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A Multidimensional Cesáro Type Identity and Applications

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Abstract

We generalize a well-known identity due to Cesáro and get some asymptotics on some multivariate gcd-sum functions. Several examples when the functions are completely multiplicative are given.

1 Introduction

In 1885, Cesáro [2] proved the following identity.

Lemma 1. *Let f be an arithmetic function and φ denote Euler's totient function. For every positive integer n, we have*

$$\sum_{i=1}^{n} f(\gcd(i,n)) = (f \star \varphi)(n).$$

This follows from

$$\sum_{i=1}^{n} f(\gcd(i,n)) = \sum_{d \mid n} f(d) \sum_{\substack{k \leqslant n/d \\ \gcd(k,n/d)=1}} 1 = \sum_{d \mid n} f(d) \varphi\left(\frac{n}{d}\right) = (f \star \varphi)(n)$$

where $(f \star g)(n) = \sum_{d \mid n} f(d) g(n/d)$ is the usual Dirichlet convolution product. It should be mentioned that such an identity also occurs with some other convolution products where the summation is over some subset of the set of the divisors of n. For instance, Tóth [3] showed that

$$\sum_{i \in \mathrm{Reg}(n)} f(\gcd(i,n)) = (f \odot \varphi)(n)$$

where $f \odot g$ is the unitary convolution product and where the notation $i \in \mathrm{Reg}(n)$ means that $1 \leqslant i \leqslant n$ and there exists an integer x such that $i^2 x \equiv i \pmod{n}$.

In this work, we intend to generalize Lemma 1 to higher dimensions and present some applications to various common number-theoretic functions.

The following result is easily deduced in a similar fashion to Lemma 1.

Proposition 2. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : (\mathbb{Z}_{\geqslant 1})^r \longrightarrow \mathbb{C}$ be an arithmetic function of r variables. For any $n \in \mathbb{Z}_{\geqslant 1}$*

$$\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n), \dots, \gcd(i_r,n)\right) = \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1, \dots, d_r\right) \varphi(n/d_1) \cdots \varphi(n/d_r).$$

Proof. Similarly as in Lemma 1, we have

$$\begin{aligned}
& \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n), \dots, \gcd(i_r,n)\right) \\
= \; & \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1, \dots, d_r\right) \sum_{\substack{k_1 \leq n/d_1 \\ \gcd(k_1, n/d_1)=1}} \cdots \sum_{\substack{k_r \leq n/d_r \\ \gcd(k_r, n/d_r)=1}} 1 \\
= \; & \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1, \dots, d_r\right) \varphi(n/d_1) \cdots \varphi(n/d_r)
\end{aligned}$$

as announced. ∎

2 Applications

2.1 Main result

Theorem 3. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : \mathbb{Z}_{\geqslant 1} \longrightarrow \mathbb{C}$ be a completely multiplicative function such that, for any prime number p, $|f(p)| \leqslant 1$. Set $\theta := \max_{p \,\mathrm{prime}} |1 - f(p)|$ and assume $\theta r \geqslant 1$. Then, for any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n)\right) \cdots f\left(\gcd(i_r,n)\right) = C_r(f)\, x^{r+1} + O\left(\left(x \log^{\theta} x\right)^r\right)$$

where

$$C_r(f) := \frac{1}{r+1} \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1 - \left(1 - \left(\frac{f(p)}{p}\right)^{\alpha}\right) \frac{1 - f(p)}{p - f(p)}\right)^r\right). \tag{1}$$

This result immediately yields the following examples.

Corollary 4. *Let $r \in \mathbb{Z}_{\geqslant 1}$.*

1. *For any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \frac{1}{\gcd(i_1,n)\cdots\gcd(i_r,n)} = C_r x^{r+1} + O\left((x\log x)^r\right),$$

where

$$C_r = \frac{1}{r+1} \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\left(1-\frac{1-p^{-2\alpha}}{p+1}\right)^r\right).$$

2. *Let λ be the Liouville function. Then for any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \lambda\left(\gcd(i_1,n)\right)\cdots\lambda\left(\gcd(i_r,n)\right) = C_r x^{r+1} + O\left(x^r(\log x)^{2r}\right),$$

where

$$C_r = \frac{1}{r+1} \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\left(1-\frac{2\left(1-(-1)^\alpha p^{-\alpha}\right)}{p+1}\right)^r\right).$$

3. *Let ℓ be an odd prime number, $(\cdot/\ell)$ be the Legendre symbol modulo ℓ and define* [1]

$$\Omega_\ell(n) := \sum_{\substack{p \mid n \\ (p/\ell)=-1}} 1 \quad \text{and} \quad \lambda_\ell(n) := (-1)^{\Omega_\ell(n)}.$$

Then for any x sufficiently large

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \lambda_\ell\left(\gcd(i_1,n)\right)\cdots\lambda_\ell\left(\gcd(i_r,n)\right) = C_r x^{r+1} + O\left(x^r(\log x)^{2r}\right),$$

where

$$C_r = \frac{1}{r+1} \prod_{(p/\ell)=-1} \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\left(1-\frac{2\left(1-(-1)^\alpha p^{-\alpha}\right)}{p+1}\right)^r\right).$$

4. *Let $q > 1$ be an integer and χ be a real non-principal Dirichlet character modulo q. Then for any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \chi\left(\gcd(i_1,n)\right)\cdots\chi\left(\gcd(i_r,n)\right) = C_r x^{r+1} + O\left(x^r(\log x)^{2r}\right),$$

where

$$\begin{aligned} C_r &= \frac{1}{r+1} \prod_{\chi(p)=-1} \left(1-\frac{1}{p}\right) \left(1+\sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1-\frac{2\,(1-(-1)^\alpha p^{-\alpha})}{p+1}\right)^r\right) \\ &\quad \times \prod_{\chi(p)=0} \left(1-\frac{1}{p}\right) \left(1+\frac{1}{p}\left(1-\frac{1}{p}\right)^{r-1}\right). \end{aligned}$$

2.2 Average order of certain multiplicative functions

The proof of Theorem 3 uses the following tool, which is of independent interest.

Proposition 5. *Let F be a multiplicative function for which there exists $\kappa \geqslant 1$ such that, for all prime powers p^α*

$$|(F \star \mu)(p^\alpha)| \leqslant \kappa p^{-\alpha}.$$

Then

$$\sum_{n \leqslant x} F(n) = x \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty} \frac{F(p^\alpha)}{p^\alpha}\right) + O\left((\log x)^\kappa\right).$$

Proof. Let $G := F \star \mu$. The function G is multiplicative and satisfies

$$|G(n)| \leqslant \frac{\kappa^{\omega(n)}}{n} \quad (n \in \mathbb{Z}_{\geqslant 1}).$$

For all $x \geqslant 1$, we then have

$$\sum_{d \leqslant x} \frac{|G(d)|}{d} \leqslant \sum_{d \leqslant x} \frac{\kappa^{\omega(d)}}{d^2} \ll 1,$$

so that the series $\sum_{d \geqslant 1} G(d)/d$ is absolutely convergent and one may check that

$$1+\sum_{\alpha=1}^{\infty} \frac{G(p^\alpha)}{p^\alpha} = \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty} \frac{F(p^\alpha)}{p^\alpha}\right) \quad (p \text{ prime}).$$

Hence

$$\begin{aligned} \sum_{n \leqslant x} F(n) &= \sum_{n \leqslant x} (G \star \mathbf{1})(n) = \sum_{d \leqslant x} G(d) \left\lfloor \frac{x}{d} \right\rfloor \\ &= x \sum_{d \leqslant x} \frac{G(d)}{d} + O\left(\sum_{d \leqslant x} |G(d)|\right) \\ &= x \sum_{d=1}^{\infty} \frac{G(d)}{d} + O\left(\sum_{d \leqslant x} |G(d)| + x \sum_{d > x} \frac{|G(d)|}{d}\right) \\ &= x \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^\alpha}\right) + O\left(R(x)\right) \end{aligned}$$

say, and by partial summation, we finally get

$$\begin{aligned} R(x) &\ll \sum_{d \leqslant x} |G(d)| + x \int_x^\infty \left(\sum_{d \leqslant t} |G(d)| \right) \frac{\mathrm{d}t}{t^2} \\ &\ll \sum_{d \leqslant x} \frac{\kappa^{\omega(d)}}{d} + x \int_x^\infty \left(\sum_{d \leqslant t} \frac{\kappa^{\omega(d)}}{d} \right) \frac{\mathrm{d}t}{t^2} \\ &\ll (\log x)^\kappa + x \int_x^\infty (\log t)^\kappa \, \frac{\mathrm{d}t}{t^2} \ll (\log x)^\kappa \end{aligned}$$

as required. □

2.3 Proof of Theorem 3

Let f satisfy the hypotheses of Theorem 3 and define

$$F_r(n) := \left(\frac{(\varphi \star f)(n)}{n} \right)^r \quad (n, r \in \mathbb{Z}_{\geqslant 1}) \,.$$

We start with the following lemma.

Lemma 6. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : \mathbb{Z}_{\geqslant 1} \longrightarrow \mathbb{C}$ be a completely multiplicative function such that, for any prime p, $|f(p)| \leqslant 1$. Set $\theta := \max\limits_{p\,\text{prime}} |1 - f(p)|$. For any prime powers p^α and any $r \in \mathbb{Z}_{\geqslant 1}$*

$$|(F_r \star \mu)(p^\alpha)| \leqslant \frac{\theta r}{p^\alpha}.$$

Proof. We first note that

$$\left| 1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right| \leqslant 1 \quad (p \text{ prime}, \; \alpha \in \mathbb{Z}_{\geqslant 0}) \,. \tag{2}$$

Now for any prime powers p^α, straightforward computations show that

$$F_r(p^\alpha) = \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right)^r$$

so that

$$\begin{aligned} & |(F_r \star \mu)(p^\alpha)| \\ = \; & \left| \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right)^r - \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^{\alpha - 1} \right) \frac{1 - f(p)}{p - f(p)} \right)^r \right|. \end{aligned}$$

Note that

$$|(1-a)^r - (1-b)^r| = \left|(b-a)\sum_{k=0}^{r-1}(1-a)^k(1-b)^{r-k-1}\right| \leqslant r\rho^{r-1}|b-a|$$

whenever $|1-a|,\ |1-b| \leqslant \rho$ $(\rho > 0)$ so that, using (2)

$$|(F_r \star \mu)(p^\alpha)| \leqslant r\left(\frac{|f(p)|}{p}\right)^\alpha \left|1 - \frac{1}{f(p)}\right| \leqslant \frac{\theta r}{p^\alpha}|f(p)|^{\alpha-1} \leqslant \frac{\theta r}{p^\alpha}$$

as asserted. □

We now are in a position to prove Theorem 3.

Proof of Theorem 3. From Lemma 6, we may apply Proposition 5 to the function F_r and get

$$\sum_{n \leqslant x} F_r(n) = (r+1)\, C_r(f)\, x + O\left((\log x)^{\theta r}\right)$$

where $C_r(f)$ is defined in (1). Now from Proposition 2, we get

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f(\gcd(i_1, n)) \cdots f(\gcd(i_r, n)) = \sum_{n \leqslant x} (\varphi \star f)^r(n) = \sum_{n \leqslant x} n^r F_r(n)$$

and we conclude the proof using partial summation. □

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