant q\gamma_{\ell+1}\theta_{\min}(y\gamma_{\ell+1})$. We remark that $\sum_{j=1}^{\ell+1}|\log\gamma_j| \leqslant (\ell+1)\log 2 \leqslant 2\ell\log 2$ which implies that $t \mapsto (\ell\log t + \displaystyle\sum_{j=1}^{\ell+1}\log\gamma_j)/t$ is decreasing for $t \geqslant 4e$. From this and Equation (76) defining D_ℓ, we infer that

$$D_\ell \leqslant \frac{\theta(q\gamma_{\ell+1})}{q} - \frac{\ell\log q + \sum_{j=1}^{\ell+1}\log\gamma_j}{q}. \tag{80}$$

Since $q \geqslant y$, the sequence Γ being y-admissible is a fortiori q–admissible. Then, by using Equation (75) of Proposition 36, Equation (80) gives $0 < qD_\ell \leqslant \log h(n)$, from which, by using (41) and (40) to get upperbounds for $\log h(n)$, we obtain

$$q = P^+(h(n)) \leqslant \min(b, b')\sqrt{n\log n}. \tag{81}$$

Inequalities (77) and (78) follow from (79) and (81). ☐

5.1 The optimal y-admissible G-sequence

The upper bound (75) in Proposition 36 leads us to construct y-admissible G-sequences with γ_j as large as possible. This is the subject of this section.

Let y be $\geqslant 12$ and $(\gamma_0 = 0, \gamma_1 = 1/2, \gamma_2)$ be a G-sequence y–admissible of length 1. By (73), $\gamma_2 = \dfrac{1+\alpha_1}{2}$; thus, the largest value of γ_2 is obtained by giving to α_1 the largest possible value. By Remark 33, we have $m_1 = 1$ so that, by Remark 34, the sequence $(\gamma_0 = 0, \gamma_1 = 1/2, \gamma_2)$ is y–admissible if and only if $\alpha_1 \leqslant \frac{1}{4}\eta_2\big(\frac{y}{4}\big)$. So, we get the largest value for γ_2 by setting $\alpha_1 = \frac{1}{4}\eta_2\big(\frac{y}{4}\big)$ and $\gamma_2 = \frac{1+\alpha_1}{2}$.

Now let $(\gamma_j)_{0\leqslant j\leqslant \ell+1}$ be a G-sequence y–admissible of length ℓ, that we want to extend. Equation $\gamma_{\ell+1} = \beta_{\ell+1}^2$ determines $\beta_{\ell+1}$. Equality $\gamma_{\ell+2} = \dfrac{1+\alpha_{\ell+1}}{2}$ shows that the largest value of $\gamma_{\ell+2}$ is got by choosing $\alpha_{\ell+1}$ as large as possible. We set $m = 1$ and we try

$$\alpha_{\ell+1} = \beta_{\ell+1}\eta_{m+1}(y\beta_{\ell+1}). \tag{82}$$

– If $\alpha_{\ell+1} \leqslant \alpha_\ell$ our construction fails because it is not possible to satisfy $\gamma_{\ell+1} = \dfrac{1+\alpha_\ell}{2} < \dfrac{1+\alpha_{\ell+1}}{2} = \gamma_{\ell+2}$.

– If $\alpha_{\ell+1} > \alpha_\ell$ let us consider $I_{\ell+1} = (\alpha_{\ell+1},\, \beta_{\ell+1}]$. If this interval meets at most m intervals among $J_0, J_1, \ldots, J_\ell$, we finish by choosing $\gamma_{\ell+2} = (1+\alpha_{\ell+1})/2$. If $I_{\ell+1}$ meets $m' > m$ intervals among $J_0, J_1, \ldots, J_\ell$, we have to choose again $\alpha_{\ell+1}$ by using formula (82) with m replaced by $m+1$.

More formally this construction is described in Algorithm 1. This algorithm is not guaranteed to terminate; however, for $y = 1853.18$, it gives the y–admissible G-sequence of length 10 which we will use in Section 6.1.

Algorithm 1 Computes $\alpha_{\ell+1}, \beta_{\ell+1}, \gamma_{\ell+2}$ from $\gamma_{\ell+1}$, α_ℓ and y

$\beta_{\ell+1} = \gamma_{\ell+1}^2, \quad m = 1$
Repeat
  $\alpha_{\ell+1} = \beta_{\ell+1}\eta_{m+1}(y\beta_{\ell+1})$
  if $\alpha_{\ell+1} \leqslant \alpha_\ell$ **return** FAIL
  else $m = m+1$
until $(\alpha_{\ell+1},\, \beta_{\ell+1}]$ meets at most m intervals $(0,\, \gamma_1], \ldots, (\gamma_\ell,\, \gamma_{\ell+1}]$
$\gamma_{\ell+2} = (1+\alpha_{\ell+1})/2$

5.2 Uniform G-sequences

The theoretical study of optimal G-sequences does not seem easy. In this section we introduce the *uniform* G-sequences, less efficient for numerical computation, but simpler to study.

Definition 38. Let η be a real number, $0 < \eta < 1$. We define $\gamma_0 = 0$ and, for $j \geqslant 1$,

$$\gamma_j = \gamma_j(\eta) = \frac{1 + \eta\gamma_{j-1}^2}{2}. \tag{83}$$

Remark 39. Let us note that $\gamma_j(\eta)$ is an increasing function of j and η. Note also that

$$\alpha_j = 2\gamma_{j+1} - 1 = \eta\gamma_j^2 = \eta\beta_j. \tag{84}$$

Lemma 40. *With the notation of Definition 38, let us write $\varepsilon = \varepsilon(\eta) = 1 - \eta$, and $L_\varepsilon = \lim_{j\to+\infty} \gamma_j$. then*

$$L_\varepsilon = \frac{1}{1+\sqrt{\varepsilon}} \qquad \text{and, for } j \geqslant 0, \qquad L_\varepsilon - \gamma_j \leqslant L_\varepsilon(1-\sqrt{\varepsilon})^j \leqslant (1-\sqrt{\varepsilon})^j. \tag{85}$$

For each integer $\ell \geqslant 1$, the sequence $(\gamma_j(\eta))_{0\leqslant j\leqslant \ell+1}$ defined by (83) is a G-sequence. We call it the uniform *G-sequence of parameter η and length ℓ.*

Proof. The proof is by induction. We have $\gamma_0 = 0$, $\gamma_1 = \dfrac{1+\eta\gamma_0^2}{2} = \dfrac{1}{2}$, and, for $j \geqslant 1$,

$$\gamma_{j+1} = \frac{1+\eta\gamma_j^2}{2} < \frac{1+\gamma_j^2}{2}.$$

Moreover $\gamma_{j+1} = f(\gamma_j)$ with $f = t \mapsto (1 + (1-\varepsilon)t^2)/2$. Function f is increasing for $t \geqslant 0$ and has two fix points which are $\dfrac{1}{1+\sqrt{\varepsilon}}$ and $\dfrac{1}{1-\sqrt{\varepsilon}}$. Since $\gamma_0 < \gamma_1$, the sequence (γ_j) is increasing with limit $L_\varepsilon = \dfrac{1}{1+\sqrt{\varepsilon}}$. Moreover, since f' is increasing,

$$\begin{aligned} L_\varepsilon - \gamma_j &= f(L_\varepsilon) - f(\gamma_{j-1}) < f'(L_\varepsilon)(L_\varepsilon - \gamma_{j-1}) = (1-\sqrt{\varepsilon})(L_\varepsilon - \gamma_{j-1}) \\ &\leqslant (1-\sqrt{\varepsilon})^j(L_\varepsilon - \gamma_0) = L_\varepsilon(1-\sqrt{\varepsilon})^j. \end{aligned}$$

For each $\ell \geqslant 1$, the conditions of Definition 72 are satisfied and $(\gamma_j)_{0\leqslant j\leqslant \ell}$ is a G-sequence. ☐

Throughout this section η is a positive real number satisfying $0 < \eta < 1$, $\varepsilon = 1 - \eta$, (γ_j) is the uniform G-sequence of parameter η, and $I_j = (\alpha_j,\, \beta_j]$, $J_j = (\gamma_j,\, \gamma_{j+1}]$ are the intervals associated with this G-sequence (cf. Definition 72).

Lemma 41. *Let us write $u_j = L_\varepsilon - \gamma_j$. Then $(\gamma_{j+1} - \gamma_j)$ is a decreasing sequence, and, for every j we have*

$$\gamma_{j+1} - \gamma_j = \sqrt{\varepsilon}u_j + \frac{u_j^2}{2}(1-\varepsilon).$$

In particular $\sqrt{\varepsilon}(L_\varepsilon - \gamma_j) < \gamma_{j+1} - \gamma_j$.

Proof. Indeed, since $\gamma_j = L_\varepsilon - u_j = \dfrac{1}{1+\sqrt{\varepsilon}} - u_j$, we have, with $f(t) = (1-(1-\varepsilon t^2))/2$,

$$\begin{aligned}
2(\gamma_{j+1} - \gamma_j) &= 2f(\gamma_j) - 2\gamma_j = 1 + (1-\varepsilon)\gamma_j^2 - 2\gamma_j \\
&= 1 + (1-\varepsilon)\left(\frac{1}{1+\sqrt{\varepsilon}} - u_j\right)^2 - \frac{2}{1+\sqrt{\varepsilon}} + 2u_j \\
&= 2\sqrt{\varepsilon}u_j + (1-\varepsilon)u_j^2 > 2\sqrt{\varepsilon}u_j = 2\sqrt{\varepsilon}(L_\varepsilon - \gamma_j).
\end{aligned}$$

□

Lemma 42. *There is no pair (i,j) such that $(\gamma_i,\, \gamma_{i+1}] \subset (\alpha_j,\, \beta_j]$. Therefore each interval I_j meets at most two intervals J_i.*

Proof. Let us suppose $(\gamma_i,\, \gamma_{i+1}] \subset (\alpha_j,\, \beta_j]$. Then

$$\alpha_j \leqslant \gamma_i < \gamma_{i+1} \leqslant \beta_j = \gamma_j^2$$

and

$$L_\varepsilon - \gamma_i > L_\varepsilon - \gamma_j^2 > L_\varepsilon - L_\varepsilon^2 = L_\varepsilon(1 - L_\varepsilon) = \sqrt{\varepsilon}L_\varepsilon^2. \tag{86}$$

By Lemma 41, we also have

$$\sqrt{\varepsilon}(L_\varepsilon - \gamma_i) < \gamma_{i+1} - \gamma_i \leqslant \beta_j - \alpha_j = \varepsilon\beta_j = \varepsilon\gamma_j^2 < \varepsilon L_\varepsilon^2$$

which implies $L_\varepsilon - \gamma_i < \sqrt{\varepsilon}L_\varepsilon^2$, contradicting (86). □

6 Estimates of $P^+(h(n))$

In this section, we study the behavior of $P^+(h(n))$ in terms of n. The results are similar to those obtained in [4] for $P^+(g(n))$, except for the numerical value of the constants.

6.1 Maximum of $P^+(h(n))/\sqrt{n \log n}$ for $n \geqslant 4$.

Theorem 43. *For $n \geqslant 4$ we have*

$$\frac{P^+(h(n))}{\sqrt{n\log n}} \leqslant \frac{P^+(h(170))}{\sqrt{170\log(170)}} = 1.38757162\cdots, \tag{87}$$

the maximum being attained only for $n = 170$ with $h(170) = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 41 = N_{10} \times 41$ and $P^+(h(170)) = 41$.

Proof. We use Proposition 37 with $n_0 = 150\,000$, $a = 1.386$ and $y = 1853.18$. Using Algorithm 1 we compute the first 10 terms of the optimal y–admissible G-sequence. We get intervals

j	α_j	β_j	γ_{j+1}	$\{i\}$	D_j
1	$0.2390\cdots$	$0.2500\cdots$	$0.619515\cdots$	0	$0.582374\cdots$
2	$0.3722\cdots$	$0.3837\cdots$	$0.686121\cdots$	0	$0.641498\cdots$
3	$0.4569\cdots$	$0.4707\cdots$	$0.728463\cdots$	0	$0.677646\cdots$
4	$0.5150\cdots$	$0.5306\cdots$	$0.757531\cdots$	1	$0.701222\cdots$
5	$0.5569\cdots$	$0.5738\cdots$	$0.778493\cdots$	1	$0.724454\cdots$
6	$0.5882\cdots$	$0.6060\cdots$	$0.794120\cdots$	1	$0.737558\cdots$
7	$0.6094\cdots$	$0.6306\cdots$	$0.804703\cdots$	$1,2$	$0.745702\cdots$
8	$0.6285\cdots$	$0.6475\cdots$	$0.814257\cdots$	2	$0.750926\cdots$
9	$0.6435\cdots$	$0.6630\cdots$	$0.821764\cdots$	2	$0.754179\cdots$
10	$0.6554\cdots$	$0.6752\cdots$	$0.827725\cdots$	2	$0.755943\cdots$

Column $\{i\}$ contains the values i for which I_j meets J_i, so that m_j is the number of integers appearing in the j^{th} line of the column $\{i\}$.

This gives $D_{10} = 0.755943\cdots$ and, with (77), $b = 1.386614\cdots > a$ which proves that $P^+(h(n)) < 1.3867\sqrt{n\log n}$ for $n \geqslant 150\,000$. The computation of $h(n)$ for $4 \leqslant n \leqslant 150\,000$ shows that the maximum is obtained only once, for $n = 170$. ☐

Lemma 44. *Let q be a prime factor of $h(n)$ and $\eta \leqslant \eta_3(q/4)$; let $\Gamma = (\gamma_j)_{0\leqslant j\leqslant \ell+1}$ be the uniform G-sequence of parameter η. This sequence is q–admissible, and*

$$\theta(q\gamma_{\ell+1}) - \ell\log q \leqslant \log h(n). \tag{88}$$

Proof. By Lemma 42, each interval I_j meets at most 2 intervals J_j. Thus, the integer m_j introduced in Definition 32 satisfies $m_j \leqslant 2$ for every j. Equality (84), the inequality $\beta_1 = \dfrac{1}{4} \leqslant \beta_j$ and the fact that η_3 is nondecreasing give

$$\alpha_j = \eta\beta_j \leqslant \beta_j\eta_3(q/4) \leqslant \beta_j\eta_3(q\beta_j).$$

Thus Inequality (74) holds with $m = 3$ and $y = q$, and Lemma 35 shows that the sequence Γ is q-admissible. Therefore, from Proposition 36, Eq. (88) holds. ☐

Lemma 45. *Let n be an integer and q the largest prime factor of $h(n)$. When n tends to infinity, we have*

$$q = P^+(h(n)) \leqslant \log h(n)(1 + O(\varepsilon)), \tag{89}$$

with $\varepsilon = \varepsilon_1 + \sqrt{\varepsilon_2} \to 0$ and

$$\varepsilon_1 = \max_{x\geqslant \frac{q}{2}} \left|\frac{\theta(x)}{x} - 1\right|, \qquad \varepsilon_2 = \max\left(1 - \eta_3\Big(\frac{q}{4}\Big), \Big(\frac{\log q}{\sqrt{q}}\Big)^2\right) < 1. \tag{90}$$

Proof. Let us write

$$\eta = 1 - \varepsilon_2 \quad \text{and} \quad \ell = \left\lfloor \frac{\log q}{\sqrt{\varepsilon_2}} \right\rfloor. \tag{91}$$

When $n \to +\infty$, by (10), $q = P^+(h(n))$ tends to infinity, ε_1 tends to 0, $\eta_3(q/4)$ tends to 1 (by Proposition 14), ε_2 and ε tend to 0 and ℓ tends to infinity.

Let $\Gamma = (\gamma_j)_{0\leqslant j\leqslant \ell+1}$ be the uniform G-sequence of parameter η and length ℓ. Since Γ is increasing, we have $q\gamma_{\ell+1} \geqslant q\gamma_1 = q/2$. By the definition of ε_1, it comes $\dfrac{\theta(q\gamma_{\ell+1})}{q\gamma_{\ell+1}} \geqslant 1-\varepsilon_1$, therefore, with the notation of Lemma 40,

$$\begin{aligned} \frac{\theta(\gamma_{\ell+1}q)}{q} &\geqslant \gamma_{\ell+1} - \gamma_{\ell+1}\varepsilon_1 \\ &\geqslant \gamma_{\ell+1} - \varepsilon_1 = 1-\varepsilon_1-(1-L_{\varepsilon_2})-(L_{\varepsilon_2}-\gamma_{\ell+1}). \end{aligned} \tag{92}$$

One use of Lemma 40 gives

$$1 - L_{\varepsilon_2} = 1 - \frac{1}{1+\sqrt{\varepsilon_2}} \leqslant \sqrt{\varepsilon_2}. \tag{93}$$

By (91), $\ell + 1 > \dfrac{\log q}{\sqrt{\varepsilon_2}}$; then a second use of Lemma 40 gives

$$L_{\varepsilon_2} - \gamma_{\ell+1} \leqslant (1-\sqrt{\varepsilon_2})^{\ell+1} \leqslant (1-\sqrt{\varepsilon_2})^{\frac{\log q}{\sqrt{\varepsilon_2}}} \leqslant \left(\frac{1}{e}\right)^{\log q} = \frac{1}{q}.$$

With (92), (93) and the definition of ε we get

$$\frac{\theta(q\gamma_{\ell+1})}{q} \geqslant 1-\varepsilon_1-\sqrt{\varepsilon_2}-\frac{1}{q} = 1-\varepsilon-\frac{1}{q}. \tag{94}$$

Definition (91) of ℓ gives $\ell\dfrac{\log q}{q} \leqslant \dfrac{(\log q)^2}{q\sqrt{\varepsilon_2}}$. With (94), this gives

$$\frac{\theta(q\gamma_{\ell+1})}{q} - \ell\frac{\log q}{q} \geqslant 1-\varepsilon-\frac{1}{q}-\frac{\log^2 q}{q\sqrt{\varepsilon_2}}.$$

By the definition of ε_2, for $q \geqslant 3$, we have $\dfrac{1}{q} \leqslant \dfrac{\log^2 q}{q} \leqslant \varepsilon_2 \leqslant \sqrt{\varepsilon_2}$, and therefore

$$\frac{\theta(q\gamma_{\ell+1})}{q} - \ell\frac{\log q}{q} \geqslant 1-\varepsilon-\frac{1}{q}-\sqrt{\varepsilon_2} \geqslant 1-\varepsilon-2\sqrt{\varepsilon_2} \geqslant 1-3\varepsilon. \tag{95}$$

By the definition of ε_2 and η (cf. (90)) and (91)), we have $\eta \leqslant \eta_3\left(\dfrac{q}{4}\right)$, so that Lemma 44 applies to the G-sequence Γ. Inequality (95) shows that $\theta(q\gamma_{\ell+1}) - \ell\log q$ is positive for n large enough, so that, by using (88) and (95), we may write

$$q \leqslant \frac{\log h(n)}{\frac{\theta(q\gamma_{\ell+1})}{q} - \ell\frac{\log q}{q}} \leqslant \frac{\log h(n)}{1-3\varepsilon} = \log h(n)(1+O(\varepsilon)).$$

$\square$

Proposition 46. *When n tends to infinity, $P^+(h(n)) \sim \sqrt{n \log n}$.*

Proof. Let us write $q = P^+(h(n))$. By (10), q tends to infinity with n. Using successively the prime number theorem and Inequality (11), we get

$$q \sim \theta(q) \geqslant \log h(n).$$

With (89), this gives $q \sim \log h(n)$, and ends the proof with Theorem 18. □

6.2 Asymptotic upper bound for $P^+(h(n))$

Theorem 47. *Let $P^+(h(n))$ be the largest prime factor of $h(n)$. When n tends to infinity, $P^+(h(n))$ is bounded as follows:*

(i) Without any hypothesis, there exists $a > 0$ such that

$$P^+(h(n)) \leqslant \sqrt{\mathrm{Li}^{-1}(n)} + O\left(\sqrt{n} e^{-a\sqrt{\log n}}\right). \tag{96}$$

(ii) Under the Riemann Hypothesis:

$$P^+(h(n)) \leqslant \sqrt{\mathrm{Li}^{-1}(n)} + O\left(n^{3/8} (\log n)^{7/8}\right). \tag{97}$$

(iii) Under the Riemann Hypothesis and the Cramér conjecture (Equation (30)):

$$P^+(h(n)) \leqslant \sqrt{\mathrm{Li}^{-1}(n)} + O\left(n^{1/4} (\log n)^{9/4}\right). \tag{98}$$

Proof. Let us write $q = P^+(h(n))$. By Proposition 46 we have

$$q \sim \sqrt{n \log n} \quad \text{and} \quad \log q \sim \frac{1}{2} \log n. \tag{99}$$

We shall also use the relation

$$\mathrm{Li}^{-1}(n) \sim n \log n. \tag{100}$$

We will apply Lemma 45, evaluating in the three cases the quantities ε_1, ε_2 and $\varepsilon = \varepsilon_1 + \sqrt{\varepsilon_2}$.

(*i*) By the prime number theorem (20), there is some $a_1 > 0$ such that

$$\varepsilon_1 = O\left(e^{-a_1\sqrt{\log n}}\right).$$

By Proposition 14, (*i*), we have

$$1 - \eta_3\left(\frac{q}{4}\right) = O\left(q^{-0.475}\right).$$

Thus, by (99),

$$\sqrt{\varepsilon_2} = O\big(q^{-0.2375}\big) = O\big(n^{-0.11875}\big)$$

and $\varepsilon = \varepsilon_1 + \sqrt{\varepsilon_2} = O\big(\exp(-a_1\sqrt{\log n}\big)$. Lemma 45 gives the inequality

$$q \leqslant (\log h(n))\Big(1 + O\Big(\exp(-a_1\sqrt{\log n})\Big)\Big). \tag{101}$$

By (65) there is $a_2 > 0$ such that

$$\log h(n) = \sqrt{\mathrm{Li}^{-1}(n)} + O\Big(\sqrt{n}e^{-a_2\sqrt{\log n}}\Big)$$

which, with (101) and (100), proves (96) for $a < \min(a_1, a_2)$.

(ii) Under the Riemann hypothesis, we have, by (21),

$$\varepsilon_1 = O\left(\frac{\log^2 q}{\sqrt{q}}\right) \tag{102}$$

and Inequality (33) of Proposition 14 gives

$$1 - \eta_3\left(\frac{q}{4}\right) = O\left(\frac{\log q}{\sqrt{q}}\right).$$

We thus get $\varepsilon_2 = O\left(\dfrac{\log q}{\sqrt{q}}\right)$ and, by (99)

$$\varepsilon = \varepsilon_1 + \sqrt{\varepsilon_2} = O\left(\frac{\sqrt{\log q}}{q^{1/4}}\right) = O\left(\frac{(\log n)^{3/8}}{n^{1/8}}\right),$$

so that, by Lemma 45,

$$q \leqslant \log h(n)\left(1 + O\left(\frac{(\log n)^{3/8}}{n^{1/8}}\right)\right),$$

which, with (67) and (100), yields (97).

(iii) Estimate (102) is still true, while the definition (90) of ε_2 and Equation (34) for $k = 3$ give $\varepsilon_2 = O\left(\dfrac{\log^2 q}{q}\right)$ and then

$$\varepsilon = O\left(\frac{\log^2 q}{\sqrt{q}}\right) = O\left(\frac{(\log n)^{7/4}}{n^{1/4}}\right),$$

which, by Lemma 45, (67) and (100), proves (98).

Let us remark that (98) is still true if we replace Cramér's conjecture (30) by the weaker relation $p_{i+1} - p_i = O\big(\log^4 p_i\big)$.

$\square$

6.3 Effective upper bound for $P^+(h(n))$

Theorem 48. *$P^+(h(n))$ satisfies the following inequalities*

$$P^+(h(n)) \leqslant \log h(n)\left(1+\frac{1.012}{\log n}\right) \quad (n \geqslant 138\,940). \tag{103}$$

$$P^+(h(n)) \leqslant \sqrt{n\log n}\left(1+\frac{\log_2 n + 1.145}{2\log n}\right) \quad (n \geqslant 233\,089). \tag{104}$$

Proof. The computation of $P^+(h(n))$ and $\log(h(n))$ for $2 \leqslant n \leqslant 10^{10}$, shows that Inequalities (103) and (104) are satisfied for these values of n.

Let us suppose $n > 10^{10}$, and that $q = P^+(h(n))$ satisfies

$$q > \log h(n)\left(1+\frac{1}{\log n}\right). \tag{105}$$

Since $n \geqslant 10^{10} > 7387$, it results from (39) that

$$q > \sqrt{n\log n}\left(1+\frac{1}{\log n}\right) = \sqrt{n}\left(\sqrt{\log n}+\frac{1}{\sqrt{\log n}}\right).$$

Since $t \mapsto t + 1/t$ for $t \geqslant 1$ is increasing this gives

$$q > \sqrt{n}\left(\sqrt{\log 10^{10}}+\frac{1}{\sqrt{\log 10^{10}}}\right) \geqslant 5.006923\sqrt{n} > 500\,692.$$

Therefore,

$$q \geqslant 500\,693, \qquad 0.25\,q \geqslant 125\,173 \tag{106}$$

and

$$\log(0.25\,q) > \log(0.25 \times 5.00693\sqrt{n}) > \log(\sqrt{n}) = 0.5\log n. \tag{107}$$

Here we define η and ε by

$$\varepsilon = \frac{0.4822}{\log^2 n} \quad \text{and} \quad \eta = 1-\varepsilon. \tag{108}$$

From (108) we deduce

$$\frac{0.6944}{\log n} \leqslant \sqrt{\varepsilon} \leqslant \frac{0.6945}{\log n} \tag{109}$$

and, also, with $n \geqslant 10^{10}$,

$$\varepsilon < 0.00091 \quad \text{and} \quad \eta = 1-\varepsilon > 0.99909. \tag{110}$$

From the table [3, Tabulation de delta3], we get $\delta_3(85\,991) = 0.120544\cdots$, which, with the fact that δ_3 is nonincreasing (cf. §3.6) and (106), implies $0.12055 \geqslant \delta_3(85\,991) \geqslant$

$\delta_3(125\,173) \geqslant \delta_3(0.25q)$. Therefore, Definition (108) of ε and the lower bound (107) give

$$\varepsilon = \frac{0.4822}{\log^2 n} \geqslant \frac{0.12055}{(\log(0.25\,q))^2} \geqslant \frac{\delta_3(0.25\,q)}{(\log(0.25\,q))^2}. \tag{111}$$

From (111), with (37), (where we take $x = y = 0.25\,q$), we get

$$\eta = 1 - \varepsilon \leqslant 1 - \frac{\delta_3(0.25\,q)}{(\log(0.25\,q))^2} \leqslant \eta_3\,(0.25\,q)\,.$$

The uniform G-sequence of parameter η satisfies the hypothesis of Lemma 44. We apply this lemma, choosing

$$\ell = \lfloor 2.5\,\log n \log_2 n \rfloor \geqslant \lfloor 2.5\,\log 10^{10} \log_2 10^{10} \rfloor = 180. \tag{112}$$

We have seen (cf. Remark 39) that $\gamma_j = \gamma_j(\eta)$ is an increasing function of η and j. Since, by (110), $\eta > 0.99909$ and $\ell \geqslant 180$ we have $\gamma_{\ell+1} \geqslant \gamma_{180}(0.99909) > 0.9705$. By using (106) and (107), we get

$$q\gamma_{\ell+1} \geqslant 0.9705 \cdot 500\,693 \geqslant 485\,922 \quad \text{ and } \quad \log q\gamma_{\ell+1} \geqslant \log 0.25\,q \geqslant 0.5 \log n.$$

The table of θ_d-champion numbers given in [3, Tabulation de thetad] yields $\theta_d(485\,922) \leqslant 0.3644$, so that, from (36), we get

$$\frac{\theta(q\gamma_{\ell+1})}{q\gamma_{\ell+1}} \geqslant 1 - \frac{\theta_d(485\,922)}{(\log q\gamma_{\ell+1})^2} \geqslant 1 - \frac{0.3644}{(\log q\gamma_{\ell+1})^2} \geqslant 1 - \frac{1.4576}{(\log n)^2}. \tag{113}$$

Using $n \geqslant 10^{10}$, this gives

$$\frac{\theta(q\gamma_{\ell+1})}{q} \geqslant \left(1 - \frac{1.4576}{\log^2 n}\right)\gamma_{\ell+1} \geqslant \left(1 - \frac{0.0634}{\log n}\right)\gamma_{\ell+1}. \tag{114}$$

Now we have to get a lower bound for $\gamma_{\ell+1}$. From (85), using (109) and (112), we deduce

$$\begin{aligned}
\gamma_{\ell+1} &\geqslant \frac{1}{1+\sqrt{\varepsilon}} - \left(1-\sqrt{\varepsilon}\right)^{\ell+1} \geqslant 1 - \sqrt{\varepsilon} - \left(1 - \frac{0.6944}{\log n}\right)^{\ell+1} \\
&\geqslant 1 - \sqrt{\varepsilon} - \left(1 - \frac{0.6944}{\log n}\right)^{2.5\,\log_2 n \log n} \\
&\geqslant 1 - \sqrt{\varepsilon} - \left[\left(1 - \frac{0.6944}{\log n}\right)^{\frac{\log n}{0.6944}}\right]^{2.5\,\times 0.6944\,\log_2(n)} \\
&\geqslant 1 - \frac{0.6945}{\log n} - \frac{1}{(\log n)^{2.5\times 0.6944}} \geqslant 1 - \frac{0.794}{\log n}.
\end{aligned}$$

With (114), this gives

$$\frac{\theta(q\gamma_{\ell+1})}{q} \geqslant \left(1 - \frac{0.0634}{\log n}\right)\left(1 - \frac{0.794}{\log n}\right) \geqslant 1 - \frac{0.8574}{\log n} + \frac{0.05034}{\log^2 n}. \tag{115}$$

Since $\dfrac{\log t}{t}$ is a decreasing function of t for $t \geqslant e$, and $q \geqslant \sqrt{n \log n}$, we have

$$\frac{\log q}{q} \leqslant \frac{1}{2}\frac{\log(n\log n)}{\sqrt{n\log n}}.$$

Using(112) and the fact that $t \mapsto (\log t)^{3/2}\log_2 t \log(t\log t)/\sqrt{t}$ is decreasing for $t \geqslant 10^{10}$, this gives

$$\ell\frac{\log q}{q} \leqslant 1.25\ \log n \log_2 n \frac{\log(n\log n)}{\sqrt{n\log n}} \leqslant \frac{0.114}{\log n}. \tag{116}$$

Since the inequality $\log n > 0.9714$ holds, Formula (88) of Lemma 44, with (115) and (116) give

$$q \leqslant \frac{\log h(n)}{1 - \dfrac{0.9714}{\log n} + \dfrac{0.05034}{\log^2 n}}.$$

On the interval $0 \leqslant X \leqslant 1/\log 10^{10}$, the fraction $\dfrac{0.9714 - 0.05034X}{1 - 0.9714X + 0.05034X^2}$ is increasing and less than 1.012. This implies, for $n \geqslant 10^{10}$,

$$\frac{1}{1 - \dfrac{0.9714}{\log n} + \dfrac{0.05034}{\log^2 n}} = 1 + \frac{\dfrac{0.9714}{\log n} - \dfrac{0.05034}{\log^2 n}}{1 - \dfrac{0.9714}{\log n} + \dfrac{0.05034}{\log^2 n}} \leqslant 1 + \frac{1.012}{\log n}$$

which, with (105), proves (103). Applying (40), we deduce from that

$$\begin{aligned} q &\leqslant \sqrt{n\log n}\left(1 + \frac{\log_2 n - 0.975}{2\log n}\right)\left(1 + \frac{1.012}{\log n}\right) \\ &\leqslant \sqrt{n\log n}\left(1 + \frac{\log_2 n + 1.049}{2\log n} + \frac{1.012}{\log n}\frac{\log_2 n - 0.975}{2\log n}\right) \\ &\leqslant \sqrt{n\log n}\left(1 + \frac{\log_2 n + 1.049}{2\log n} + \frac{1}{2\log n}\frac{1.012(\log_2 10^{10} - 0.975)}{\log 10^{10}}\right) \end{aligned}$$

which completes the proof of (104). □

6.4 Lower bound for $P^+(h(n))$

Lemma 49. *For all $n \geqslant 7387$, we have*

$$P^+(h(n)) \geqslant \frac{\sqrt{n \log n}}{1.000028}. \tag{117}$$

Proof. From (11) we have $\log h(n) \leqslant \theta(P^+(h(n)))$. Inequalities (39) and (24) end the proof. □

Theorem 50. *For $n \geqslant 7\,992$ we have*

$$P^+(h(n)) \geqslant \sqrt{n \log n}\left(1 + \frac{\log_2 n - 1.18}{2 \log n}\right), \tag{118}$$

and, for $n \geqslant 21$

$$P^+(h(n)) \geqslant \sqrt{n \log n}. \tag{119}$$

Proof. First, we compute $h(n)$ and $P^+(h(n))$ for $n \leqslant 10^8$ and we verify that (118) is true for $7\,992 \leqslant n \leqslant 10^8$ and false for $n = 7\,991$, and that (119) is true for $21 \leqslant n \leqslant 10^8$ and false for $n = 20$.

Let us note that for $n > 10^8$ Eq. (119) is implied by Eq. (118), since $\log_2 n > \log_2 10^8 = 2.913 \cdots > 1.17$.

Thus it remains to prove that Eq. (118) is true for $n > 10^8$. Let us write $q = P^+(h(n))$. From Eq. (11), we have $\log h(n) \leqslant \theta(q)$. With Eq. (38) this gives

$$\sqrt{n \log n}\left(1 + \frac{\log_2 n - 1.16}{2 \log n}\right) \leqslant \log h(n) \leqslant \theta(q). \tag{120}$$

(i) If $n_0 = 10^8 < n \leqslant n_1 = 6.6 \times 10^{21}$, by Eq. (87), we get

$$q \leqslant 1.388\sqrt{n \log n} \leqslant 8 \times 10^{11}. \tag{121}$$

Thus, by Eq. (24) we have $\theta(q) < q$, which, with Eq. (120), implies inequality (118).

(ii) If $n > n_1 = 6.6 \times 10^{21}$. By Eq. (117), $q = P^+(h(n))$ satisfies

$$q > 5.7 \times 10^{11},$$

and, by Eq. (25),

$$\theta(q) - q \leqslant 0.05\frac{q}{\log^2 q} < \frac{0.05}{\log(5.7 \times 10^{11})}\frac{q}{\log q} < 0.002\frac{q}{\log q}.$$

With Eq. (120) this gives

$$\sqrt{n \log n}\left(1 + \frac{\log_2 n - 1.16}{2 \log n}\right) < q + 0.002\frac{q}{\log q}. \tag{122}$$

Using Eq. (117) we have $q > \sqrt{n}$ and $\log q > (\log n)/2$. With Eq. (87), we get

$$\frac{q}{\log q} < \frac{1.388\sqrt{n\log n}}{\log q} < \frac{1.388\sqrt{n\log n}}{\frac{1}{2}\log n} = 5.552\frac{\sqrt{n\log n}}{2\log n}.$$

By noticing that $0.002\times 5.552 < 0.02$, we deduce from Eq. (122) that, for $n \geqslant 6.6\times 10^{21}$, we have

$$q = P^+(h(n)) > \sqrt{n\log n}\left(1 + \frac{\log_2 n - 1.18}{2\log n}\right),$$

which ends the proof of Theorem 50.

$\square$

Theorem 51. *(i) There exists two positive constants C and a such that, for $n \geqslant 2$,*

$$P^+(h(n)) \geqslant \sqrt{\mathrm{Li}^{-1}(n)} - C\sqrt{n}\exp(-a\sqrt{\log n}). \tag{123}$$

(ii) Under the Riemann hypothesis, there exists a positive constant C' such that

$$P^+(h(n)) \geqslant \sqrt{\mathrm{Li}^{-1}(n)} - C'n^{1/4}(\log n)^{9/4}. \tag{124}$$

Proof. This is similar to the proof of [4, Theorem 9.2] and follows by using Eqs. (20), (22), and (11), and the lower bounds of $\log h(n)$ given in Eqs. (65) and (67). $\square$

6.5 Comparison of $P^+(h(n))$ and $\log h(n)$

Lemma 52. *There exist infinitely many prime numbers p such that $\theta(p) < p$ and infinitely many prime numbers p such that $\theta(p) > p$.*

Proof. This is [4, Lemma 10.1], based on Littlewood's oscillation theorem. $\square$

Theorem 53. *There exist infinitely many values of n such that $P^+(h(n)) > \log(h(n))$ and infinitely many values of n such that $P^+(h(n)) < \log(h(n))$.*

Proof. Let us consider the values n belonging to the sequence (σ_k). For such an n, $n = p_1 + p_2 + \cdots + p_k$, we have (cf. Eq. (8))

$$h(n) = N_k, \quad \log h(n) = \theta(p_k), \quad P^+(n) = p_k,$$

so that $P^+(h(n)) - \log h(n) = p_k - \theta(p_k)$. By Lemma 52, there are infinitely many values of k for which this quantity is positive, and infinitely many values of k for which it is negative. $\square$

Proposition 54. *Let p_{k_0} be the smallest prime such that $\theta(p_{k_0}) > p_{k_0}$ holds, and n_0 the smallest integer such that $P^+(h(n_0)) < \log h(n_0)$. Then we have*

$$\sigma_{k_0-2} + p_{k_0} \leqslant n_0 < \sigma_{k_0}. \tag{125}$$

Proof. From Eqs. (6) and (8), we get

$$\log h(\sigma_{k_0}) = \theta(p_{k_0}) > p_{k_0} = P^+(N_{k_0}) = P^+(h(\sigma_{k_0}),$$

which proves the upper bound $n_0 \leqslant \sigma_{k_0}$.

Now, let n satisfy $n < \sigma_{k_0-1}$ so that $k = k(n)$ satisfies $k < k_0 - 1$, $\sigma_k \leqslant n < \sigma_{k+1}$, $\theta(p_k) < p_k$ (as $\theta(p_k)$ is transcendental, it cannot be equal to p_k) and $\theta(p_{k+1}) < p_{k+1}$.

If $\sigma_k \leqslant n < \sigma_{k+1} - p_k$, by Proposition 7 (i), we get

$$\log h(n) = \log N_k = \theta(p_k) < p_k = P^+(N_k) = P^+(h(n)).$$

If $\sigma_{k+1} - p_k \leqslant n < \sigma_{k+1}$, by Eq. (9) and Proposition 7 (ii), we have

$$\log h(n) \leqslant \log N_{k+1} = \theta(p_{k+1}) < p_{k+1} \leqslant P^+(h(n))$$

which shows $n_0 \geqslant \sigma_{k_0-1}$.

It remains to study the case $\sigma_{k_0-1} \leqslant n < \sigma_{k_0-2} + p_{k_0}$. By Proposition 7 (i), we get $\log h(n) = \theta(p_{k_0-1}) < p_{k_0-1} = P^+(N_{k_0-1}) = P^+(h(n))$. □

Corollary 55. *For* $n \leqslant 11\,896\,693\,289\,932\,185\,243\,249$*, we have* $\log h(n) < P^+(h(n))$.

Proof. The value of p_{k_0} is still unknown. From Eq. (23), we know that $p_{k_0} \geqslant p_{k_1} = 800000000047$, the smallest prime exceeding $8{\cdot}10^{11}$. Therefore, we have $p_{k_1-2} = 799999999889$, $\sigma_{p_{k_1-2}} = 11896693289132185243203$ and

$$n_0 \geqslant \sigma_{k_1-2} + p_{k_1} = 11\,896\,693\,289\,932\,185\,243\,250.$$

□

Remark 56. There are 3 272 numbers $\leqslant 10^6$ such that $P^+(g(n)) > \log g(n)$ (cf. [4, §10]).

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